THE NEW

ROYAL ROAD TO GEOMETRY,

AND

FAMILIAR INTRODUCTION TO THE MATHEMATICS.

PART I.

ELEMENTS OF GEOMETRY ABRIDGED.

Containing the whole Subſtance of Euclid's firſt ſix, the eleventh and twelfth Books; with many other, uſeful and valuable, Theorems; treated in the moſt brief, eaſy, and intelligent manner; being an attempt to render the knowledge of that moſt uſeful and neceſſary Science more general.

With Notes interſperſed, critical, explanatory, and inſtructive.

BY THOMAS MALTON, SEN.

To which is annexed, an Appendix, on the Theory of Menſuration of Superficies and Solids, as deduced from the Elements; with ſome Properties of Ellipſes demonſtrated.

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M DCC XCIII.

MVSEVM
BRITAN
NICVM

TO THE

PUBLIC.

GENTLEMEN.

THE firſt Edition of the Royal Road to Geometry having been well received, and approved by you, I therefore requeſt your Sanction to this new, enlarged, and much improved Edition; which, I flatter myſelf, will be found more deſerving your Patronage and Protection. 'Tis to You I dedicate it; to the Public I look for that Encouragement and Support which it may merit; I requeſt no more, I have no right to expect more; nor can all I might be diſpoſed to ſay in its behalf have any weight, ſo as to biaſs, much leſs to impoſe on the Judgment of the candid and impartial Public.

I ſhall, I muſt, therefore, leave to the Deciſion, to the Tribunal of the Public to ſtamp a Merit on it, or otherwiſe, as it may be found deſerving; not doubting of Candour, I am confident that their Judgment will be impartial and free from Prejudice.

In conformity with Cuſtom, I was perſuaded to dedicate the firſt Impreſſion of this Work to a Gentleman whom I knew nothing of, not ſo much as by Reputation, not perſonally; conſequently, I could have no Inducement to be laviſh in Encomiums on his great Endowments. I believe he was a Friend to the Perſon who adviſed me, being at the Head of that Department, in which he held a valuable Employment; and, that 'twas in his Power to ſerve, and

be a Friend to me. I acquieſced in it, accordingly; what could I leſs? being perſuaded, that it is the firſt Motive, and perhaps the beſt, if not the only Reaſon that can be given, for Dedications in general.

I look on all, or the far greater part of Dedications to great Men, or Men in Power, or Truſt, as fulſome Panegyrics on the many amiable Virtues and extraordinary Qualities, which the Right Honourable, or Moſt Noble Perſon might be imagined to inherit; who, having done ſome trifling, public Action, by which the Author happened to be benefited, though without any Expence to the public ſpirited Gentleman, or friendly Intention to the Author, muſt therefore (though his Virtues never ſhone very conſpicuouſly) be a moſt worthy, if not a truly benevolent Man.

I ſhall allow them to put the beſt, or what Conſtruction they pleaſe on ſuch Acts of Friendſhip, and give the Adulation due, to ſuch exalted Virtues. I neither ſeek for nor expect the Favour, or Patronage, of Individuals; but, from the Public, at large, I truſt that I ſhall find a Reward for my Study and Labour, in an Endeavour to render the firſt, the moſt valuable, and generally uſeful of all the mathematical Sciences eaſy and familiar; by adapting it to the Capacities of all who have any reliſh for, or a Deſire to be acquainted with, and converſant in Geometry.

I am, with due Reſpect,
The PUBLIC's obedient, humble Servant,
THOMAS MALTON.

PREFACE.

THE reception which the firſt Impreſſion of this Work met with, from the Public, might have been productive of a ſecond, long ago; no greater proof of which need be given, than the great demand there has been for it, of late years, and not one to be had, at any price, but by accident, for theſe ſeven years paſt, or more: The price was Half a guinea, ſtitched, whilſt moſt other octavo works, on the Subject, are but five or ſix ſhillings, bound. It was hoped that the firſt Edition, being the firſt eſſay of an obſcure perſon, wholly unknown to the world, in the field of Science, would not be too haſtily cenſured by thoſe more converſant in mathematical ſcience, on account of his having dared to go out of the beaten track, ſomewhat; at leaſt, before they had given it a candid peruſal, and fair inſpection; which having paſſed that public Ordeal, without cenſure, it is now preſumed, that this ſecond Edition, on a greatly improved plan, needs no apology from the Author, for obtruding it on the Public in a new dreſs, at this time; whoſe unſettled ſituation for ſeveral years paſt, and his determination to reprint it on a plan, which would reduce the price (conſidering it as two volumes) have prevented a ſecond Edition making its appearance ſooner.

Having made (by ſelf-application) ſome proficiency in Geometry, and other mathematical Sciences my favourite ſtudies, I am much ſurprized at the deficiency of our common Schools, for the cultivation of Youth who are intended to fill the middle Sphere of life, in mechanic Trades, &c. Probably, the blame may, with propriety, be attributed to the Legiſlature, for not ſetling a better and more uſeful mode of common Tuition; and an enquiry into the abilities of thoſe who undertake to cultivate

the Minds of Youth, of both ſexes. They, almoſt in general, purſue (in boy's ſchools) one common plan or track of learning. After the firſt and neceſſary branches, Reading, Writing, and Arithmetic (which, indeed, might be acquired in half the time it uſually is); if the Pupil make a progreſs through Arithmetic in any reaſonable time, the next ſtep is the grammar of the Latin tongue, through which he often labours to little purpoſe.

It may reaſonably be aſked, why this Time is ſo ſpent? which might be employed to better purpoſe. For, what have Mechanics to do with Latin, more than a Porter or Carman has with Logic? it may indeed make him a pedantic Coxcomb, but can never be of real uſe in his Profeſſion; even, ſuppoſe he make a tolerable proficiency, it can anſwer no other purpoſe but to ſet him above his Employment, without being of any ſervice in it.

On the contrary (ſuppoſing no particular avocation intended for the ſtudent) if, inſtead of Latin, Geometry and Menſuration, &c. were introduced in all public, common Schools, I would aſk any perſon, who has conſidered theſe things, and their uſes in life, which is the moſt likely to turn to the pupil's advantage? Is there a mechanic profeſſion in which Geometry or Menſuration may not be of ſome uſe? in ſome particular ones, it is well known to be of the greateſt, the foundation of it; and yet, although the Youth were particularly intended for that Profeſſion, it was, perhaps, never once ſo much as thought on; until, by too late experience, he finds the want of it: I mean all ſuch Trades as relate particularly to Building, in general. Had ſome Builders, whom I have known, been converſant in Mathematics, or only in Plane Geometry; inſtead of ploding on in a low ſphere of Employment, they would, if their natural, mechanical genius had been properly cultivated, have filled a more elevated ſtation.

If only practical Geometry be well inculcated, it will be of more ſervice, in common life, than a proficiency in Latin can poſſibly be. If the young pupil have a genius, and diſcover a reliſh for mathematical ſcience, let him go through the Elements. Having acquired a competent ſhare of knowledge therein, it will then be time to conſider, what particular Profeſſion he either wiſhes or is deſtined for: In chooſing of which, regard ought particularly to be had to the Boy's genius and diſpoſition, which will, ere this, be diſcernable. Inſtead of aiming to drive a uſeleſs dead Language into a ſtupid Boy, let the practice of Geometry be introduced in its ſtead; there is ſomething entertaining to the Mind, better than burdening the memory with *Propria quæ maribus*, and other Rules of the Latin grammar.

Accuſtoming boys, early, to handle and uſe Compaſſes and other drawing utenſils, in delineating the diagrams, as they proceed, will make it an entertainment to them of great utility, rather than a perplexing ſtudy; and make them more readily perceive the Demonſtration; which, under a proper tutor, they would ſoon have a reliſh for, and then they would proceed with pleaſure: beſides, it is an introduction to Drawing. A foundation being laid in Geometry, they are then qualified to purſue any branch of the Mathematics, ſuitable to the Profeſſion they are intended for; ſuch as Menſuration, Trigonometry, Navigation, Gunnery, Fortification, Architecture (naval, military, or civil) Surveying, &c. In ſhort, all the uſeful and neceſſary employments, in the mechanic Arts, have their foundation in this moſt neceſſary Science; which, being acquired, will, moſt probably, make its poſſeſſor ſtrike out of the common and vulgar track, and make him eminent and diſtinguiſhable, in whatever profeſſion he is caſually fixed in; as he will have laid a ſolid and permanent foundation, in theory, whereon may, very probably, be erected a laſting monument to his future Fame.

In the works of ſome able Geometricians, I find it treated in a manner ſcarcely intelligible to a beginer, from the too great uſe of Symbols and appearance of Algebra; others would be more approved, if they were not ſo prolix, dwelling too much on ſelf-evident Propoſitions, or tedious Diſſertations to little purpoſe, on mere Axioms †. Some have ſo mutilated the whole Elements ‡, others ſo curtailed them §; ſome have ſo varied the Order, according to Euclid, whilſt others have ſcarce preſerved any order at all; inſomuch that, when a ſtudent has acquired ſome knowledge of it, from them, and he afterwards meets with Euclid's Elements, according to Keil (or Cunn) Profeſſor Simſon, Barrow, and others, he can ſcarcely think it the ſame Subject. But what is, moſt of all, to be lamented; the number of trivial and puerile Abridgments, of late years, by pedantic, ſelf-ſufficient Teachers; who imagine, that juſt ſo much as they are capacitated to teach is ſufficient for any perſon to know, which induces them to expoſe their trifling Works, and themſelves, to the world's Cenſure. I ſay, it is much to be lamented, that a Boy of genius, and talents for the ſtudy of it, ſhould be obliged to wade through their ill digeſted Publications; by which, he may, probably, ever after, be limited in his ideas of it, and his acquiſitions in Geometry protracted, by a partial knowledge obtained from them; according to the contracted notions, or caprice of his ſuperficial Tutor, or greatly defective Author.

As the knowledge I have acquired, in ſuch ſtudies, has been entirely from Books, without other aſſiſtance, I may, perhaps, have ſeen deficiencies or ſuperfluities in them, better than thoſe who have ſtudied under a Tutor; who are

† Cunn (or Keil) Profeſſor Simſon, Williamſon (a late production) &c. tranſcribers of Euclid. ‡ Emmerſon, T. Simpſon, and others. § Tacquet, Pardie, Le Clerc, &c.

apt to give their aſcent, too often, on credit. Perhaps I ſhall be thought raſh, or vain, in preſuming to cenſure, as uſeleſs and redundant, ſeveral Propoſitions in that able and firſt of Geometricians, Euclid; and with his ſtating the Premiſes of others, particularly, the 7th of Book I. I will venture to affirm, that very few, who are not already tolerably verſed in Geometry, can conceive what the propoſition means, or attempts to prove; who, from inſpection, only, of a proper Figure, might be fully convinced of the truth of the premiſes.

I have endeavoured to render every Propoſition clear in the premiſes, ſo as to be intelligible to any capacity; and have omited none that are uſeful, or neceſſary for attaining the moſt eſſential. I will be bold to aſſert, that thoſe who cannot, from this work, acquire a competent knowledge of all that is requiſite, in common Geometry, will never be able to attain it at all. Where there is not a Capacity adequate to the taſk, they had better deſiſt from the undertaking. An equal talent is not given to all; ſuch as have not a talent, and a natural propenſity to it, muſt not expect to acquire any tolerable ſhare of knowledge, in Geometry, 'tis aiming at impoſſibilities.

But, not to deter any from the ſtudy of it, I would obſerve to them, that, Geometry is nothing more than the elements and principles of logical reaſoning, of Common ſenſe; without which, the ordinary affairs in life could not be tranſacted; and have the appearance of difficulty and intricacy, only from the artificial arrangement of thoſe ſimple and familiar Truths, in regular order; and being ſomewhat diſguiſed, by dreſſing it with a few technical terms, which will ſoon become familiar to them.

I ſhall ever be of opinion with Tacquet, and ſome others, that, to attempt demonſtration, of Propoſitions which are ſelf-evident, is involving Truth, which, in itſelf, is clear and conſpicuous, in darkneſs, or perplexity. I have always found more difficulty in demonſtrating, to

another perſon, ſelf-evident Propoſitions, than the moſt intricate of others; and, when done, have only confuſed a more perfect Idea which the pupil had of it, from inſpection of the Figure. If the knowledge of ſeveral ſimple Properties of figures be acquired, which are neceſſary to illuſtrate, to enforce, or prove ſuch as are more complex, is not the eaſieſt method the moſt eligible? barely to be told ſeveral properties of Triangles is ſufficient for attaining the reſt. Some Propoſitions are merely Corollaries; certain conſequences deducible from preceeding propoſitions.

The 16th and 17th Propoſitions, Book I, I deem entirely uſeleſs; for ſince (in the 32nd) the external angle is proved to be equal to the two remote angles of the triangle; and, ſince all the three angles, of every triangle, are there proved equal to two right ones; it ſeems abſurd to prove, before hand, that any two of the angles are leſs than two right angles, and that the external angle, is greater than either of the remote ones; as if one ſhould undertake, firſt, to prove that five is greater than either two, or three, which, ſoon after muſt be proved equal to them both. I have, therefore, made free, to alter the diſpoſition of the Elements of the firſt Book; that, by a different arrangement, omiting ſome, and tranſpoſing the places of others, thoſe more ſublime Propoſitions, which depend on them, may be briefly and clearly demonſtrated.

Dr. Keill, in the Preface to his tranſlation of Commandine, and alſo Mr. Cunn, think it unpardonable in Tacquet, to omit the demonſtrations of the 5th Book; Cunn aſſerts, that not one Demonſtration in the 6th, 11th and 12th, can be obtained without it. But, I muſt beg his pardon, for differing from him in opinion, and am, myſelf, a living witneſs againſt ſuch an aſſertion; for, I do aver, that, without any other demonſtration of the 5th Book, than what Tacquet has delivered, I was enabled to go through all the other; and, without vanity, I think

I underſtand them; but, of that, let this Treatiſe bear witneſs. I never had patience to go through the tedious demonſtrations, delivered in 25 Propoſitions, of the 5th Book of his Euclid; had it firſt fallen in my way, I ſhould (I verily believe) have lain it aſide before I had got through a fourth part of it; yet, I muſt acknowledge, that, Tacquet is much too brief. I cannot think it abſolutely neceſſary, in order to attain a competent knowledge in Geometry, and of Proportion in particular, to tread exactly in the ſame ſteps with Euclid; therefore, I have made free to deviate, where I preſume to think it for the better, a readier and clearer way of inveſtigating.

My aim has been to bring the whole Elements into leſs compaſs, by abridging it of every redundancy, and avoiding that tedious prolixity, which is in many Authors, on the ſubject; in dwelling too long on ſuch propoſitions as are clear in themſelves; by that means, to render the ſtudy of Geometry more pleaſant and entertaining. 'Tis enough to deter any one from the purſuit of a Science, who finds, in the rudiments of it, ſo much perplexity; for, I am ſenſible, that, unleſs a perſon has a ſtrong inclination to it, and a good natural capacity, 'tis not a pleaſing ſtudy, at the firſt, until they begin to feel the Truths it inculcates; and am therefore of opinion, that, the eaſier it is made, in the begining, 'tis the more engaging, for young ſtudents to perſevere.

Can there need demonſtration, that any two Sides of a Triangle, are greater than the third? is there a perſon ſo ignorant as not to know it? it is implanted in us by nature; every common Porter knows it, or practices it every day. Who ever ſaw one of them traverſe two ſides of a Square, when he could croſs the diagonal? and why is it, but, becauſe he knows it to be ſhorter than the two Sides? Is it not obvious, that the greateſt angle of every triangle, muſt be oppoſite to the greateſt ſide? and can there be any need to demonſtrate the converſe? that the

greateſt ſide ſubtends, or is oppoſite to the greateſt angle; is not one the conſequence of the other? it is trifling to no purpoſe; all converſe Propoſitions may be Corollaries to the former. If two ſides of a triangle, equal to two ſides of another triangle, contain a greater or leſs angle, the remaining ſide will be greater or leſs? Theſe, and ſeveral more, are obvious and clear, from a bare inſpection of the Figure, nay almoſt without it; 'tis enough to I told they have ſuch properties, which will be readily granted, as Axioms, being ſelf-evident; not to exhauſt the patience of the ſtudent, with a tedious demonſtration of what he ſaw clearer before; if the thing be ſeen or known, what need there more? is it intended to perplex, only, where it can be of no uſe? to diſcourage the beginer, before he is able to perceive the ſublime Truths it contains? Yet, I do not omit theſe entirely, becauſe the whole Elements depend on them; but I have endeavoured to treat them in as ſimple a manner as the nature of the Subject will admit of; if it be conciſe, I doubt not I ſhall be excuſed, having ſaid enough.

BUT, as I think I have ſaid enough, in this place, I ſhall ſtraightway proceed to the Subject; through which, if the Reader be inclined to follow, with intention to learn what it contains, I am much miſtaken if he loſe his labour; and for ſuch as peruſe it only with intention to cavil, I hope they will be diſappointed, and find but little to cavil at. I am not ſo vain as to ſuppoſe it is without defect, or that it will pleaſe all, for that I know is impoſſible; but, if I have made it intelligible, and uſeful to common capacities, it is what I aimed at, and ſhall reſt ſatisfied in the ſuppoſition that my labour is not entirely loſt.

Correct and well adapted Schemes are of conſequence, in which, I have been particular; and have carefully reviſed the Letter preſs, ſo that, I hope, few errors have eſcaped my obſervation. For, Errors, in miſplacing the References to the ſchemes, and ſometimes omiting them entirely (which is

better of the two) is almoſt unpardonable in mathematical works; having frequently experienced the perplexity it occaſions; eſpecially, when the ſubject is new to the ſtudent. But, if any ſhould remain unnoticed, I hope the candid reader will impute it to human fallibility, and not quarrel with it on that account, for I am confident he will not meet with many.

The alterations made in this ſecond Edition are conſiderable, in the arrangement of the Elements very little; but, on the whole, it is much improved. In the ſecond Book, more general Demonſtrations are given to the firſt eight Theorems; the fourth is abridged of what is merely practical, being transferred to the ſecond Part. The doctrine of Proportion (in the fifth) is better defined and illuſtrated, the Theory is the ſame, nearly; the theory of Menſuration, in the Appendix, is greatly improved. There are little or no alterations in the other Books, ſave the tranſpoſition of the 17th and 19th Theorems, of the firſt; excepting the abridgements of ſome, and more elegant demonſtrations of others.

Having made it in two diſtinct Books, theoretic and practical, great part of what was introductory, in the former Work, is transfered to the ſecond Part, in which are many additional valuable Problems, of great uſe to Mechanics, in general, ſelected from various other works, and otherwiſe collected; nothing trivial has any place there, though ſome of them are merely ſpeculative; at leaſt, their uſes are not immediately apparent, to me, others may find them. There are many to whom it may be of great uſe, who either have not leiſure or inclination to go through, or a ready comprehenſion of the elementary part, which diſcourages them from proceeding; but, thoſe who are not inclined to take the whole on credit will readily obtain demonſtration, from the Theorems referred to.

ADVERTISEMENT.

IT may appear ſomewhat ſtrange, to call ſo copious a Work as this, an Abridgment; which, on examination, and comparing with thoſe who have tranſcribed Euclid, will be found to be a great abridgment of the Elements; both in the number of Theorems and length of the Demonſtrations.

But, they will find in this Work, particularly in the 2nd, the 3rd, the 6th, 7th and 8th Books, ſeveral valuable Theorems (in the whole, forty nine) which are not Euclid's; ſome of which are elementary, and neceſſary to be known.

The practical Part, with the Introduction thereto, is now made an entire Work, of itſelf, with a ſuitable Appendix.

The Applications, Notes, and Remarks, neceſſary to a young Student, and the length of ſome (in the fifth Book particularly) have ſwelled the Work, conſiderably. The much greater number of Terms defined, various ways of demonſtrating ſome Theorems, the Preamble to each Book, alſo, the manner of printing, in order to have the Figure in view, ſo that, many Pages are not near full; all, which, have concured to extend the Work, greatly; but, although the Elements are much abridged, it contains the whole in Subſtance. The full and perfect knowledge of all, and more than is contained in Euclid, may be acquired in a third part of the Time, and with more Eaſe, Pleaſure, and Satisfaction to the Learner (having no *Aſſes Bridge** to go over, but the Road ſmooth and even); conſequently, it may, with great propriety, be called an Abridgment.

Having deviated greatly from Euclid, I have, in order to make it generally uſeful, put his Number after mine, to each Propoſition; alſo, by means of the Index, is ſhewn where to find any Propoſition of Euclid, which, in caſe of Reference, in other Works, may be readily turned to. I have well conſidered and digeſted every Propoſition, have carefully reviſed them over and over, with the ſtricteſt attention, and preſume to think that I have not omited the ſubſtance of any Propoſition which will ever be refered to, by Authors in any other Science.

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* The 7th Prop. I think (ſome ſay the 5th) of the firſt of Euclid, has, in the Seminaries of Erudition, been long diſtinguiſhed by the Appellation of the *Aſſes Bridge*, on which, many have foundered, and never got over; nor been able to advance one ſtep further.

AN INDEX,

SHEWING where to find the Propositions of Euclid, in the First, the Third, the Fifth, and Sixth, the Eleventh, and Twelfth Books; in cases of Reference, to Euclid, by Authors in the Mathematics.

In the Second and Fourth Books, the Propositions follow in the same Order as in Euclid.

N. B. The Problems, of Euclid, as refered to, are amongst other select Problems, in the second Part. (*S.* I. Section 1st. &c.)

Note. The Numbers in the First Column are Euclid's; opposite to them is shewn where each Proposition may be found in this Work, whether Problem or Theorem.

Book I.

Prop.	Theo.
1 Prob. 14 } 2 Prob. 2 } 3 Prob. 3	*S.* I.
4	8.
5	9.
6 — Cor. 3.	9.
8	7.
9 Prob. 12 } 10 Prob. 10 } 11 Prob. 6 } 12 Prob. 8	*S.* I.
13	1.
14 — Cor. 1.	1.
15	2.
16 and 17 —	10.
18	12.
19 — Cor. 1.	12.
20	13.
21	14.
22 Prob. 15 } 23 Prob. 4 }	*S.* I.
24 & 25 are in Cor. to	8.
26	11.
27 and 28 —	4.
29 Cor. 1 & 2.	4.
30	5.
31 Prob. 5.	*S.* I.
32	10.
33 — Cor. to	15.
34	15.
35, 36, 37 and 38. }	18.
39 & 40. Coroll.	
41	19.
42 Prob. 2.	*S.* VI.
43	17.
44 Prob. 18 } 45 Prob. 3 }	VI.
46 Prob. 16.	*S.* I.
47	20.
48 Corollary.	

Book III.

Prop.	Theo.
1 Prob. 1.	*S.* III.
2 - Ax.	5.
3	1.
4	2.
5 & 6 - Ax.	3.
7	5.
8	6.
9 — Cor. to	5.
10 - Ax.	4.
11 and 12 —	7.
13 - Ax.	6.
14	3.
15	4.
16	8.
17 Prob. 4.	III.
18 — Cor. 3.	8.
19 — Cor. 2.	8.
20	9.
21	10.
22	11.
23 & 24 are useless	
25 Pr. C. to 2.	III.
26 and 27 —	10.
28 } 29 } Cor. 2, of	3.
30 Prob. 10.	*S.* I.
31	12.
32	13.
33 Prob. 6 } 34 Prob. 5 }	III.
35	14.
36	16.
37 Converse.	

Book V.

Prop.	Theo.
1	1.
2, 3, 4, 5 and 8 of no use at all.	
7 - Ax.	4.
9 - Ax.	5.
10 - Ax.	6.
11 - Ax.	13.
12	2.
14 - Ax.	12.
15 - Ax. 7 &	8.
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AN

INTRODUCTION

TO

GEOMETRY.

DEFINITIONS of the TERMS peculiar to, and uſed in any Science, with explanatory Notes and Remarks on them, where it is neceſſary to enlarge, claim the firſt Place. Explanations of all the Abbreviations made uſe of in the Work, the next; ſome, uſed here, are explained in Grammars, and other ſchool Books, yet may not be impertinent, or ſuperfluous: Nothing ſhould be omited, that may facilitate, and elucidate the Subject treated on.

IF I have any pretence to Merit in this Work, it is in the attempt to render an apparently intricate, yet generally uſeful and moſt neceſſary, Science attainable to ordinary Capacities; and, at the ſame time, I hope, not exceptionable to thoſe of acuter talents.

I apprehend, that it will not be leſs acceptable to any, for being eaſy to be attained; to write only for Proficients is to little purpoſe. By ſuch I may, in ſome caſes, be thought rather prolix, yet, I preſume, not tedious; a repetition is ſometimes neceſſary to young Minds, and is more agreeable, in general, than turning back, which they are too frequently obliged to do; it being impoſſible to retain, in memory, all that is paſſed over on the firſt peruſal.

GEOMETRY.

ACCORDING to its original derivation, GEOMETRY ſignifies to meaſure the Earth; from GEO, the Earth, and meter, or metior, to meaſure.* It is a Science which contemplates continued QUANTITY, EXTENSION, or MAGNITUDE, abſtractedly conſidered; it teaches the uſe and properties of LINES, and ANGLES, of FIGURES, SURFACES, and SOLIDS.

GEOMETRY is in two parts, ſpeculative and practical; it demonſtrates the powers and properties of Lines, Figures, &c. in ſpeculation; from which is deduced the practical part, for various uſes, for the benefit of mankind, in mechanic Arts.

EUCLID has judiciouſly divided the Subject into Books, or Sections; each of which, treats of different Figures, or different properties of Figures, the power of Lines, Proportion, &c. which ſome Authors have thought proper to deviate from, though, I think, without a juſt reaſon for ſo doing.

It treats, in the firſt, third, fourth, and ſixth Books, of Plane Figures, and thence is called Plane Geometry; and afterwards, in the 11th and 12th of Euclid, the 7th and 8th of theſe Elements, of Planes and Solids.

Each Book contains ſundry Propoſitions; from which are deduced Corollaries, Scolia, &c. the ſignification of all which I ſhall firſt beg leave to explain or define, before I proceed with the Definitions of the more eſſential Terms, which are the ſubject of Geometry.

* Geometry is ſuppoſed to owe its origin to the overflowing of the Nile, in Egypt; by which, their Land marks being obliterated, or removed, it became neceſſary to have recourſe to Science, for determining the boundaries of each Perſon's property, when the Waters had ſubſided.

A DEFINITION is the defining or explaining, the true and full ſignification of any Term, Name, or technical Word, peculiar to, or made uſe of, in that Science of which we are about to treat; the ſignification of which not being underſtood, generally.

A PROPOSITION is either a THEOREM, propoſed to be proved or demonſtrated, contemplatively; or, it propoſes ſomething to be done, problematically or mechanically.

A CONVERSE PROPOSITION is the contrary of the other; that, which, in the foregoing, was the Concluſion, drawn from the Premiſes of it.

e. g. If a Triangle has two equal Sides, the Angles which they ſubtend are alſo equal.

The CONVERSE is, that if a Triangle has two equal Angles, the Sides ſubtending them are alſo equal.

A THEOREM is a ſpeculative PROPOSITION; an affirmation of certain properties, equality, or other proportion relative to Quantity, or Figure, propoſed to be conſidered, and demonſtrated.

A PROBLEM is a PROPOSITION, which propoſes ſomething to be done, practically or mechanically.

An AXIOM is a ſelf-evident PROPOSITION, which does not requìre to be demonſtrated. See Axioms, P. 31 & 32.

A LEMMA is a PROPOSITION, as it were by the bye, or out of courſe; ſerving, previouſly, to prepare the mind for better comprehending the Demonſtration of the following Propoſition.

I do not make uſe of Lemmas in this Work, as ſome Authors do; for, if there be a neceſſity for a Lemma, I ſee no reaſon why that Lemma is not as much a Propoſition as any other. The 7th and 16th Propoſitions of the firſt Book of Euclid, may be called Lemmas, for they are redundant Propoſitions. In various mathematical Subjects, Lemmas are frequently neceſſary, when no Propoſition can be found, in Euclid, that is directly to the point; but, in Geometry, I think them inconſiſtent.

A COROLLARY is a neceſſary CONSEQUENCE deducible from the Propoſition, already demonſtrated.

A SCOLIUM is a REMARK, or uſeful Leſſon derived from the preceding PROPOSITION.

A POSTULATE is a PETITION, or REQUEST, which is required to be granted. See POSTULATES, P. 30.

HYPOTHESIS. Whatever is ſuppoſed, or premiſed, in a PROPOSITION, is called the HYPOTHESIS or PREMISES of it; from which ſome certain Conſequence is deduced, as affirmed, and afterwards demonſtrated; called the THESIS, or AFFIRMATION.

e. g. If a Right Line, cuting two Right Lines, makes equal Angles with them both, thoſe Lines are parallel.

Here, the HYPOTHESIS is, that the Angles are equal; the CONSEQUENCE, that the Lines are parallel.

SUPPOSITION. In demonſtrating ſome THEOREMS, it is neceſſary to have recourſe, frequently, to ſuppoſe ſuch and ſuch things, which are not ſo in reality; by the abſurdity of the CONSEQUENCES, ariſing from ſuch a SUPPOSITION, a concluſion is drawn, and the Demonſtration is made evidently to appear.

Such kind of Demonſtration is called *reductio ad abſurdum*, i. e. proving it to be abſurd, or impoſſible to be, on that or any other ſuch Suppoſition; which, not being direct and poſitive, is, to many, unſatisfactory; yet, if rightly conſidered, is full, though not direct Demonſtration.

CONSTRUCTION is the contriving and conſtructing, geometrically, i. e. drawing Lines and Figures, neceſſary for making the Demonſtration appear; and muſt

always be of ſuch Figures as are already well underſtood, their Properties being previouſly demonſtrated.

DEMONSTRATION. When any thing is propoſed, or affirmed in a PROPOSITION, the Caſe is firſt ſtated and prepared, by drawing ſuch Lines, or forming ſuch a CONSTRUCTION as is neceſſary, by means of which it is to be demonſtrated; that is, the truth of the Aſſertion is made to appear, obvious, and without the leaſt doubt remaining; the performance, or operation of which, is called the DEMONSTRATION.

The three laſt Terms, being common Words in the Engliſh Tongue, may, by ſome, be thought impertinent; but, notwithſtanding the common acceptation of them is almoſt univerſal, yet, the application of them, in Geometry, requires to be explained.

DEFINITIONS

Of the eſſential and operative TERMS.

The TERMS of ART, to be defined in any Science, are Names, arbitrarily given, by the firſt Authors, or others, to certain Objects, Symbols, Figures, Marks, or Characters, poſſeſſing certain Properties, or Relations, in reſpect of Figure, Poſition, Situation, Quality, &c.

Operative Terms are, generally, technical Words, peculiar to that Science, though perhaps applicable to others; which are not of common uſe, in Language, or have different ſignifications.

The following Definitions are frequently referred to, hereafter, for *illuſtration, or proof* of what is advanced in the Propoſitions.

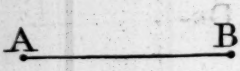
When any Figure, &c. which we are contemplating, is found to poſſeſs ſuch or ſuch Qualities, or Properties, as characterize it, we affirm it to be ſuch a Figure, as anſwers to them; or, in contemplating any Figure, given in the Premiſes, we affirm that it has ſuch or ſuch Properties, arbitrarily, by the Definition of it; and therefore, it requires no Demonſtration.

DEFINITIONS.

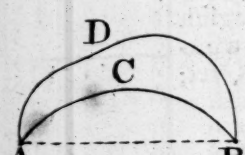
DEF. 1. A POINT is rather to be imagined than conceived, or underſtood, to be without dimenſions; therefore indiviſible, even in thought.

In Plane Geometry, it is a Mark, ſuppoſed to be made with a ſharp-pointed Needle, or fine Pen. As A.

DEF. 2. A LINE has length only, without other dimenſions, of breadth or thickneſs; ſeeing it is generated, in Idea, by the motion or flowing of a Point, which has no Dimenſions.



DEF. 3. A RIGHT LINE is the ſhorteſt that can be drawn between two given Points (A and B) uſually called a *ſtreight* Line.



DEF. 4. A CURVE, or CURVED LINE, is any other than a Right Line; either regular, as ACB; or irregular, as ADB.

Curve Lines are of various kinds, as circular, elliptic, parabolic, &c. each of which has particular properties peculiar to them, and ſome in common.

DEF. 5. A SURFACE, or SUPERFICIES, is the outer coat or boundary of Bodies, ſolid and fluid, having no regard to their ſubſtance; and therefore, it has but two dimenſions, *viz.* length and breadth. As AB and CD.

SURFACES are various; as Plane, Convex, and Concave.

Irregular Surfaces are ſuch as are not, uniformly, any of the three, but may be compounded of them all.

A convex Surface is one that is externally round or protuberant, as AB.

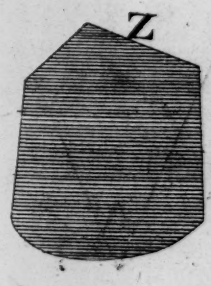
The curve of a Circle, &c. is convex, outwardly.

A concave Surface is one that is round internally, or hollow, ſuch as the inſide (X) of a round Veſſel, whither globular or cylindrical; the outſide is convex.

The curve of a Circle towards the Center is concave.

DEF. 6. A PLANE is a perfectly even and regular Surface, which is neither convex nor concave, in any part; to which, if a Right line (the edge of a ſtreight Ruler) be any how applied, it will touch the Plane in every Point.

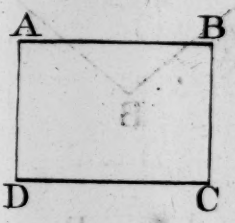
Or, if any two Points, in a Plane, be joined by a Right line, the whole Line is in that Plane.



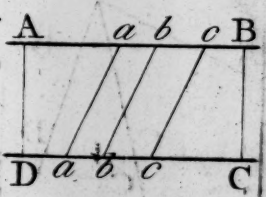
N. B. A PLANE may be of any Shape or Figure, in reſpect of its bounds or limits. As Z.

A PLANE may be conceived to be generated by the direct motion of a Right line, laterally; or whirled around on any Point in it.

If the Right line AB be moved, directly to CD, there will be generated the Plane ABCD.



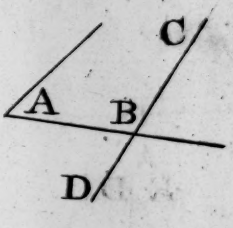
DEF. 7. PARALLEL. Right Lines are Parallel to each other, which, if produced infinitely, at either or both extremes, being in the ſame Plane, would never meet. As AB and CD.



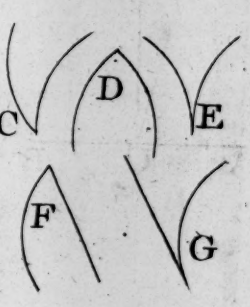
N. B. Parallel Lines are equidiſtant in every part; between which, all Right Lines, as *aa*, *bb*, and *cc*, being alſo parallel, are equal. Their diſtance aſunder is AD, perpendicular to both.

The ſame holds true in parallel Planes.

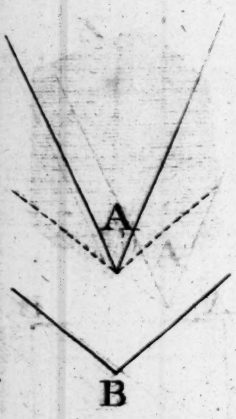
DEF. 8. A PLANE-ANGLE is formed by the meeting, or cuting of two Lines in the ſame Plane, ſo as not to fall into and conſtitute one Line. As AB and CD.



ANGLES are either right-lined, as A or B; curved, as C, D, and E; or mixed, as F and G; which are compounded of a Right Line and a curved or crooked one.



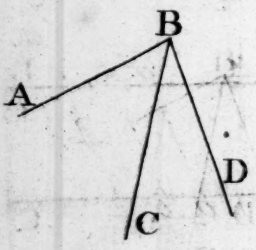
N. B. ANGLES are neither increaſed nor diminiſhed by the length of the Lines which form them; for the Angle B is greater than the Angle A, notwithſtanding the Lines which form the Angle A, are longer than thoſe of the Angle B; but, the Lines forming the Angle A are more inclined to each other.



2. Imagine the dotted Lines to repreſent the Angle B, to which the Angle A is applied; it is evident, that the Angle A is leſs than the Angle B, having a greater inclination of its Sides, or the aperture between them leſs; therefore a leſs Angle may always be contained in a greater.

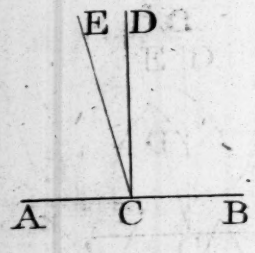
3. If the Sides of any Angle (as B) were lengthened infinitely, the Angle is not varied, if the inclination of the Lines remains the ſame; but, if you ſuppoſe them moveable, on the angular point, like a folding Rule, and are parted farther aſunder, ſo that the Sides of A approach towards the dotted Lines, their inclination will then be leſs; and conſequently, the Aperture or Opening between them greater; and therefore, the Angle is ſaid to be greater.

DEF 9. VERTEX is the angular Point in which two Lines, forming a Plane Angle, meet, and touch each other.



N. B. A ſingle Angle is uſually deſcribed by one Letter only, or other Character, as A or B, (Fig. X) but, if three or more Lines meet in a Point, then three Letters are uſed, to ſpecify the Angle ſpoken of; as ABC, CBD, or ABD; of which the middle Letter always denotes the Vertex; and means, the Angle formed by the Lines AB and BC, or CB and BD, &c. wherefore, the middle Letter, denoting the Angle, is underſtood to be twice named.

DEF. 10. PERPENDICULAR. A Right Line is perpendicular to another, when it does not incline to the other on either Side; but makes the Angle, on one Side, equal to the other. See D.14.

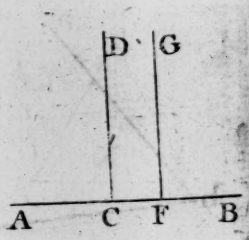


Thus; if the Angles ACD, DCB, are equal to one another, then CD is perpendicular to AB.

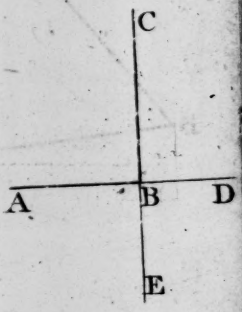
N. B. No more than one Perpendicular can be drawn from the ſame Point in a Right Line, on the ſame Side, and in the ſame Plane.

For, if any other Line, as CE, be drawn from the ſame Point, C, it cannot be a Perpendicular, but will incline to AB; and, the inclination is on the ſame Side of the Perpendicular CD; i. e. the Angle ACE is the Angle of Inclination of the Lines AB and CE.

N. B. 2. If another Right line, as FG, be drawn in the ſame Plane, from any Point, F, perpendicular to AB, on either Side, it will be parallel to CD. And conſequently, any number of Lines, perpendicular to AB, will be parallel among themſelves.



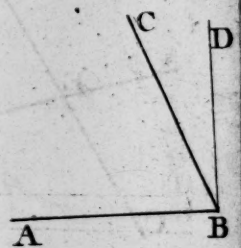
DEF. 11. A RIGHT ANGLE is formed by the meeting of two Right Lines which do not incline to each other; but, either is perpendicular to the other. As ABC.



N. B. If either Side of a Right angle be produced, drawn out beyond the Vertex, there is neceſſarily generated another Right angle; Conſequently, if both Sides are ſo drawn out, there will be generated four Right angles.

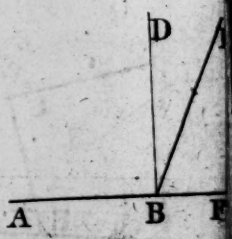
ABC is a Right angle; if AB or CB, lengthened towards D or E, make an equal Angle CBD, or ABE. If both are produced, EBD is a fourth Right angle.

DEF. 12. An ACUTE ANGLE is leſs than a Right angle.



If CB, meeting AB, falls on one Side of a Perpendicular, at that Point; the Angle ABC, being leſs than the Right angle ABD, is called Acute.

DEF. 13. An OBTUSE ANGLE is greater than a Right angle.



If BE falls on the other Side of the Perpendicular, BD; the Angle ABE is Obtuſe.

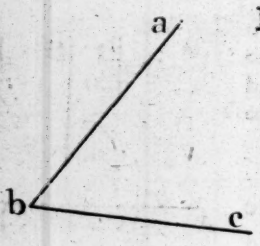
Acute and Obtuſe Angles are alſo called Oblique.

N. B. The difference, CBD, between an Acute angle, ABC, and a Right one, ABD, is called the COMPLEMENT of the Angle ABC.

And, if either Side of an Obtuſe angle, as AB, be produced, the Angle EBF is the Complement of the Obtuſe angle; or its deficiency to two Right Anes, ABD, DBF.

DBE is its SUPPLEMENT, to the Right angle ABD.

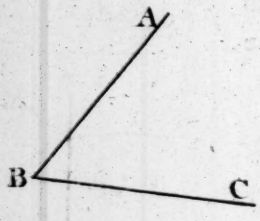
N. B. 2. All Right Angles are equal; but, Acute and Obtuſe angles are infinitely various.



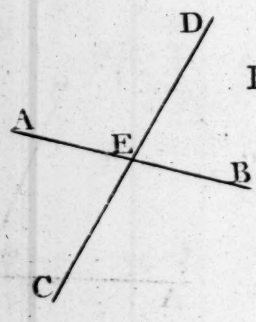
DEF. 14. EQUAL ANGLES. Angles are equal, when the Lines, which form them, have the ſame Inclination each to the other, reſpectively.

In the Angles a b c, ABC; if the Vertex, b, of the one, be applied to the Vertex, B, of the other, in ſuch wiſe, that the Side a b falling on AB, c b alſo coincides with CB; then, there is the ſame inclination of c b to a b as of CB to AB, and the Angles, a b c, ABC, are equal.

N. B. The length of the Lines, forming the Angle, is not conſidered in the equality of Angles.



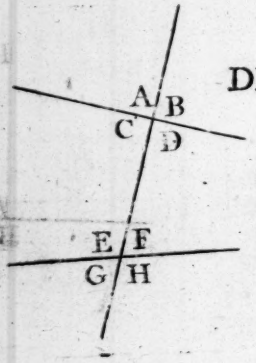
ANGLES have other Denominations, which are derived to them only from their Situation, in reſpect of each other; yet ſtill retain the general appellation of Right, Acute, or Obtuſe angles. Such are the following.



DEF. 15. VERTICAL and CONTIGUOUS ANGLES.

If two Lines, AB and CD, cut and croſs each other, there are made four Angles, at the Point, E, of their mutual Interſection; either two of which, AED, CEB, or AEC and DEB, touching at their Vertices, only, are called VERTICAL ANGLES.

Any other two, as AEC, AED, or AEC and CEB, &c. having one Side, AE or CE, common to both Angles, are called Contiguous or Adjoining ANGLES.



DEF. 16. ALTERNATE ANGLES, and others.

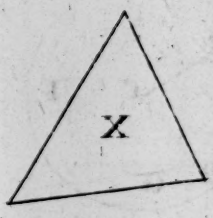
If a Line croſſes or interſects two Lines, there are made eight Angles, A, B, C, D, &c. of which C and F, alſo E and D, between the two Lines (one contiguous to each) on different Sides of the cuting Line, are called ALTERNATE ANGLES.

C and E, alſo D and F, are called INTERNAL ANGLES on the ſame Side.

E and A, F and B, C and G, alſo D and H, are called INTERNAL and OPPOSITE ANGLES, on the ſame Side.

DEF. 17. A PLANE FIGURE is a Space bounded on all ſides by one or more Lines in the ſame Plane.

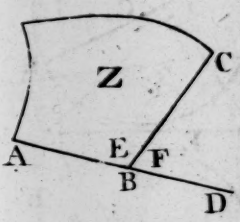
And every Line (between two adjacent Angles) forming and bounding a Figure, is called a SIDE of that Figure.



N. B. If a PLANE FIGURE be bounded by Right lines, only, it is called a right-lined FIGURE; as X. And if it be formed of Right lines and curved, it is called a mixed FIGURE. As AC, or Z.

DEF. 18. INTERNAL and EXTERNAL ANGLES.

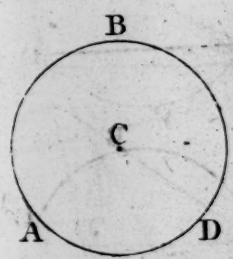
If any Side of a PLANE FIGURE be drawn out beyond the Figure, as AB to D, the Angle E, or ABC, and every Angle within the Figure, is INTERNAL; and the Angle F, or CBD, without the Figure, is called an EXTERNAL ANGLE.



DEF. 19. A CIRCLE is the ſimpleſt and moſt perfect of all Plane Figures, therefore the firſt.

It is bounded by one regular and uniform curved Line, falling again into itſelf; which is called the CIRCUMFERENCE of the Circle.

The curved Line ABD is the CIRCUMFERENCE; the Space, included within it, is the CIRCLE.

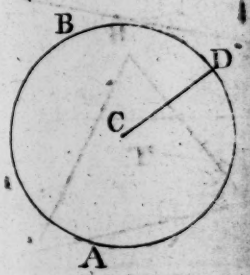


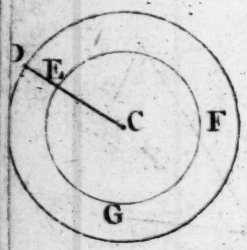
DEF. 20. CENTER of a CIRCLE, or CENTRE, is a Point in the middle of a Circle, or the middle Point of a CIRCLE; which is equally diſtant, every way, from the CIRCUMFERENCE. As C.

N. B. The geneſis of a Circle is thus underſtood.

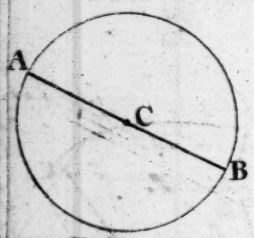
If a Line (CD) be conceived to be revolved quite around, on one extreme (C) fixed to a Point; the other extreme (D) will, in its revolution, deſcribe the Circumference of the Circle, ABD; and the Line CD, having gone over the whole ſpace, has generated the Circle, bounded by that Circumference.

Hence, it is evident that all Right lines, from the Center of a Circle to the Circumference, are equal.





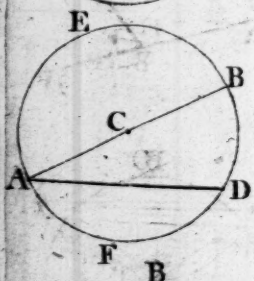
N. B. 2. Equal Right lines deſcribe equal Circles; but, if another Line, CE, or, if any Point, E, be aſſumed in the Line CD, another Circle will be deſcribed, on the ſame Center, and in the ſame ſpace of time; whoſe Circumference, FFG, is to that of the other Circle, deſcribed by the Point D, in proportion to the Lines CE and CD, by which they were generated.



DEF. 21. DIAMETER, of a CIRCLE, is a Right line drawn through the Center, and terminated on both ſides by the Circumference. As AB.

Half the Diameter of a Circle is called the RADIUS. As AC or CB.

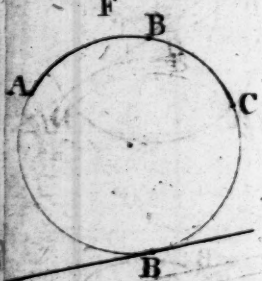
N. B. Every Diameter divides the Circle, and alſo the Circumference, in two equal Parts,



DEF. 22. A SEGMENT of a CIRCLE is any portion cut off by a Right line.

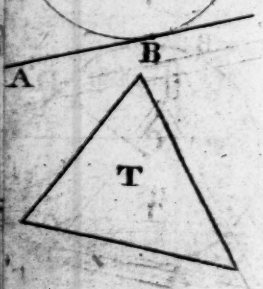
CHORD, or SUBTENSE is the Line cuting a Circle. As AD, making two Segments, AED and AFD.

N. B. A DIAMETER is alſo a CHORD LINE.



DEF. 23. A SEMICIRCLE is a SEGMENT made by a DIAMETER, AB. As AEB, or AFB.

Therefore, the Segment AED, which is greater than a SEMICIRCLE, is called a GREATER SEGMENT; and AFD a LESSER SEGMENT.



DEF. 24. An ARK, or ARCH, is any portion of the CIRCUMFERENCE of a Circle. As AB, BC, or ABC.

DEF. 25. A TANGENT is a Right line drawn without a Circle, and touching it in a Point only. As AB, touching the Circle in B; which is called the POINT OF CONTACT.

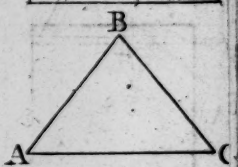
DEF. 26. A TRIANGLE is a Plane Figure bounded by three Lines; and containing as many Angles. As T.

N. B. Not leſs than three Right lines can include a Space and form a Figure; wherefore, a Triangle is the firſt of all right-lined Figures.

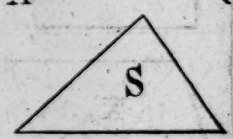
TRIANGLES are of various Kinds, as follow.

DEF. 27. 1. An EQUILATERAL TRIANGLE has all its three Sides equal, to one another. As E.

DEF. 28. 2. An ISOSCELES TRIANGLE has only two equal Sides. AB and BC.

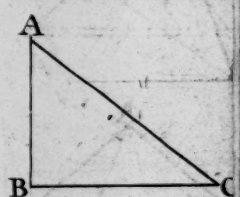


DEF. 29. 3. A SCALENE TRIANGLE has all its Sides unequal. As S.



DEF. 30. 4. A RIGHT-ANGLED TRIANGLE is one that has a Right Angle. B.

2. The Side AC, oppoſite the Right Angle, is called the HYPOTHENUSE.



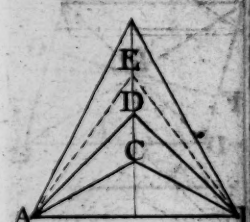
DEF. 31. 5. An OBTUSE ANGLED TRIANGLE is one which has an Obtuſe Angle. C.

N. B. The two laſt are not diſtinct ſpecies of Triangles, but only a particular kind; which ſtill come under the general Denomination of Iſoſceles or Scalene.

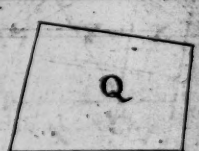
An Iſoſceles or Scalene Triangle may be either right or obtuſe angled, or have all its Angles acute.

The Triangle ADB is right-angled; ACB is obtuſe-angled; and AEB has all its Angles acute; yet, they are all Iſoſceles.

So likewiſe, the Figures of the two laſt Definitions are Scalene Triangles.

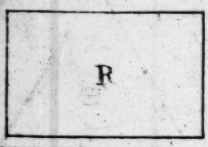


DEF. 32. A QUADRILATERAL, or QUADRANGLE, is a PLANE FIGURE which has four Sides, and four Angles. As. Q.



Theſe are ſynonimous Terms; the firſt expreſſing it by the number of its Sides, the other by its Angles.

DEF. 33. A PARALLELOGRAM is a QUADRILATERAL, whoſe oppoſite Sides are parrallel.

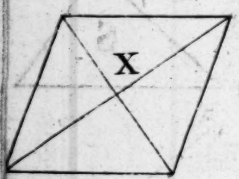


DEF. 34. A RECTANGLE is a PARALLELOGRAM, whoſe Angles are all Right ones. As R.*

DEF. 35. A SQUARE is a RECTANGLE, whoſe Sides are all equal to one another. S.

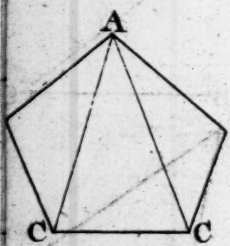
N. B. All RECTANGLES are PARALLELOGRAMS.

DEF. 36. A RHOMBUS is a PARALLELOGRAM, whoſe Sides are all equal, and its Angles not Right ones. As X.



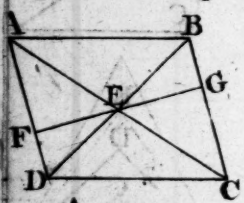
2. If the Sides of a PARALLELOGRAM are not all equal, and the Angles not Right ones, it is called a RHOMBOIDES.

DEF. 37. A DIAGONAL is a Right line drawn between any two Angles, that are not contiguous, in any right-lined or mixed Figure; i. e. from one Angle to the other, within it. As AC.



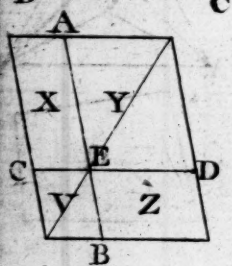
N. B. In PARALLELOGRAMS, the DIAGONAL is uſually called a DIAMETER; becauſe it paſſes through the Center, the middle Point (E) where the two Diagonals (AC and BD) interſect; and, as in a Circle, it divides the Parallelogram into two congruous Figures.

Any Right line paſſing through its Center, which terminates in the Sides, is a DIAMETER. As FG.



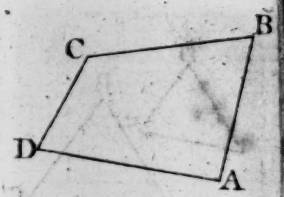
D. 38. COMPLEMENTS of a PARALLELOGRAM.

If any Point, as E, be taken in the Diagonal of a Parallelogram, and, through that Point, two Right lines are drawn parallel to the Sides, both ways (AB and CD) it will be divided into four Parallelograms, V, X, Y, and Z; the two, X and Z, which touch the Diameter, in the Point E, only, are called the COMPLEMENTS; which, with either of the other, about the Diameter, taken together (as XYZ or XVZ) is called a GNOMON.



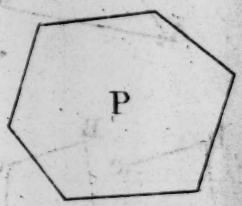
* That ſuch a Parallelogram can exiſt may be diſputed; but, the oppoſite Sides being parallel, it is eaſy to conceive, that if there be one Right angle, they are, neceſſarily, all Right.

DEF. 39. A TRAPEZIUM is an irregular four-sided Figure.



Wherefore, every QUADRILATERAL which is not a Parallelogram is, conſequently, a TRAPEZIUM. As ABCD.

DEF. 40. A POLIGON. All right-lined or other Plane Figures, having more than four SIDES, have the general appellation of POLIGONS, ſignifying many Sides. As P.



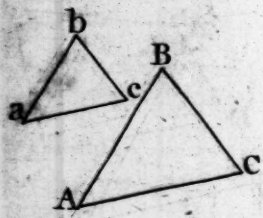
N. B. Ordinate or regular POLIGONS are ſuch as have all their SIDES and ANGLES equal; about which a Circle may be circumſcribed, whoſe Circumference ſhall paſs through every Angle of the Poligon; and, a Circle may alſo be inſcribed, which ſhall touch every Side. As, at a, b, &c.

POLIGONS have various Names, derived to them from the Number of their Sides; as follow.

1. A PENTAGON has five Sides. No. 1.
2. A HEXAGON has ſix Sides. No. 2.
3. A HEPTAGON has ſeven Sides.
4. An OCTAGON has eight Sides. No. 3.
5. A NONAGON has nine Sides.
6. A DECAGON has ten Sides.
7. A DUODECAGON has twelve Sides.
8. A QUINDECAGON has fifteen Sides.

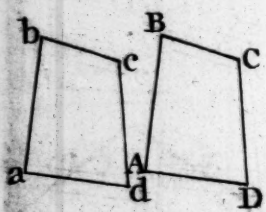
Theſe eight are the moſt eſſential, in Geometry, and the moſt uſeful amongſt Mechanics.
To ſpecify every kind of Poligon would be infinite.

DEF. 41. PERIMETER is the ſum or meaſure of all the Sides of a POLIGON, or other right-lined or mixed Figure, in one Sum; which is ſometimes called its Circumference, or PERIPHERY.



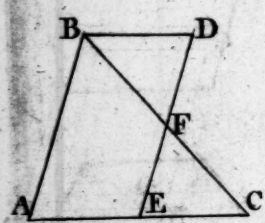
DEF. 42. EQUIANGULAR FIGURES are ſuch as having an equal number of Angles, the Angles of one are alſo equal, reſpectively, to the Angles of the other; each to its correſponding Angle.

As, a equal A, b equal B, and c equal C.



DEF. 43. CONGRUOUS FIGURES are ſuch as have all their ANGLES equal, reſpectively; alſo, the SIDES, which contain equal Angles, or which lie between equal Angles, are equal.

If the Angle a be equal A, b equal B, c equal C, and d equal D; and, if the Side ab is equal AB, bc equal BC, cd equal CD, and ad to AD; then, the Figures, abcd, ABCD, are congruous.



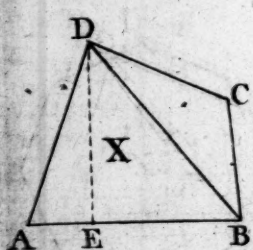
DEF. 44, EQUAL FIGURES are ſuch as have an equal AREA. See Def. 47.

N. B. If two Figures (tho' of different Denominations) have an equal Area, they are called equal Figures. So, the PARALLELOGRAM ABDE is equal to the TRIANGLE ABC; the Triangles BDF, FEC, being equal. CONGRUOUS FIGURES are equal and ſimilar.

DEF. 45. BASE, of a PLANE FIGURE, is the Side on which it is ſuppoſed to ſtand.

As AC, or AB; Fig. X.

N. B. It may be any Side, at diſcretion; but is generally applied to the lower Side, or that which is next towards us.

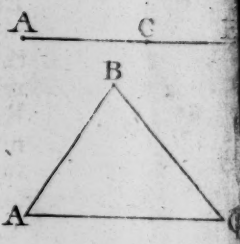


DEF. 46. ALTITUDE, of a FIGURE, is its perpendicular height from the Baſe.

ED, perpendicular to the Baſe, AB, is the Altitude of the Trapezium ADCB; or of the Triangle ADB.

DEF. 47. AREA of a PLANE FIGURE, or other Surface, is its ſuperficial Contents; i. e. the meaſure, or quantity of Space contained within the Bounds of the Figure, expreſſed in ſquare Feet, Yards, or any other known meaſure, of length.

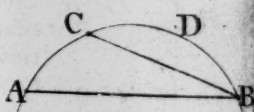
DEF. 48. SEGMENT of a LINE, is any portion of the Line. As AC, or CB, of the Line AB.



DEF. 49. To SUBTEND. The Side of a Triangle which is oppoſite to any Angle, is ſaid to ſubtend, or ſtand under that Angle.

Thus, AB ſubtends the Angle C, AC ſubtends the Angle B, and BC ſubtends the Angle A.

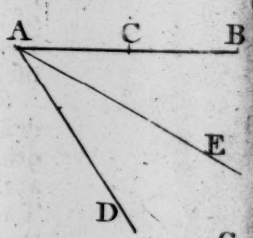
Likewiſe, the CHORD or SUBTENSE AB, ſubtends the Ark ACDB; and, CB ſubtends the Ark CDB.



DEF. 50. To BISECT, is to cut or divide a Line or Angle, &c. into two equal Parts.

Thus, AB is biſected in the Point C; and the Angle BAD, is biſected by the Line AE.

To TRISECT, is to cut equally into three Parts.

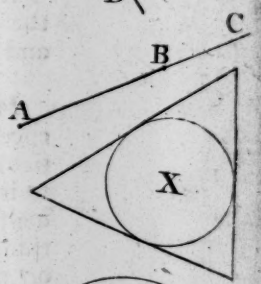


DEF. 51. To PRODUCE is to *draw* out a Line or Plane, to lengthen or extend it, at pleaſure.

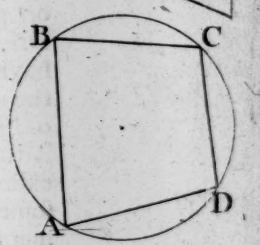
Thus, the Line AB is produced to C.

DEF. 52. To DESCRIBE a Line, or Figure, is to draw a Line, Circle, or other Figure.

DEF. 53. To INSCRIBE, is to draw a CIRCLE, or other Figure, touching every Side of another Figure, internally, as X; or, whoſe Angles ſhall all be in the Circumference of a Circle (as ABCD) or touching the Sides of any other Figure.



DEF. 54. To CIRCUMSCRIBE, is to deſcribe a Circle or other Figure paſſing through all the Angles of another Figure. As ABCD.



DEF. 55. EXTREMES or BOUNDS.

The EXTREMES of a LINE are POINTS; the Bounds or Extremes of a SURFACE are LINES; and, the Extremes of a SOLID are SURFACES.

N. B. A SPHERE, or GLOBE, has but one Surface; alſo, a CIRCLE is bounded but by one Line, without begining or end; conſequently, they have not EXTREMES.

SOME Readers will, perhaps, think I have already deviated from the Plan propoſed in the Preface, to abridge the Elements of Euclid. The time ſpent on the Definitions will not be loſt, having made ſeveral of them uſeful Leſſons; and am perſuaded, that, the better they are underſtood, the progreſs, in the Subject they relate to, will be thereby facilitated. But, I muſt inform the Reader, that 'tis not the length of the Definition, itſelf, but the number of Terms defined, and Notes to ſeveral, which have ſwelled the bulk of them; having near twice the Number which ſome Authors have, and, in my opinion, not one ſuperfluous.

However, not to diſcourage the young Student, I adviſe him not to burthen his memory too much, at once; for there is not the leaſt neceſſity that he ſhould retain them all, before he proceeds further; 'tis enough, at firſt, to read them over with attention, and underſtand them clearly; they will ſoon become familiar to him, by frequent uſe, in his ſtudy of the Science. I alſo particularly adviſe him, when he meets with any Term hereafter, of which he has not a clear Idea, to turn immediately back to the Definition of it; he may depend on it, that the time will not be uſeleſſly employed.

Several of the Terms here uſed, are abſolutely neceſſary to be defined, though they are not elementary. Such are the 4th, the 9th, the 14th, 15th, 16th, and 18th, the 36th, the 41ſt, 44th, 47th, and 48th, and the laſt ſix, ſave one, are merely operative. The 49th, 50th, 51ſt, and 52d, are as neceſſary to be defined, as the 53rd and 54th. The 55th, and laſt, is in three, the 3rd, 6th, and 13th of Euclid; yet they are in no wiſe elementary.

IT is, or ought to be, the deſign of every Author, on any Science, to make his Book a perfect Tutor; conſequently, no particular Term, made uſe of in that Science, and which is peculiar to it, ſhould be left undefined: for, we ſhould ſuppoſe the Student to be entirely ignorant of all that relates to it. Can any thing be more abſurd, than to propoſe biſecting a Line or Angle, or producing a Line, &c. to talk of Alternate Angles, &c. of Diagonals, Complements, Subtenſes, and ſubtending, without having firſt defined what is meant by them? I am ſurpriſed at the omiſſion of the Definition of a Poligon, and the various kinds of Poligons; the forming, inſcribing, and circumſcribing of which, is the principal ſubject of the 4th Book. They are frequently called by the Terms Pentagon, Hexagon, &c. but are not, by ſome, defined. In the 23rd Definition they are, in general, called many-ſided Figures; 'tis an ungeometrical Term, and is never, uſed after, but other Names are aſſumed.

Euclid, himſelf, has not defined a Parallelogram, that moſt uſeful and neceſſary Figure. I admire Mr. Stone's Apology for that omiſſion, as being unwilling to increaſe the Number, unneceſſarily; and ſays, that Euclid has ſufficiently defined it in the 34th Propoſition. Mr. Stone indeed has, but not Euclid; for why elſe

has Keill defined it before the Propoſition? But, if it be thought moſt eligible to define that Term, where it is firſt uſed, why are not others ſo defined? why are not all, as well as it?

And yet, Euclid has, in Mr. Stone's Opinion, given ſeveral unneceſſary Definitions; viz. the 3rd, the 6th, and 13th of his Euclid, which certainly are ſo; the 9th I think ſo too, alſo the 20th. He likewiſe, thinks the 18th, 26th, the 32nd and 33rd, uſeleſs. In reſpect of the 26th (the 29th of this) I cannot ſay it is eſſentially neceſſary; yet, as Scalene Triangle is a general Term, including all that are neither Equilateral nor Iſoſceles, I cannot think it redundant.

The 29th, of which he ſays nothing, is really ſuperfluous, viz. "when all the Angles are acute, it is called an acute-angled Triangle." I know of no properties peculiar to ſuch an acute-angled Triangle, that is not common to every Triangle; for every Triangle has, neceſſarily, two acute Angles. An equilateral Triangle is included in that Definition, and ſo are the Iſoſceles frequently.

Why is not a three-ſided Figure (Def. 21.) called a Triangle, as it is always called afterwards? we might as well go on, to four, five, or fifteen ſided Figures. An Oblong (Def. 31.) for a Rectangle, is ungeometrical, and never called ſo after. The Terms Rhombus and Rhomboides, 32. and 33. (Def. 36. of this) are, entirely, uſeleſs, being ſignified in a Parallelogram, which alſo includes Squares and other Rectangles; all, which, are only particular ſpecies of Parallelograms; the Term Rhomboides (including all unequal ſided and acute angled) is of no uſe in Geometry.

I can by no means agree with Mr. Stone, in thinking the 18th Definition, of a Semicircle, needleſs; I alſo think, the Radius as neceſſary to be defined as the Diameter. In ſpeaking of Angles in a Segment of a Circle, Prop. 31. of the 3rd Book (the 12th of this) viz. "the Angle in a Semicircle is a Right one;" ſince there is a neceſſity for calling it ſomething, I think a Semicircle more elegant and expreſſive than half a Circle, which does not confine it to any particular form; provided it has half the Area, it is half a Circle; whereas, the Semicircle is always underſtood to be contained under a Diameter and half the Circumference.

Mr. Stone is too dogmatical in his Remarks. I cannot be of opinion, that the manner in which Euclid has compiled his Elements, is the beſt that can poſſibly be, becauſe Euclid lived two thouſand Years ago. 'Tis not to be imagined that the mathematical Sciences were at their *ne plus ultra* in his Days, for it is notorious they were not; then, why ſhould the Elements of Geometry? I am not ſo wedded to antiquity, as to think the Antients were infallible; nor do I think that all Euclid's Definitions are neceſſary, or that he has omited none that are ſo: Such, therefore, as I preſume to think uſeleſs, I have rejected, and have ſupplied their places with others, which I think more neceſſary to be defined.

ABBREVIATIONS, &c. EXPLAINED.

&c. *Et cætera*. (And all the reſt.) When, what is ſuppoſed to follow may be readily conceived.

i. e. *Id eſt*. THAT IS. When, what has been ſaid requires to be further explained.

viz. *Videlicet*. TO WIT; or, that is to ſay. When any thing advanced is given in Groſs, which is more particularly ſpecified, as follows after.

e. g. *Exempli gratia*. FOR INSTANCE; or, for the ſake of example. When an Example is to be given of what is advanced.

N. B. *Nota Bene*. MARK WELL. That is, take particular notice of that Paragraph.

Q. E. F. *Quod erat faciendum*. Which was required to be done.

Q. E. D. *Quod erat demonſtrandum*. Which was to be demonſtrated.

DEM.	Demonſtration.	Hyp. AB.	Hypothenuſe AB.
COR.	Corollary.	Perp.	Perpendicular.
SCHOL.	Scholium, or Remark	Dia.	Diameter.
		Diag.	Diagonal.
APP.	Application, or Uſe.	Trap.	Trapezium.
		Pent.	Pentagon, &c.
Par. AB.	Parallelogram AB.		

Rect. AB. Rectangle AB, or ABCD, &c.

R. Ang. ABC. Right Angle ABC, &c.

Line AB. Denotes Right Line AB, &c.

Note. When it is required to join any two Points, it is meant, that a Right Line be drawn between them, i. e. from one Point to the other.

ABBREVIATIONS, by way of Reference.

Poſt. 1. or 2. Refers to the firſt or ſecond Poſtulate; where it is requeſted that the thing may be granted.

Def. 6. or 7. Refers to the ſixth or ſeventh Definition in
Or, Def. 6. 3. the general Introduction. But, when there
or 6. 5. &c. are two numbers, as in the Margin, it refers to the 6th Definitions of the 3rd or 5th Book.

Ax. 3. Refers to the third Axiom for Demonſtration, ariſing from ſelf-evident properties of things.

Pr. 1. or 2. &c. Refers to the firſt or ſecond Problem, for the conſtructing of ſome Figure, &c.

P. 1. or 2. &c. Refers to the firſt or ſecond Propoſition of that Book, for proof of the Aſſertion.

P. 2. 3. &c. To the ſecond Propoſition of the third Book.

C. 2. 4. 1. To the 2d Corollary, of the 4th Propoſition of the firſt Book.

Hyp. That the thing is ſo by the Hypotheſis, or is ſuppoſed in the Premiſes.

Sup. That it is ſo by Suppoſition, only.

Con. That it is ſo by Conſtruction; i. e. the thing was formed or made ſo.

Conſ. conſequently. Th. therefore. Wh. wherefore, it is ſo.

Note. When there is but one Number within Parentheſes or otherwiſe, as (15.) &c. it refers to the fifteenth Theorem of that Book. But if there be two Numbers, as (9. 1.) or (12. 3.) &c. it refers to the 9th Prop. of the firſt Book, or to the 12th of the third, &c.

Explanation of the Marks and ſymbolic Characters uſed in this and other mathematical Works.

= Note of Equality.

Thus; A=B, ſignifies that A and B are equal; and is thus read, A is equal to B; frequently, A equal B.

+ Note of Addition, Plus, or more.

Thus A+B, ſignifies the Sum of A added to B. A and B repreſenting any Quantity, whatever. Let A be 5, and B, 3; then A+B=8.

− Note of Subtraction; Minus, or leſs.

Thus A−B, ſignifies the ſum of B taken from A. A−B=2, muſt be read, A, leſs B, is equal to 2.

× Of Multiplication. A×B, denotes A multiplied into B, or B into A.

A×B=15. A multiplied into B is equal 15.

÷ Of Diviſion. A÷B implies A divided by B.

Let A be 6 and B 2; then, A÷B=3.

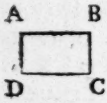
: :: : Note of Equality of Proportion.

Thus, A:B::C:D, ſignifies, that A bears the ſame Proportion to B, as C to D. 6:9::2:3.

∺ Continued Proportion, or geometrical Progreſſion.

Thus, A:B:C:D∺ ſignifies, that A has the ſame Proportion to B, as B has to C, and, as C to D. In Numbers, thus, 1:3:9:27∺

□ Square. Thus, AB□, is, the Square of a Line A−B.



Rectangle. Thus, AC, or BD▭; or ABC, means a Rectangle under the Lines AB and BC; i. e. AB×BC. ABC=BD□, muſt be read, Rectangle ABC equal BD Square; i. e. to the Square of BD.

2ABC; or 2AB▭; muſt be read, two Rectangles ABC, or twice the Rect. AB; and, 2 or 3 AB□, muſt be underſtood, twice or thrice the Square of AB.

△ Triangle. Thus, △ABC, ſignifies the Triangle ABC.

ELEMENTS OF GEOMETRY.

BOOK I.

THE firſt Book of Euclid's Elements treats of the properties of Right lines; and of Angles, formed by the interſections of Right lines, &c. which, being entirely unconnected with Figures, in this Work, are firſt treated on. Next, it treats of right-lined Figures; firſt, of Triangles, the moſt uſeful and extenſive of all. Then is ſhewn the affinity between Triangles and Parallelograms; which prepares the way for the laſt, grand Propoſition, the 47th of Euclid, which concludes this Book.

THE Order in which they are treated, according to Euclid, ſeems to me not very regular; begining firſt with Figures, then Lines and Angles; again, Figures and Lines, alternately. The fourth Propoſition begins the properties of Triangles, in general; the 5th introduces the Iſoſceles, a particular ſpecies of Triangles, merely (in this place) for the Demonſtration of the 7th; and that, is of no other uſe but to demonſtrate the 8th, which may be clearly demonſtrated (according to Proclus) without it. In the 24th and 25th, the ſame general properties of Triangles is again introduced, which are evidently deducible from the fourth. For, having, in the fourth, proved, that, in two Triangles, if there are found two Sides, in each, reſpectively equal, and

that the Angles, contained between the equal Sides, are alſo equal, the remaining Sides are equal; no Perſon converſant with Plane Angles can want Demonſtration, that if, in the Caſe of equality of the Angles, the Sides oppoſite thoſe Angles are equal; if one of the Angles be greater or leſs, the ſide oppoſite will alſo be greater or leſs? and, *vice verſa* (in the 25th) if the Side, or Baſe, lying between the equal Sides, be greater or leſs, the Angle it ſubtends muſt neceſſarily be ſo too; they are ſuch neceſſary Conſequences, that I would not heſitate to take them for granted.

THIS Book is encumbered with Demonſtrations of ſeveral Propoſitions, which need none; for, what is evident, in itſelf, needs no Demonſtration; and ſuch Propoſitions, as may eaſily and clearly be deduced from preceding ones, are as well made Corollaries to them. Such are all converſe Propoſitions; I think it muſt perplex and embarraſs the Minds of Youths, to demonſtrate what is ſelf-evident, and not very eſſential. I have, therefore, greatly abridged, and altered the Elements of Euclid's firſt Book; yet, I am much miſtaken, if I have not retained all that is eſſential in it; and which, I think, may be much eaſier and more regularly acquired.

Here are but twenty theoretic Propoſitions in this firſt Book, three of which number (the 3rd the 6th and 16th) are not in Euclid; whereas (excluſive of the Problems) Euclid has thirty-four Theorems, twice the number, of the remaining ſeventeen, in this Tract.

THE 7th of Euclid is not only uſeleſs, in itſelf, but the Demonſtration is intricate, to a beginer; and ſince the 8th, which depends on it, is deducible from the 4th, I ſee no uſe for it at all. In reſpect of the 4th (my 8th) I muſt own I cannot think the Demonſtration of it geometrical; but, ſuch as it is, has been generally accepted; and although

it is not ſatisfactory to many, yet, I have not ſeen even an attempt at any other kind of Demonſtration; which is ſufficient conviction that no other can be given. As it is the firſt Propoſition on the properties of Figures, which might well paſs for an Axiom,* the Demonſtration, being purely mental, is ſatisfactory; though it ſuppoſes a manual application of one to the other, to make it appear.

It is not poſſible, for writers on any Science, to treat it ſo as to pleaſe all; every Author has his peculiar Notions, and is prejudiced in favour of his own mode of treating the Subject. I think it more regularly introductory to treat, firſt, of Lines and Angles, before Figures; the 13th of Euclid I have, therefore, made the firſt; for, on the foundation of it, depends, in a great meaſure, the whole firſt Book. I am the more confirmed in it, as Mr. T. Simſon has done the ſame; which, I do aver, I had not ſeen or knew of, before my plan was ſettled. According to Euclid, the Properties of parallel Lines, cut by another, cannot be obtained without having recourſe to Triangles, and is the reaſon why the 27th, and the following, are not introduced ſooner, which are previouſly neceſſary to demonſtrate the 32nd: in which, the 16th and 17th are more fully and perfectly demonſtrated: wherefore, Euclid was obliged to have recourſe to the 16th, as a Lemma, in demonſtrating ſeveral Propoſitions previous to them. But, for what uſe the 17th is introduced, I am at a loſs to deviſe, it being never once referred to, in the whole Elements; it might (if neceſſary) be a Corollary after the 32nd; but, we can never be at a loſs to know, that any two Angles of a Triangle are leſs than two Right angles, having full Demonſtration that the three Angles of every Triangle are, together, equal to two Right ones.

In the 18th of this, is contained the 35, 36, 37 and 38th of Euclid; including alſo, in a Corollary, the 39th and 40th.

* Mr. T. Simſon has made it an Axiom.

This Property of Parallelograms and Triangles is very extenſive; but the whole is included in the firſt part of the 18th, the 35th of Euclid. To dwell ſo long on one Property, in ſix Propoſitions, becomes tedious, if not trifling; for where is the difference, whether Parallelograms, or Triangles, have the ſame or an equal Baſe?

THE alterations, in the firſt Book, are very conſiderable; the manner of demonſtrating is more conciſe, the clearneſs of it muſt be left for others to determine. I do not doubt but the judicious, candid, and impartial Reader will concur with me, or not diſapprove, entirely, the liberty I have taken. I thought it incumbent on me to advertiſe him of the alterations I have made, and, by pointing them out, he may, with more eaſe, form a judgment of them.

THE firſt ſix Propoſitions, and the Corollaries deduced from them, contain all the properties of Right lines, and Angles formed by their Interſections; which are eſſentially neceſſary to be known, before we can obtain a knowledge of the properties of Plane Figures. But, I think it ſcarce worth the loſs of time to demonſtrate them, they being in a manner ſelf-evident; inſomuch that, the greateſt part of them may be, and are, by ſome, given as firſt Principles, or ſelf-evident Propoſitions.

1. By Def. 10. one Right line, ſtanding perpendicularly on another, makes the Angle, on either ſide, equal to the other, and are, therefore, Right angles.

The Line CD, being common to both Angles, ACD and DCB; the two other Sides, AC, CB are, conſequently (the Angles being right ones) in one Right line, ACB. And, it is clearly evident, that any other Line, or number of Lines, as CE, CD, CF, at the ſame Point C, muſt make all the Angles, ACE + ECD + DCF + FCB, equal to

the two Right angles (ACD + DCB) Hence, the firſt Propoſition and its Corollaries are evidently clear and manifeſt.

2. AGAIN; (by Def. 11.) if the Perpendicular, CD, be produced, towards H, there is generated equal Angles, to the oppoſite, on either Side; for, they are all Right angles, conſequently equal, by the 9th Axiom.

So likewiſe, if any other, inclined Line, be produced, as EC, or FC, it will, neceſſarily, produce equal Angles, to the oppoſite, on the other ſide of AB; as ACG equal FCB, and GCB equal ACF; and which, together, are alſo equal to two Right angles.

For, if EC biſects the Right angle ACD, CI will alſo biſect the oppoſite, HCB; and, if CF triſects the Right angle DCB; the oppoſite Angle, ACH, will alſo be triſected, if FC be produced, towards G; and ſo, in whatever proportion DCB is divided, by FC, the continuation of FC will, neceſſarily, divide the oppoſite, ACH, in the ſame Proportion; wherefore, the Angle GCH is equal to DCF, and ACG equal FCB; alſo, GCH added to HCB is equal DCF added to ACD; and ACG added to GCB is equal to FCB added ACF; i. e. they are each equal to two Right angles; and conſequently, all the Angles about the Point C are equal to four Right angles. Hence, it is plain, that the firſt and ſecond Propoſitions, with their Corollaries, are clearly deducible from the 10th and 11th Definitions.

3. IT is alſo, I preſume, eaſy to be conceived, that two or more parallel Lines muſt neceſſarily have the ſame Inclination to any Right line which cuts them all; the Idea of Paralleliſm ſeems to indicate it; which being known, or underſtood, all the reſt follow of courſe, viz. that the Alternate angles are equal to one another, and the two internal Angles, on each Side, are equal to two Right angles.

For, ſuppoſe the Right lines AB, CD, and EF, to be parallel amongſt themſelves, and all cut by any Right line, GE; making, with them, the Angles x, y, z; and, ſuppoſe AB to be moved, directly, to CD, ſtill keeping parallel to CD, it muſt neceſſarily coincide with CD; conſequently, the Angle BAG will remain the ſame, and, at laſt, coincide with ACD. And if AB be moved directly forward, to EF, ſtill keeping parallel to its firſt poſition, it will alſo coincide with EF, and the Angle GAB, or ACD, with CEF.

Hence it is plain, that the Angles x, y and z, made with GE, are all equal to one another; and, being equal to one another, the Right lines AB, CD, and EF are conſequently parallel between themſelves.

4. AGAIN; ſince the Angles x, y and z are equal to one another; and the oppoſite Angles, v, u, w, are alſo equal between themſelves, and to the other (as it was made evident in the foregoing) conſequently, the Alternate angles, v and y, u and z, &c. are equal; for, they are all equal to one another, as above (the Lines AB, CD, and EF being parallel); and, converſely, if the Alternate angles are found to be equal, the Lines, forming them, are conſequently parallel.

5. ALSO; ſince a Right line cuting another Right line makes the Agles, on both Sides, equal to two Right angles (by the firſt) it is evident, that, if the Angles x, y and z are equal to one another, and the adjoining Angles x added to A, and y added to C, &c. are each equal to two Right angles; conſequently, A, C, and E are alſo equal between themſelves, and the internal Angles (between each two Lines) A added to y, and C added to z, are alſo, each two, equal to two Right angles; by the 6th Axiom.

6. AND, becauſe A added to v, and y added to o are each equal to two Right angles; and A added to y, on one Side GE, are equal to two Right angles; conſ. v added to o, on the other Side GE, are, alſo, equal to two Right angles. Therefore, if two parallel Right lines, AB and CD, &c. are cut by any other Right line, GE, the internal Angles, on each Side, are equal to two Right angles; and, conſequently, if a Right line cuts two Right lines, which are not parallel (as CD and HI) they will meet on that Side where the two internal Angles (ACD, CAI) on the ſame Side, are leſs than two Right angles; which, is Euclid's 12th Axiom.

Now, ſince all theſe Properties of Right lines, and of Angles formed by Right lines, are manifeſt, from what has been advanced already, there is, I think, but little occaſion for Demonſtration; nevertheleſs, in conformity to the Antients, I have formed them into Propoſitions, for the ſatisfaction of the ſcrupulous, and for the ſake of reference, in other Works, as well as in this; yet, I am clearly of opinion, that this brief extract is ſufficient, towards attaining all the neceſſary knowledge inculcated by the firſt ſix Propoſitions. It will, at leaſt, be of uſe to the young Student, in giving him a clear Idea of the Properties therein demonſtrated; which, by being more familiarized to them, will appear more evident and conſpicuous; the neceſſary knowledge, contained in them, will be deeper rooted and more ſecurely eſtabliſhed, and the Pupil, better prepared to proceed with the Demonſtrations of the Properties of right lined Figures.

POSTULATES.

POSTULATES are *Requeſts*, that certain fundamental Principles or Elements, in a Science (being extremely ſimple in themſelves) may be granted; as it is not poſſible to give perfect and indiſputable Demonſtration, if inſiſted on.

The following are, therefore, *requeſted* to be granted.

1. That there may be a perfect PLANE; and that it may be extended at pleaſure.
2. That a RIGHT LINE may be drawn between any two given Points; i. e. from one Point to the other.
3. That a RIGHT LINE may be produced at pleaſure; i. e. that a finite Line may be continued or lengthened.
4. That a CIRCLE may be deſcribed on any Center, and with any given Radius.
5. That one LINE, or FIGURE, may be applied to, or laid upon another Line or Figure, as required.

Theſe Poſtulata muſt be granted, at leaſt in Idea, or all Geometry falls at once to the Ground; for, if there cannot be a Plane, a Right line, and a Circle, the whole Elements of Geometry are to no purpoſe; as it will be impoſſible to form a Conſtruction, whereby we may demonſtrate the eſſential Properties of Figures, in general; conſequently, if no Demonſtration can be given, there is an end of the Science, having no Data to build on.

In every mathematical or phyſical Science, there is a neceſſity for ſome Data or firſt Principles to be given, whereon to frame Hypotheſes, in order to demonſtrate the Theorems; and the more ſimple thoſe Principles are, the better; becauſe there is leſs cauſe to diſpute them. But, nevertheleſs, if they are diſputed, they are more difficult to demonſtrate, for being the moſt ſimple; becauſe, there is no reverting back to any thing more ſo; and conſequently, no Demonſtration can be given. That the thing is ſo. of itſelf, is arbitrary, notwithſtanding there is no poſſibility of denying it; therefore, the more ſimple the firſt Principles are, the readier the aſſent will be given; and Demonſtration of the moſt complex Propoſition will be eaſier obtained, more firmly ſupported, and, conſequently, the whole Science, which is built on thoſe Principles, more ſolid and permanent, and more ſecurely eſtabliſhed.

AXIOMS.

AXIOMS (as already defined) are manifeſt and ſelf-evident Propoſitions, clear in themſelves, and therefore do not require to be demonſtrated: Nevertheleſs, an Illuſtration will not, I preſume, be thought ſuperfluous, as it may be neceſſary to ſome.

I. THINGS which correſpond, coincide, and mutually agree with each other, in every Part, are equal. And if the wholes be equal, the halves are alſo equal.

Thus, A is equal B; if every correſponding part of the Figure A is coincident with and equal to B.

II. The WHOLE is equal to all its Parts; conſ. it is greater than a Part of the ſame Thing.

Thus, AD=AB+BC+CD.

III. If two, or more, THINGS, or Quantities, are each equal to the ſame third Thing or Quantity, they are equal between themſelves.

Thus, if A=C, and if B=C, then A=B.

IV. Things or Quantities, which are each equal to half, or a third (or any other portion) of the ſame third Thing, or equal Things, are equal to one another.

Thus, if A and B are each equal to half of C; A and B are equal, to each other.

V. THINGS, or Quantities, which are each double, or triple, &c. of the ſame third Thing, or equal Things, or Quantities, are equal to one another.

Thus, if D and E are each equal twice or three times F, D is equal to E.

VI. If to equal Things be added equals, the Sums, or wholes, are equal.

If to the equal Rectangles A and B, be added equals, C and D; then, A+C=B+D.

VII. If from equal Things be taken away equals, the remainders will be equal.

If from the equal Rectangles, A and B, be taken away equals, C and D, the remainders of A and B are equal.

VIII. If to or from unequal Things be added or taken away equals, the Sums or Remainders will be unequal.

This neceſſarily follows from the two laſt.

IX. All RIGHT ANGLES are equal to one another.

The Right angle ABC is equal to the Right angle CBD, or BDE.

For, CB, being perpendicular to AD, does not incline to it on either Side; wherefore, the Angles are equal, and they are Right angles; by Def. 10, & 11.

X. Two or more Right lines, being perpendicular to the ſame Right line, and being in the ſame Plane, are parallel.

The Angles ABC, BDE, being Right angles, are equal, by the laſt: Then, ſince neither of them inclines to AD, BC is parallel to DE.

THEOREM I. 13. Euclid.

If one Right line cuts another Right line, at any Point apart from its Extremes, it will make two Angles; which are either both Right angles, or, both together, equal to two Right angles.

Let CD be a Right line, cuting AB in the Point C, making two Angles, ACD, DCB; I say, they are either Right angles, or both together equal to two Right angles.

If CD be perpendicular to AB the thing is manifeſt; by Def. 10.

But if CD does not make Right angles with AB; let CE be perpendicular to AB.

DEM. Now, ACE, ECB are Rt. Angles (Sup.) th. equal. But, ACD=ACE+ECD; alſo DCB=ECB−ECD. Wherefore, as ECD, the exceſs of the obtuſe Angle, ACD, to the Right angle ACE, is alſo the deficiency of the acute Angle DCB to the Right angle ECB; being taken from one and added to the other, conſequently ACD+DCB is equal to ACE+ECB.— - Ax. 2. Therefore, they are equal to two Right angles. Q.E.D.

COR. 1. If from the ſame Point in a Right line (as C in the Line CD) two Right lines CA, CB are drawn, on contrary Sides and in the ſame Plane, making Right angles, or Angles equal to two Right angles, with the given line, they will conſtitute one Right line, ACB.

2. Two Angles at the ſame Point, and on the ſame Side of a Right line, being equal, they are Right angles.

Wherefore, when two Angles are contiguous or adjoining, if one of them, ACE, be a Right angle, the other, ECB, is alſo a Right one, and if one of them, ACD, be obtuſe, the other, DCB, is conſequently acute.

3. Two, three, or more Right lines meeting in the ſame Point, and on the ſame Side, of another Right line, make all the Angles, together, equal to two Right angles.

For, ACF+FCD+DCB=ACE+ECB.

4. When one Right line makes Angles with another, if one of the Angles be known the other is alſo known.

For, it is the difference between the Angle known and two Right ones. As, ACD=ACE+ECB−DCB.

THEOREM II. 15. Euclid.

Vertical or oppoſite Angles, made by the Interſection of two Right lines, are equal to one another.

AB and CD interſect in E; then, the Angles AEC, DEB, are equal; alſo, AED equal to CEB.

DEM. The Right line AE, cuting the Right line CD, makes the Angles AEC, AED, together, equal to two Right angles. - - - - - - - Th. 1.

And, becauſe DE cuts the Right line AB, the Angle AED+DEB, are alſo equal to two Right ones.- by the ſame.

Therefore, AEC+AED=AED+DEB. - - Ax. 9.

And if there be taken away the common Angle AED, there will remain AEC equal to DEB. - - Ax. 7.

In the ſame manner, AED may be proved equal to CEB.

OR, it may be demonſtrated after another manner, thus.

About E, the point of Interſection, as a Center, deſcribe a Circle (with any Radius at diſcretion) cuting the two Lines in A, C, B, and D. Then, the Ark AD+DB=CB+BD.

For they are, each two, equal to the Semi-circumference of a Circle. - - - - See N. B. Def. 21.

Wh. taking away the Ark DB, there remains AD=CB; and, the Angles AED, CEB, having equal Arks, are equal.

Cor. Hence, if two Right lines interſect each other, the Angles they make are equal to four Right angles.

2. All the Angles which can be made about a Point are equal to four Right angles.

THEOREM III.

If a Right line, cuting two Right lines, makes all the Angles, on both Sides, equal amongſt themſelves, the two Lines are parallel.

Let the Right line EF cut two Right lines, AB and CD, making the Angles AEF, EFC, BEF, and EFD equal to one another.

Then AB is parallel to CD.

Dem. Becauſe the Angle AEF is equal to BEF, by Hyp. EF. is perpendicular to AB - - Def. 10.
And, becauſe EFC=EFD, EF is, alſo, perpend. to CD.
Wherefore, ſince EF cuts both Lines, AB and CD, perpendicularly, the Angles AEF, EFC, &c. (being equal) are Right angles; and, AE, EB, CF, and FD are perpendicular to EF. - - - Def. 11.
But, AE, EB is the ſame with AB, and CF, FD, with CD.
Therefore, AB and CD, being both perpendicular to EF, are parallel. - - - Ax. 10.

Cor. From hence it is manifeſt, that, any Right line, cuting two parallel Lines, makes the internal Angles, on each Side, equal to two Right angles.

For, if they are cut perpendicularly, as EF, it is clear.

And, ſince any other Line (FG) cuting them both, makes the adjoining Angles (a and b, c and d) with each Line (AB and CD) equal to two Right angles; - - - - Th. 1. conſ. all the internal Angles (a, b, c, and d) which it makes with them both, are equal to four Right angles. Ax. 2.

Now, ſince AB is parallel CD (Theo.) and the two internal Angles, on one Side EF, are equal to thoſe on the other Side (Hyp.) it is evident, that the internal Angles on one Side are, alſo, equal to thoſe on the other Side, of any Right line (FG) cuting them both; wh. $a+c=b+d$.

But, all the four internal Angles are equal to four Right angles; conſequently, the internal Angles, $a+c$, and $b+d$, on each Side, are, each two, equal to two Right angles.

COR. 2. If two Right lines, which are not parallel, be cut by another Right line, they will meet on that Side where the internal Angles are leſs than two Right ones.

For, if $b+d$ be leſs than two R. angles $a+c$ is greater; becauſe, $a+b+c+d=$ four Right angles. - - - Th. 1.

Wherefore, ſince AB is inclined to CD, they will, if produced, meet on that ſide EF, towards B and D, on which the Angles, $b+d$, are leſs than two Right ones.

Becauſe, the Angles, $b+d$, on that ſide EF, are, together, leſs than $a+c$, on the other Side.

If this Theorem, and the Corollaries deduced, cannot be admited, as perfect Demonſtration, it may be ſuppoſed a Lemma to the next; being an illuſtration, at leaſt, of Euclid's 12th Axiom; which, in my opinion, cannot be received as ſuch, by any Perſon as yet unacquainted with Geometry. From that conſideration, it may paſs with the moſt ſcrupulous; my deſign is to make Geometry attainable to common Capacities.

THEOREM IV. 27. Euclid.

If a Right line, cuting two Right lines, makes the alternate Angles equal, to one another; or, ſhall make the external Angle equal to the internal and oppoſite; the two Lines are parallel.

Let FG be a Right line, cuting the two Right lines AB and CD, and making the Angles AEF, EFD, equal to one another. Then, the two Lines, AB and CD, are parallel.

For, if the Angles AEF, EFC, be not equal to two Right angles, they are either greater or leſs; ſuppoſe them greater; then, the Lines, AB and CD, will meet on the other Side, towards B and D.

DEM. The Ang. BEF + EFD, is leſs than AEF + EFC. Sup.
And the Angle AEF + BEF = EFC + EFD; - Ax. 9.
for, they are, each two, equal to two Right angles. Th. 1.
But, the Ang. AEF = EFD (Hyp.) wh. BEF = EFC. Ax. 7.
Conſequently, BEF + EFD = AEF + EFC, - Ax. 6.
i. e. they are, on each ſide, equal to two Right angles.
Th. AB is par. to CD; as they do not incline on either Side.

2. Let the Angle GEB be equal to EFD.
Now, the Ang. EFD = GEB (Hyp.) and, AEF = GEB. 2.
Wherefore, AEF is equal to EFD - - - Ax. 3.
But, GEB + BEF are equal to two Right angles, - 1.
conſ. BEF + EFD are alſo equal to two Right angles.
By the ſame, AEF + EFC are equal to two Right angles.
Therefore AB is parallel to CD.

HENCE, a Right line (as CD) may be drawn through a given Point, F, parallel to AB. For, drawing a Line, FG, through F, at pleaſure, cuting AB, by making the Angle EFD equal AEF, FD, i. e. CD, is parallel to AB.

COR. 1. Conv. If a Right line cuts two Right lines, which are parallel, the alternate Angles are equal to one another. For, AEF + BEF are equal to two Right angles. Th. 1. Alſo, BEF + EFD are equal to two Rt. angles. C. 1. 3. Wherefore, AEF = EFD; and BEF = EFC. Ax. 3.

COR. 2. If a Right line cuts two parallel Lines, it makes equal Angles with them both. viz. GEB equal to GFD.

THEOREM V. 30. Euclid.

If two, or more, Right lines are parallel to any other Right line, they are parallel between themſelves.

Let the two Right lines, AB and CD, be both parallel to the ſame Right line, EF.

I ſay, that AB is parallel to CD.

Let any Right line (AG) cut them all; being produced, if neceſſary.

DEM. Becauſe AB is parallel to EF, and they are both cut by the Rt. line AG, the Angle BAE = FEG. - - C. 2. 4. And, for the ſame reaſon, the Angle DCG = FEG; wherefore, the Angle BAE is equal to DCG. - Ax. 3. But, thoſe Angles are internal and oppoſite; - Def. 16. therefore, AB is parallel to CD. - - - 2. Th. 4.

SCHOL. *This Theorem might well paſs for an Axiom; for, by adopting Parallelism for Things, or Quantity, 'tis the ſame as Axiom the Third.*

THEOREM VI.

If two Right lines cut each other, and, if there be two other Right lines (in the ſame Plane with the former) parallel to them, reſpectively, and alſo cut each other, they contain equal Angles.

Let the Right lines DE and DF (or *DE* and *DF*) be reſpectively parallel to AB and AC.

I ſay, the Angle FDE is equal to CAB.

Produce either Side (if it be neceſſary) as DE, cuting AC at E; and *DF* at *F*.

DEM. Now, becauſe DE is parallel to AB, and they are both cut by AC, the Angle CAB=CE*F* - - C. 2. 4.
And, becauſe DF (or *DF*) is parallel to AC, and are both cut by DE, the Angle FDE=CE*F* - - - ſame.
But, the Angle CAB=CE*F*; and FDE=CE*F* proved.
Therefore, the Angle FDE is equal to CAB. - Ax. 3.

Or; becauſe *DF* is parallel to DF, and both cut by D*F*, the Angle *EDF*=D*FG*.
And, for the ſame reaſon, FDE=D*FG*,
conſ. *EDF* is equal to FDE; - - - - - Ax 3.
and therefore, *EDF* is equal to CAB.

This valuable Theorem is not in Euclid, except the 10*th of the eleventh Book, may be compared with it; in which, the two Lines (cuting each other) are, reſpectively, in different Planes; but they are, neceſſarily, either in the ſame Plane, or in parallel Planes, and it holds equally true, in both.*

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THEOREM VII. 8. Euclid.

If the three Sides of one Triangle are equal, reſpectively, to the three Sides of another Triangle, they are alſo equiangular; i. e. the Angles oppoſite to equal Sides, are alſo equal, and the whole Triangles are congruous.

Let the Triangles acb, ACB, have all their three Sides; equal; viz. a b equal to AB, b c equal BC, and a c equal AC.

I ſay, the Angle a is alſo equal to A, b equal B, and c equal to C; and the Triangle a b c, equal to ABC.

DEM. Suppoſe the Triangle a b c applied to ABC; any Side, a b, to its equal AB, with the Point a to A. - Poſt. 5.

Then, if the Triangles are alike ſituated, a c will fall on AC, and b c on BC; the Angle a will coincide with A, b with B, and c with C.

For, becauſe the Side a b is equal to AB, the Extreme a being applied to A, b will fall on B; and, becauſe a c is equal to AC, and b c to BC, the Point c muſt neceſſarily fall on C.

If not, let c, if poſſible, fall at d, in the Ark C d, deſcribed on A, with the radius ac; wh. Ad = AC, equal ac.

But, B d is not equal to b c; for, if C e be an Ark of a Circle deſcribed, on B, with the Radius b c, being equal to BC, it will cut the Ark d d, at C, and the Line A d, at e. Wherefore, the Point c cannot fall at d, nor elſewhere but on C, in the interſecting Point of the two Arks.

Th. the Triangles, agreeing in every part, are congruous.

N. B. This Property does not hold good in any other Figure.

THEOREM VIII. 4. Euclid.

Triangles having two Sides, in each, reſpectively equal, and if the Angles, contained by thoſe Sides, are alſo equal; then, the remaining Side, of one, ſhall be equal to the remaining Side of the other; the remaining Angles, of one, equal to the remaining Angles of the other, viz. thoſe that are oppoſite equal Sides; and the whole Triangles are equal, and congruous.

Let the Triangle, a b c, have two Sides, a b, b c, equal, reſpectively, to AB and BC, of the Triangle ABC; and, let he Angle a b c be equal to ABC.

Then, the Side ac is equal to AC; the Angle a is equal A, and c equal C; and the whole Triangle abc, equal to ABC.

DEM. Suppoſe the Triangle, a b c, applied to, or laid upon, the Triangle ABC, in ſuch wiſe, that the Side a b, of the one, agrees with AB of the other Triangle; the Extreme a, with A, and b with B. - - Poſt. 5.

Then, the Side b c will fall upon BC; - by Def. 14; for, they contain equal Angles, by the Hypotheſis.

The Extreme c, of the one, will coincide with C, of the other; becauſe, b c is equal to BC.

Therefore, the remaining Sides, a c and AC, muſt neceſſarily agree, and be equal to each other; the Angle a will coincide with A, and c with C; and conſequently, the whole Triangles, a b c, ABC, are congruous; therefore, the Triangle a b c is equal to ABC. Q. E. D.

COR. Hence, if two Sides of one Triangle be equal, reſpectively, to two Sides of another Triangle, and the contained Angle, of one, be greater or leſs than the contained Angle of the other, the remaining Side will alſo be greater or leſs; and converſely.

For, ſuppoſe two Triangles, a c b, ACB, having their Sides, a b and b c, reſpectively equal, to AB and BC.

It is evident, that, if the Angle a b c was the leaſt poſſible; i. e. if the inclination of the Sides, a b, b c, was ſuch, that it is not poſſible to conceive a leſs Angle, the two Sides, a b, b c, would nearly coincide; and conſequently, the length of the remaining Side, a c, would be ſomewhat more than the difference between a b and b c; viz. a c = a b − b c, nearly.

Again. Suppoſe the Angle made by the Sides AB, BC, equal a b, b c, the largeſt poſſible, ſo as, nearly to fall into one Right line; it is manifeſt, that the remaining Side, AC, would be ſomewhat leſs than the length of both, AB and BC; i e. AC = AB + BC, nearly.

Hence it is plain, that, whether the Angle, a b c or ABC, contained between a b and b c, or AB and BC, be greater or leſs, the remaining Side, a c or AC, will alſo be greater or leſs; and converſely, the greater or leſs the Side, a c or AC, the greater or leſs is the Angle, a b c or ABC.

For, imagine the Triangle a b c applied to ABC, the Side a b to AB, the Angle a to A, and b to B; and ſuppoſe an Ark of a Circle deſcribed, from c to C, on the Center B.

Then, ſuppoſe the Angle a Bc opened or enlarged, to the ſeveral Apertures, a Bd, a Be, a BC; it is evident, that as the Angle a B c is increaſed, at a B d, &c. ſo is the remaining Side, A d, A e, &c. continually longer; from Ac, the ſhorteſt poſſible, to AC, the longeſt.

HENCE the 24th and 25th of Euclid are clear, and manifeſt.

THEOREM IX. 5. Euclid.

In Iſoſceles Triangles, the Angles at the Baſe are equal to one another; i. e. if a Triangle have two equal Sides, the Angles, oppoſite to them, are alſo equal.

Let ABC be an Iſoſceles Triangle; the Side AB equal to BC. The Angle A is equal to C.

Let BD biſect the Angle ABC, made by the equal Legs, AB, BC; it will biſect the Baſe, AC.

DEM. In the Triangles ABD, DBC, the Sides AB, BC are equal (Hyp.) and, BD is common to them both.

Now, the Sides, AB, BD, are equal CB, BD, reſpectively; and the Angle ABD is equal to DBC. Con. Th. AD=DC, and the Triangle ABD=DBC. Theo. 8 But, in congruous Triangles, oppoſite equal Sides are equal Angles. (Th. 7 and 8.) Therefore, the Angle A=C, being oppoſite the common Side, BD. Q. E. D.

COR. 1. The external Angles, made by producing equal Sides, or the Side lying between them, are alſo equal.

2. An Equilateral Triangle, is alſo equiangular.

For, oppoſite to equal Sides are equal Angles.

3. If a Triangle have two equal Angles, the Sides oppoſite them are alſo equal. And, if all the three Angles of a Triangle are equal, it is equilateral.

4. A Right line, biſecting the Angle, and the Baſe, contained between equal Sides, is perpendicular to the Baſe; and conſequently, two Right lines drawn from the ſame Point in a Perpendicular to a Right line, to Points equally diſtant from the Perpendicular, are equal.

For, the Angles ADB, BDC, being equal, are conſequently Right angles. C. 2. Th. 1.

THEOREM X. 32. Euclid.

If any Side of a Triangle be produced, the external Angle, made thereby, is equal to the two remote Angles of the Triangle.

And, the three Angles of every Triangle are, together, equal to two Right angles.

In the Triangle ABC, let any Side, as AC, be produced. Then, the Ang. BCD is equal to the two Angles, A and B. From C, draw CE parallel to AB.

DEM. Becauſe CE is parall. to AB (Con.) and they are both cut by the Right line BC, the Angle BCE = ABC. Th. 4. And, becauſe AB and CE, being parallel, are cut by the Right line AD, the external Angle, ECD = BAC, the internal and oppoſite; - - - 2. Th. 4. But, the Ang. BCE (eq. B) + ECD (eq. A) = BCD. Ax. 2.

Therefore, the external Angle, BCD, = A + B, the two remote Angles of the Triangles. Q. E. D. - Ax. 3.

2. The three Angles, A + B + BCA, are equal to two R. angles.

Having produced AC and drawn CE, parallel to AB.

DEM. The Angle BCE = ABC, and ECD = BAC; - 4. wh. BCD (eq. BCE + ECD) is equal to ABC + BAC. But, the Angle ACB, + BCD, = two Right angles. - 1.

Therefore, the three Angles, A + B + BCA, are equal to two Right angles.

This 2nd part is alſo elegantly proved, by drawing a Right line, FG, through any Angle, as B, parallel to the oppoſite Side, AC.

DEM. For, the Angle FBA = A, and GBC = BCA - 4. conſ. the Ang. A + ABC + BCA (eq. FBA + ABC + CBG) are equal to two Right angles. - - C. 3. Th. 1.

COR. 1. Hence, if one Angle of a Triangle be equal to the other two Angles, it is a Right one.

For, if either Side, containing the Right angle, be produced; the external Angle, being equal to the two remote ones, is equal to the Rt angle to which it is contiguous.

2. There can be but one Right angle, or one obtuſe Angle, in a Triangle. Becauſe, if there be one Right angle, the other two are, together, equal to a Right angle.

3. If a Triangle have one Right angle, the other two are acute, and equal to a right one; and, if the Triangle be Iſoſceles, the two Angles, at the Hypothenuſe, are each equal half a Right angle. For, they are equal. - Th. 9.

4. If the ſum of two Angles of a Triangle be known, the other is alſo known; For, it is the Complement of two Right angles. Alſo, if one Angle be known, the Sum of the other two is known.

5. If the Sum of two Angles of a Triangle, either together or ſeparately, be equal to two Angles of another Triangle, the two remaining Angles are equal.

For, the Sum of all the Angles of every Triangle is the ſame; i. e. they are equal to two Right angles.

6. The equal Angles of every Iſoſceles Triangle are acute.

For, however ſmall the Angle at the Vertex, the Angles at the Baſe are, each, equal to half the difference between that Angle and two Right angles.

7. The Angles of an Equilateral Triangle are each 60 Degs.

For, it is two thirds of a Right angle, or one third of two Right angles.

8. All the Angles of every Quadrilateral are, together, equal to four Right angles.

For, by drawing a Diagonal, it is divided into two Triangles; and all the Angles, of each Triangle, are equal to two Right angles; conſequently, the ſum of all the Angles, of both, are equal to four Right angles; and they are equal to all the Angles of the Quadrilateral.

Note. The two following Theorems are alſo deducible from the laſt Propoſition; which, not being elementary, are not numbered amongſt the other Theorems.

THEO. 1. *All the internal Angles, of every right lined Figure, are equal to twice as many Right angles, wanting four, as the Figure has Sides.*

Every right lined Figure, may be reſolved into as many Triangles, as the Figure has Sides, wanting two; by drawing Diagonals, from any Angle, to all the oppoſite Angles.

In the Irregular Hexagon ABCDEF, let there be drawn the Diagonals AC, AD, and AE; which divides it into four Triangles, ABC, ACD, &c.

DEM. Becauſe (by the Theorem) the three Angles of every Triangle are equal to two Right angles, the Sum of all the Angles of the four Triangles, ABC, ACD, &c. is equal to eight Right angles.

And, they are equal to all the internal Angles of the Hexagon. Therefore, all the Angles of the Hexagon are equal to eight Right angles. But, a Hexagon has ſix Sides, and twice ſix is twelve; by taking away four, there will remain eight Right angles; which are equal to all the Angles of the Figure.

Or; by aſſuming any Point (as F) within the Figure ABCDE, and drawing Lines to every Angle, from that Point, FA, FB, &c. it will be divided into as many Triangles, as the Figure has Sides.

Now, the Angles of all the five Triangles are equal to ten Right angles. - - - - - - Theo. 10.

But, one Angle, of each Triangle, is at the Point F, within the Figure; and all the Angles about a Point are equal to four Right angles (Cor. 2. 2.) conſequently, the remaining Angles, of the Triangles, are equal to all the Angles of the Figure; which, in a Pentagon, are equal to ſix Right angles.

COR. Hence, the Sums of all the Angles of every right lined Figure, having an equal number of Sides, are equal.

THEO. 2. *All the external Angles, made by producing every Side, of any right lined Figure, are equal to four Right angles.*

For, ſince the internal and contiguous external Angle, BAE+EAG, &c. are equal to two Right angles (Th. 1.) the Sum of all the internal and external Angles, together, are equal to twice as many Right angles, as the Figure has Sides.

But, all the internal Angles are equal to as many, wanting four; conſequently, the external Angles are equal to four.

COR. The ſums of all the external Angles (taken together) of every right lined Figure, are equal.

SCHOL. *This Property of right lined Figures is really ſurprizing; for, the external Angles of a Triangle are equal to all the external Angles of any Poligon, whatever, and of any number of Sides.*

The general utility of the knowledge of this Property of Triangles, in the 10th Theorem, is very extenſive. It is, perhaps, the moſt perfect Demonſtration that can be given; and we may conceive an Idea of its extenſiveneſs, from the Corollaries and Theorems already deduced from it.

THEOREM XI. 26. Enclid.

Triangles having two Angles, and one Side, reſpectively equal to two Angles, and a correſponding Side; whether it be the Side, between the equal Angles, or either of the other, which are oppoſite to equal Angles; the remaining Sides and Angles ſhall be equal, and the Triangles are congruous.

The firſt part of this Propoſition, viz. when the equal Sides lie between the equal Angles, is ſo very obvious, that it might have been made a Corollary to the 8th, from which it is eaſily deduced.

Dem. For, AB, equal DE, being applied to DE (Poſt. 5.) if the Angle A be equal D, and B equal E (the equal Sides, AB, DE, lying between) then will AC fall on DF, and BC on EF. - - - Def. 14.

Conſ. the Angle C, will fall on F, and the Triangle ABC coincide, or agree in every reſpect, with DEF.

2. Let ABC and DEF be two Triangles, whoſe Angles A and C, of the one, are, reſpectively, equal to D and F, of the other; and let the Side AB, of the one, be equal to DE, of the other.

Then, the Triangles, ABC, DEF, are congruous.

Dem. Firſt, if the Side AC be not equal to DF, it is either greater or leſs; ſuppoſe it leſs, and take DG equal AC, and draw EG.

Then, becauſe AB=DE (Hyp.) and DG=AC, by Suppoſition; the Sides, AB, AC, are reſpectively equal to DE, DG; and the Angle BAC=EDG; - - Hyp. wh. EG=BC, and the Triangle DEG=ABC - - - 8. Conſequently, the remaining Angles, of the one, are equal to the remaining Angles of the other, which are oppoſite equal Sides; wherefore, the Angle DGE is equal ACB. And, DFE=ACB (Hyp.) th. DGE=DFE. - Ax. 3.

But, DGE is greater than DFE; - - - Th. 10. for it is an external Angle of the Triangle FEG; it is, therefore, both equal and greater, which is abſurd. Therefore, AC is not leſs than DF.

After the ſame manner, it may be proved not greater; conſequently, AC is equal to DF.

Th. the Triangle ABC is equal to DEF; and congruous.

AGAIN; if it be affirmed, that BC may be either greater or leſs than EF; ſuppoſe it greater; then,

Take BH equal EF, and join AH.

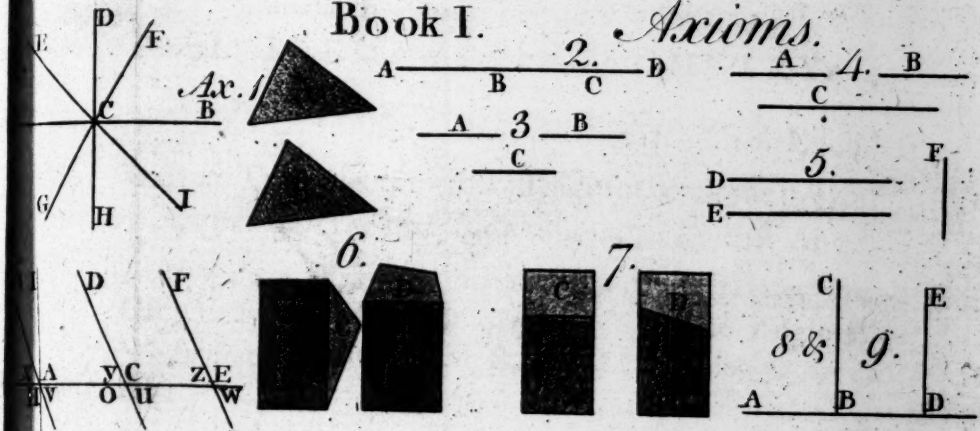
Then, the Angle BAC, being equal to EDF, is greater than the Angle BAH; which deſtroys the Hypotheſis. Wherefore, BC cannot be greater or leſs than EF.

This Theorem concludes the general properties of Triangles, which I have kept as near together as poſſible; by which properties, peculiar to Triangles, only, we can, with certainty, affirm them to be congruous, conſequently equal.

Firſt, when all the three Sides, of one Triangle, are equal, reſpectively, to the three Sides of another Triangle. (Th. 7.)

2nd. When there are found only two Sides reſpectively equal to two Sides, and containing equal Angles between thoſe Sides. (8.)

3dly. When two Angles, of one, are reſpectively equal to two Angles of another; and one Side of each Triangle, either lying between them, or oppoſite equal Angles, alſo equal; by the laſt,


Ax. 1
2.
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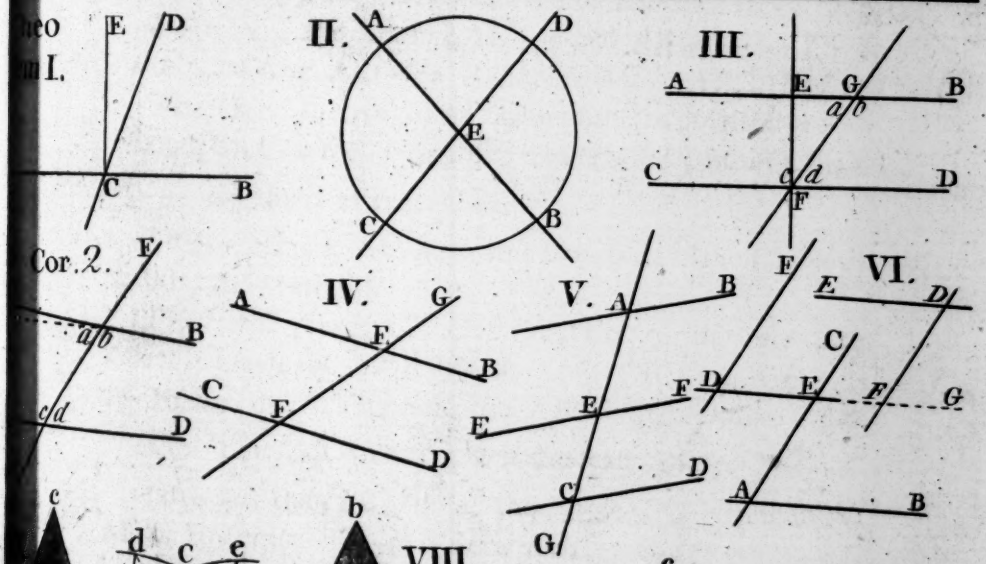
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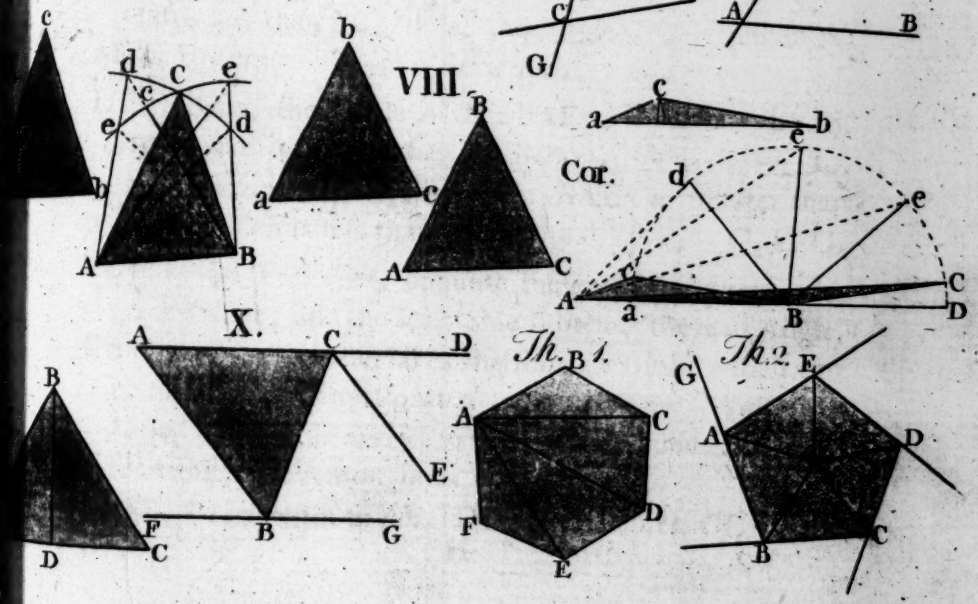
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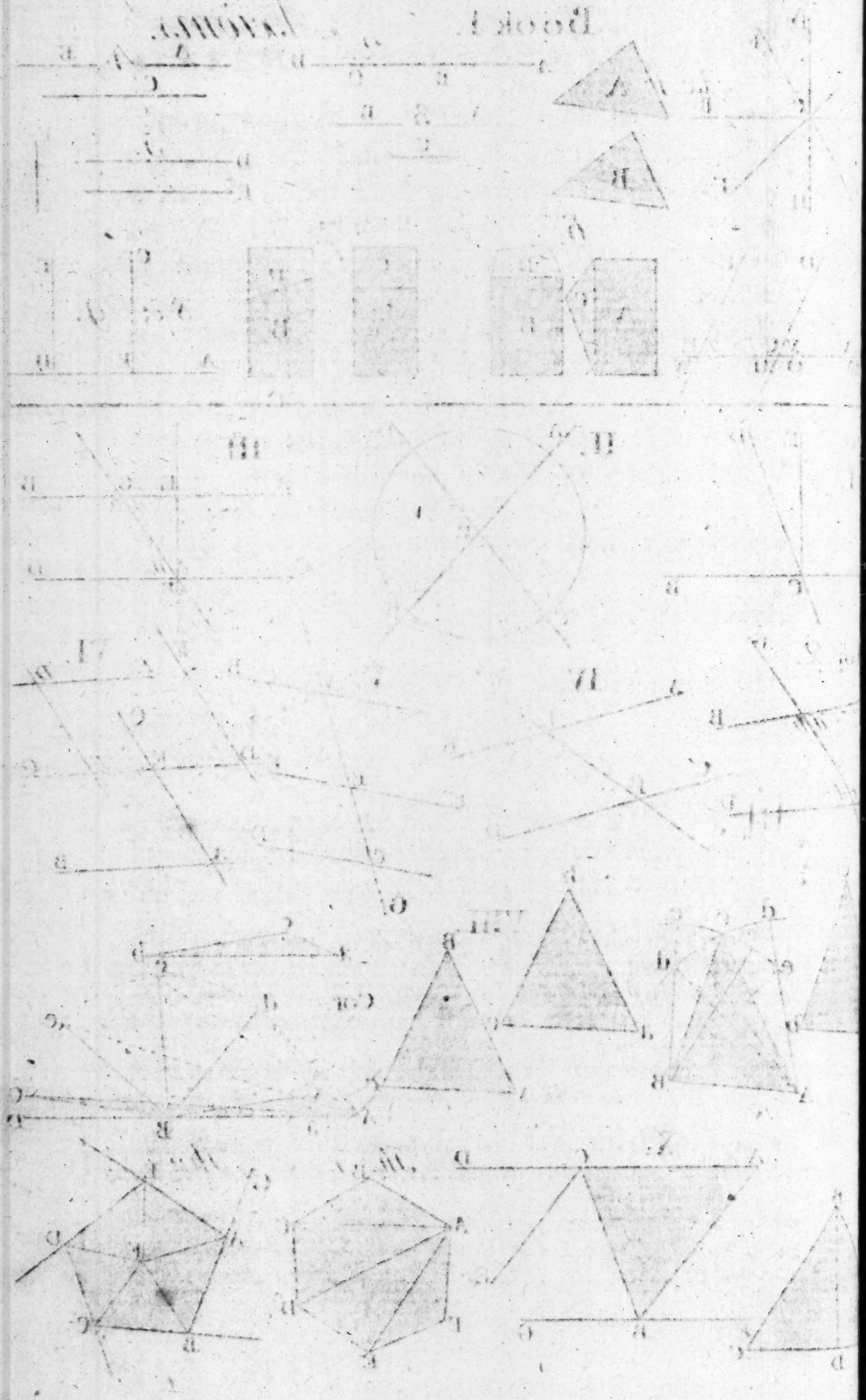


THEOREM XII. 18. Euclid.

In every Triangle, the greateſt Side ſubtends the greateſt Angle; and the leaſt Side ſubtends the leaſt Angle.

ACB is a Triangle, whoſe Side AC is greater than CB; I ſay, the Angle B is greater than the Angle A. Suppoſe AB biſected in E, and ED perpendicular to AB; join DB.

DEM. In the Triangle ADB, the Side AD=DB; - C. 4. 9.
wherefore, the Angle BAD is equal ABD. - - Th. 9.
But, the Ang. ABD (eq. BAD) is leſs than ABC. Def. 14.
Therefore, the Angle ABC (oppoſite to AC) is greater than BAC, which is oppoſite to a leſs Side, BC. Q.E.D.

Or thus, after Euclid.

By Hypotheſis, AC is greater than CB. Make CD equal to CB, and draw DB.

Then, the Angle CBD=CDB. Th. 9. But the Ang. CDB is greater than CAB. 10. And, CBD (eq. CDB) is leſs than ABC. Th. the Angle ABC is greater than CAB.

2nd. The leaſt Side ſubtends the leaſt Angle.

AB is leſs than BC; then, the Angle C is leſs than A. Make BE equal BA, and draw AE.

DEM. Now, the Angle AEB=BAE. - - - Th. 9.
But, AEB is greater than ACB. - - - - - 10.
Th. BAC, which is greater than BAE, is greater than ACB, which is leſs than AEB (equal EAB) Q.E.D.

COR. 1. In Triangles, oppoſite the greateſt Angle is the greateſt Side; and the leaſt Side ſubtends the leaſt Angle.

COR. 2. A Perpendicular is the ſhorteſt Line, which can be drawn from any Point to a Right line. And that Line, which falls neareſt to the Perpendicular, is ſhorter than the more remote ones.

Let CD be perpend. to AB. Draw, at pleaſure, CE & CB

Now, ſince CDE is a Right angle, CED is acute. C.3.10. Therefore, CE, which ſubtends the Right angle, is greater than CD. - - - - - - - - - - Theo.

2ndly. Becauſe the Angle CED is acute, CEB is obtuſe; conſ. CB is greater than CE. - Theo. Therefore, CE, which is neareſt to the Perpendicular, CD, is leſs than CB, or CF, which is more remote from it.

THEOREM XIII. 20. Euclid.

In every Triangle, any two Sides, taken together, are greater than the remaining Side.

Let AB be the greateſt Side of the Triangle ACB.
I ſay, the Sides AC, CB, are, together, greater than AB.
Produce AC to D; make CD equal CB, and join BD.

DEM. Then, becauſe CD=CB, the Ang. CBD=CDB 9.
But, the Angle ABD is greater than CBD, equal CDB.
Then, ſince in the Triangle ADB, the Angle ABD is greater than ADB; the Side AD is greater than AB - 12.
But AD=AC+CB (Con.) therefore, AC+CB, is greater than AB. Q. E. D.

THEOREM XIV. 21. Euclid.

If, from any Point within a Triangle, two Right lines are drawn to the extremes of any Side; thoſe Lines, together, are leſs than the other two Sides of the Triangle, but they contain a greater Angle.

In the Tri. ACB, aſſume any Point, E, and draw EA, EB.
I ſay, that AE added to EB, is leſs than AC added to CB.
And, the Angle AEB is greater than ACB.
Produce AE to D.

DEM. In the Triangle ACD, the Sides, AC, CD, are greater than AD. - - - - - - - - - - - - 13.
Add DB to both; and AC+CD+DB (eq. AC+CB) is greater than AD+DB. - - - - - - - Ax. 8.

Again. In the Triangle EDB, the Sides, ED, DB, are greater than EB.

Add AE to both; and AD + DB is greater than AE + EB.

But, AC + CB is greater than AD + DB, - - proved.

Therefore, AC + CB is ſtill greater than AE + EB.

2ndly. The Angle ADB is greater than ACB, and AEB is greater than ADB; - - - - - - - - 1. Th. 10. conſequently, AEB is ſtill greater than ACB. Q.E.D.

For, the Angle ADB is external, in reſpect of the Triangle ACD; and AEB, in reſpect of BED.

Cor. Hence; if two Triangles have one Side, in each, equal to the other, or have one Side common; and, if the remaining Sides, of the one, be leſs, reſpectively, than the remaining Sides of the other, they will contain a greater Angle.

THEOREM XV. 34 Euclid.

In every Parallelogram, the oppoſite Sides and Angles are equal. And, it is cut into two equal Triangles by a Diagonal.

Let ABCD be a Parallelogram.

The Side AB is equal to DC, and AD is equal to BC.

Alſo, the Angle D is equal to B, and A equal to C.

Draw either Diagonal; as AC.

Dem. By Def. 33. AB is parallel to DC; and they are both cut by AC; wh. the Angle BAC = ACD. - Th. 4.

And, becauſe AD is parallel to BC, and are both cut by AC; the Angle DAC = ACB; and AC is common; wherefore, the Triangles, ABC, ADC, are congruous - 11.

Therefore, the Side AB = DC, and AD = BC. - Def. 43.

Alſo, the Angle B = D (Th. 11.) &, DAB = BCD - Ax. 6.

(For, BAC = ACD, and DAC = ACB (Th. 4.) as above; and conſ. BAC + CAD = ACD + ACB.) - - Q.E.D.

Cor. Hence, it is manifeſt, if Right lines be drawn between the extreme Points of two Right lines, which are equal and parallel (ſo as not to croſs each other) they are alſo equal and parallel.

As AD and BC, joining the Points A & D, B & C.

Hence, the 33d of Euclid is evident.

THEOREM XVI.

The two Diagonals of a Parallelogram biſect each other. And, every Diameter divides a Parallelogram into two equal and congruous Figures.

In the Parallelogram ABCD, draw the Diagonals, AC and BD. I ſay, the Diagonal AC is biſected by DB; and, DB is alſo biſected by AC, in the Point E, of their mutual Interſection.

Dem. Becauſe, AB is parallel to DC, and are both cut by AC and BD; the Ang. BAC=ACD, & ABD=BDC. 4.
But, AB=DC (15.); wh. the Triangle AEB=DEC, and alſo congruous. - - - - Th. 11.
wh. DE=EB, & AE=EC, being oppoſite equal Angles.
Therefore, the Diagonals, AC, & BD, are biſected, in E.

2ndly. The Diameter, FG, divides the Parallelogram ABCD, into equal and congruous Figures.
AGFD is equal to FGBC, and congruous.

Dem. Now, the Triangle ABC is equal to ACD. - - 15
The Angle AEG=FEC (2.) and GAE=ECF, - 4.
and AE=EC (above) wh. the Tri. AGE=EFC. - 11.
Let them be taken away; there remains AEFD=CEGB.
and conſ. △AGE+AEFD=△EFC+CEGB. - Ax. 6.
Th. the Trapezium AGFD is equal to FGBC. Q.E.D.

Cor. Every Right line, which divides a Parallelogram in two Parts, and paſſes through the Center, is biſected at the Center. As, FG is biſected in E.

THEOREM XVII. 43. Euclid.

In every Parallelogram, the Complements are equal.

In the Parallelogram ABCD; aſſume any Point, as I, in the Diameter AC; through which, draw EF and GH, parallel to AB and AD, reſpectively. I ſay, the Complements, DI and IB, are equal to each other.

DEM. AC divides the Par. DB into two Tris. ABC=ACD; And the Parallelograms AGIE and IFCH, are divided into equal Triangles AGI=AIE, and IFC=ICH. - Th. 15. Now, if from the equal Triangles ABC, ACD, there be taken away equal Triangles AGI, IFC, =AIE, ICH, there will remain, the Par. DEIH equal to IGBF.-Ax.7.

THEOREM XVIII.

Containing the 35th, 36th, 37th, and 38th of Euclid.

Parallelograms, or Triangles, having the ſame or equal Baſes, and equal Altitudes, are equal.

The Parallelogram ABCD is equal to the Parallelogram AFGD; ſtanding on the ſame Baſe AD, and having the ſame or equal Altitudes.

Alſo, the Par. ABCD is equal to EFGH, having equal Baſes, AD equal EH.

DEM. AB=CD, AF=DG (15) and, the Ang BAF=CDG; 6.
wherefore, the Triangle ABF=DCG. - - - - 8.
But, the Triangle ICF is common to both;
wherefore, the Trapezium ABCI=IFGD. - Ax. 7.
Add, to both, AID; the Par. ABCD, =AFGD. - Ax. 6.

AGAIN. The Par. ABCD, is proved equal to AFGD; for the ſame reaſon, EFGH=AFGD, FG being common. Th. the Par. ABCD=EFGH. - - - Ax. 3.

But, AD=FG and EH=FG 15; wh. AD=EH (Ax.3.) Therefore, the Par^{s}. AC and FH have equal Baſes.

2ndly. The Triangle ABD is equal half the Par. ABCD; and, AFD is equal half AFGD. - - - Th. 15.

But, ABCD=AFGD; wherefore, ABD=AFD. Ax. 1.

Alſo, EGH is equal AFD, equal ABD;

for EGD=FEG, and FEG=FDG, equal AFD.

Th. EGD=AFD, which was equal ABD. - Q.E.D.

COR. Equal Parallelograms or Triangles, having the ſame or equal Baſes, have the ſame Altitude; or, are contained between the ſame Parallels.

This Corollary (the converſe of the Theorem) contains the 39th and 40th Propoſitions of Euclid.

SCHOL. *Hence it is evident, that two Spaces may differ greatly in Circumference, yet contain the ſame Space; and alſo, that two Figures or Spaces, of equal Circuit, may contain very different Areas.*

THEOREM XIX. 41. Euclid.

Every Triangle having the ſame Baſe as a Parallelogram, and equal Altitude, is eq. half that Parallelogram.

In the Parallelogram ABCD, from any Point in the Side AB, draw EC and ED.

The Triangle DEC is equal half the Parallelogram, ABCD. Draw EF, parallel to AD and BC.

DEM. The Parallelograms, AEFD and FEBC, are biſected, by the Diag^{s} DE&EC; ADE=DEF, & FEC=EBC-15.

Wh. the Triangle DEF + FEC, = ADE + ECB. -Ax. 6.

But, the Triangle DEC=DEF+FEC. - - Ax. 2.

Therefore, the Triangle DEC=half the Par. ABCD.

OR; produce AB, make B*E* equal to AE, and draw C*E*. The Triangle B*E*C is equal AED (8.) And the Parallelogram DE*E*C is equally divided by the Diag. EC. 15.

CASE the Second.

When *E* falls without the Par. ABCD, in AB produced. Draw *EF* parallel to BC; meeting DC, produced, in *F*.

DEM. D*E* biſects the Par. A*EF*D; wh. A*E*D = D*EF*. 15. and, the Diag. C*E*, cuts the Par. CB*EF*, in B*E*C = *ECF*. Wherefore, the Triangle D*EF* − C*EF* (equal D*E*C) is equal half the Parallelogram A*F* − B*F* (equal ABCD) i. e. the Tri. D*E*C is equal half the Par. ABCD. Q.E.D.

THEOREM XX. 47. Euclid.

In every Right-angled Triangle, the Square of the Hypothenuſe is equal to the Squares of the two Sides, containing the Right angle.

Let ABC be a Right-angled Triangle, and FB, BI, Squares, of AB and BC; alſo, let AD be the ſquare of AC, the Hypothenuſe.

I ſay, the Square AD, of the Hyp. AC, is equal to the two Squares, AG and BI of the Sides, AB and BC.

From the Right angle (B) draw BK perpendicular to AC; which, produce to L, parallel to CD.

The Square AG is equal to the Rectangle AL; and, the Square BI is equal to the Rectangle KD. Draw FC & BE.

DEM. In the Triangles FCA, ABE, AF, AC are reſpectively equal to AB, AE; and they contain equal Angles. For, FAB = EAC (Ax. 9.) and, if BAC be added to both, the Angle FAC = BAE. - - - - Ax. 6. Wh. FC = BE, and the Triangle FCA = ABE. - Th. 8.

But, the Tri. FCA is equal to half the Square FB - 19. (for, they have the ſame Baſe, FA; and, they are between the ſame Parallels, GC and FA.)

And, the Tri. ABE (eq. FCA)=half the Rect. AKLE; (on the ſame Baſe, AE, & between the Parallels, BL & AE)
Therefore, ſince the Tri. FCA=ABE (and they are reſpectively equal to half the Square AG, and the Rect. AL) the Square AFGB is equal to the Rectangle AKLE - Ax. 5.
Again; the Square BI is equal to the Rectangle CKLD;

Draw AI and BD.

Then, in the Triangles AIC, CBD, the Sides IC, CA, are equal to BC, CD, reſpectively.
And, the Angle ICA=BCD (ACB being common) Ax. 6.
wherefore, the Triangle AIC=CBD. - - Th. 8.
But, the Triangle IAC, is equal half the Square CH; (having the ſame Baſe, CI and between the Pars. AH, CI)
And the Tri. CBD (eq. IAC) =half the Rect. KD - 19. (being on the ſame Baſe, DC, & between the Pars. LB, DC)
Wh. the Square BI is equal to the Rectangle KD - Ax. 5.
But, the Square AG was proved equal to the Rect. AL. and AL + KD = AD (the Square of the Hyp. AC.) Ax. 2.
Therefore, the Square AFGB + BHIC = ACDE. Q.E.D.

OR, it may be, as elegantly and more briefly, demonſtrated after the following manner.

FB and BI are Squares of the two Sides AB, BC; let the Square of AC, the Hypothenuſe, be inverted; as AEDC.

Through B, draw KL parallel to CD. Join BD & BE; and produce IH to D.

Then, the Square AG is equal to the Rectangle AL; and CH to the Rectangle CL.

DEM. Now, the Tri. AEB=half the Square AFGB - 19. having the ſame Baſe, AB, and Altitude, BG.
The Tri. AEB is alſo equal half the Rect. AELK - ſame having the ſame Baſe, AE, and Altitude, AK,
Th. the Square AG is equal to the Rect. AL - Ax. 5.

Again; the Tri. BDC = half the Square BHIC; - 19.
(on the ſame Baſe, BC, & between the Parallels DI, BC).
And the Tri. BDC is alſo equal half the Rect. KLDC;
(having the ſame Baſe, CD, and Altitude LD.)
Wh. the Square BHIC = the Rect. KLDC. - - Ax. 5.
But, the Rect. AL + LC = the Square AEDC. - Ax. 2.
Th. the Square AEDC (of the Hyp. AC) is equal to the Squares, AFGB and BHIC, of the two Sides, AB&BC.

COR. (48. Euclid.) Hence; if, in a Triangle, the Square of one Side be equal to the Squares of the other two Sides, the Angle, contained by thoſe two, is a Right angle.

I have been particular in the Demonſtration of this famous Theorem, becauſe it is ſo extraordinary in itſelf, and of ſuch ſingular uſe throughout the Mathematics. It is alſo of ſuch uſe in the following Books, that, without it, we ſhould not be able to advance much further. Pythagoras is ſaid to be the inventor of it; for joy of which, it is ſaid, that he offered a hundred Oxen to the Deity, who inſpired him with ſo noble, and ſo uſeful a Theorem.

Notwithſtanding the peculiar elegance of the former Conſtruction and Demonſtration, I cannot but prefer this, for its eaſe and ſimplicity; ſeeing that, the ſame Triangle (AEB) is equal to half the Square of AB, and to the Rectangle AL, having the ſame Baſe and Altitude of both; and, BDC the ſame, in reſpect of the Square, BI, and the Rectangle LC. But 'tis neceſſary to prove, that the Square of the Hypothenuſe inverted, thus, muſt neceſſarily cut the oppoſite Side (FG) of the largeſt Square; and alſo, that the oppoſite Side, IH, of the leſſer Square, being produced, will paſs through the Angle D of the Sq. AD; in all caſes, whatever, and proportion of the Triangle.

DEM. FAB and EAC are Right angles, therefore equal, and EAB is common; wherefore, FAE = BAC.

Then, we have FA, AE, equal to AB, AC, reſpectively, and they contain equal Angles; conſequentl., AEF is congruous with the given Triangle, ABC.

And thus it must ever be, in every Right angled Triangle whatever; and therefore, the Angle E, of the Square AD, must fall in the Side FG, of the Square FB.

And, by the same reasoning, it is evident that CDI is also congruous with ABC; and consequently, that IH, produced, must pass through the Angle D. Q. E. D.

HAVING, in these twenty Theorems, gone through the first Book of the Elements of Geometry, it will not, I presume, be impertinent, before we proceed further, to say something of its utility. It is reasonable for a person, who has gone so far, to ask, of what use are all those properties of Triangles and Parallelograms, &c.? which question, though pertinent enough, it may not be so easy to give an answer to, which shall be satisfactory.

If the study of Geometry was of no other use, than for the enuring us to Demonstration; by which we may have conviction in the quest of Truth, it would be great. Geometry is Truth itself; the manner of reasoning, geometrically, is solid and convincing. It is necessary for a Person who would be an Orator, at the Bar, &c. to study Geometry; it teaches how to arrange our Ideas, to state the Premises fairly, and to draw Conclusions from them with clearness and certainty. But, when we consider that it is an Introduction to the Mathematics, a Key to let in the first Ideas to the most extensive knowledge, we shall pursue it with avidity.

The knowledge of the properties of Triangles and Parallelograms is very extensive, and of great use in most mathematical Sciences; as Surveying, Navigation, Perspective, Astronomy, and numerous others, depend on them. The knowledge of the most simple general properties of Triangles, in Theo. 7, 8, 9, 11, 12, and 13, is necessary for the Demonstration of others, as well as for reference from other Works. The 10th, a general property, and the 20th, a particular one, are more sublime properties of Triangles; and are, perhaps, the most perfect Demonstrations that can be given; which are frequently referred to hereafter. The equality of the Complements of Parallelograms, in the 17th, is of great utility; and the properties of Parallelograms and Triangles in the 18th, very extensive. The affinity between Triangles and Parallelograms, in Theorem 19th, is the foundation of Mensuration of Superficies.

IN respect of the properties of Right lines, and Angles formed by their Intersections, in the first six Theorems, necessary for attaining the properties of Figures, I shall give one instance, as a specimen, of what great things may be effected, from the knowledge inculcated by the fourth Theorem, viz. that the alternate Angles, formed by a Right line cuting two parallel Lines, are equal.

It would hardly gain credit, with ſome, to aſſert, that 'tis poſſible from the meaſure of a ſmall portion of the Earth's Circumference to determine the whole, by means of that ſimple property. Yet, nothing is more certain; and that, without much more previous knowledge, than, that the ſame portion or ark of every Circle gives, or ſubtends, equal Angles at the Center; and conſequently, equal Angles, being formed at the Centers of Circles of any Radius, will cut off equal portions of the Circumference.

It is practicable to meaſure the diſtance between two places, on a level Plane with accuracy; and, by means of an Inſtrument, called a Perambulator, the Ground between two Places, very remote, may be truly meaſured, let the Road be ever ſo irregular. In which caſe, it muſt be obſerved, that the meaſure, ſo obtained, is not the real Diſtance between thoſe places, which can only be meaſured by a Right line (ſee Def. 3.) and which is impoſſible to be had by an Inſtrument. And, let the Ground between two remote Places be ever ſo apparently level, it is not a Plane, but a ſpherical Surface; and conſequently, the meaſure, by an Inſtrument, over the Surface, is the meaſure of an Ark of a Circle, of which the Chord Line is the true Diſtance. (See Def. 22.)

This convexity of the Earth would be obvious, to ſight, if its Surface was regular. In a perfect calm, when the ſurface of the Sea is at reſt, it is diſcernable. A Ship going from Land, gradually appears to deſcend; the Hulk of the Veſſel is ſoon loſt to ſight; and, by means of a Teleſcope, the Sails and Rigging may be ſeen, when the Hulk is apparently ſunk in the Water; which is conviction of the Earth's rotundity.

Although it is poſſible to meaſure a ſmall portion of the Earth's Circumference, with tolerable accuracy, it would be impoſſible to meaſure the whole; becauſe it is, in no part of it, entirely circumſcribed with Land; and there is no dependence on the meaſures taken at Sea; therefore it is impracticable, to obtain its true Circumference by actual meaſurement.

It is eaſy to conceive, that two Right lines, or Poles erected perpendicular to the Earth's Surface, at any tolerable diſtance aſunder, are not parallel between themſelves; becauſe, the Earth being a Globe, or globular, every Right line, or Plane, which is perpendicular to its Surface, would, if produced, paſs through its Center. Conſequently, any two Right lines, which are perpendicular, would, if produced, form an Angle at the Center; which Angle is greater or leſs, in proportion to the diſtance of the Perpendiculars from each other. Hence it is plain, from what has been ſaid, that nothing more is required, than to determine the Angle which two Perpendiculars would form at the Earth's Center, according to the Diſtance between them.

SUPPOSE a ſmall portion of the Circumference (which may be repreſented by the Circle FGH) be meaſured, from B to D, where the Earth is as nearly on a level as poſſible. Let AB repreſent a Pole, erected perpendicular to the Horizon at B; it will conſequently, tend to the Center, at C; then, the Ark BD being meaſured, due North or South, and another Pole, DE, erected perpendicular to the Horizon, at D, it will alſo tend to the Center, C, making an Angle, ACE, with AB.

The whole myſtery is to determine that Angle.

IT muſt be obſerved, that this can only be performed at or between the Tropics; becauſe, the Sun is never vertical beyond them. Suppoſe then, AB erected perpendicular, within or on either of the Tropics; ED may be either North or South of it. When the Sun is in the Meridian of that Place, the Shadow of AB will be projected towards C, the Center of the Earth; for no Shadow of it can be projected on the Surface, the Sun being perpendicular over it. At the ſame Time, any other perpendicular Line, as ED, at ſome diſtance, North or South off AB, will caſt a Shadow, as DF, in proportion to the length of the Perpendicular DE.

Now, becauſe ED is perpendicular to DF, the Angle EDF is a Right one (Def. 11.). Wherefore, the meaſure of ED being known, and the length of the Shadow DF taken, DF being conſidered as a Tangent, and DE the Radius, the Angle DEF is determinable, by Trigonometry; or, by a Scale of equal parts, having made a right-angled Triangle ſimilar to DEF.

But, EF (being conſidered as a Ray of Light) is parallel to AB; for, from the immenſe diſtance and magnitude of the Sun, in reſpect of the Earth, its Rays, and, conſequently, Shadows of Objects, on the Earth, are projected in Right lines, parallel amongſt themſelves. But, EC is a Right line cuting them both; wherefore, the Angle ACE is equal to CEF (Theo. 4.) Therefore, the Angle, at C, is determined.

The Angle BCD being known; then, as its meaſure, in Degrees, * &c. is to 360, ſo is the Ark BD to the whole Circumference; and conſequently, as BD is to the Circumference, ſo is the meaſure of BD, in Miles, &c. to *its* meaſure.

* A DEGREE is the 360th part of the Earth's Circumference; ſo that, the Equator, a Meridian (Hour Circles) or any other Great Circle, is ſuppoſed to be ſo divided; each meaſuring 69½ Miles; giving 25020 Miles for the Circumference; ſomewhat leſs than 8000 in Diameter. Wherefore, the Circumference of a Circle, of whatever Radius, being ſo divided, ſuch Diviſions are called Degrees; by which Diviſions, Angles are meaſured. And, as two Diameters, perpendicular to each other, divide the Circumference into four equal parts, each containing 90 Degrees; a Right angle is, therefore, an Angle of 90 Degrees, and is a ſtandard of Proportion for all other.

Or, having described a Circle of any Radius, make an Angle (BCD) at the Center, equal to DEF, which a Ray of Light makes with the Perpendicular DE, it will intercept a portion of the Circumference, BD, which being taken in your Compasses (begining at B or D) see how oft it is contained in the Circumference FGH; which, multiplied into the number of Miles, measured on the Ark BD, will give the measure of the Earth's Circumference, in Miles.

It is also evident, that, if a large portion of the Earth's Surface was a Plane, instead of being spherical (of which let FG be a Section) we should, at distance from I, where a Right line, CI, from the Center, is perpendicular to FG, appear as if on a reclining Plane; and the farther from the point I, the greater will be the Inclination; insomuch that, in going from I, towards F or G, (i. e. in any, or in every direction from the Center, I,) we should seem as if climbing an ascent; making, at K, the acute Angle LKG, with the Plane FG. For, a Person, standing upright at K, would tend towards the Center C, in the direction LC; and consequently, would be more inclined to FG, the farther the point K is from the Perpendicular CI.

If such extraordinary Things can be done by the knowledge of this simple Property of Right lines, only, which is, in a manner, self-evident, how much greater Matters may be expected, from a thorough knowledge of the whole? cannot but occur to the Mind of every inquisitive and speculative Enquirer after Truth; it must determine him to lose no Time in acquiring it, and think all his labour inadequate to the acquisition.

ELEMENTS OF GEOMETRY.

BOOK II.

THE ſecond Book of Elements treats of the Powers of Right lines, divided at pleaſure, and otherwiſe; which ſeems, at firſt, being ſo very different from the foregoing, obſcure to beginers; but, if they carefully attend to what is ſaid, concerning Rectangles, the difficulty will ſoon vaniſh. The method or manner of Demonſtration is, in the firſt eight Theorems, both mathematical and ocular; it is alſo, in the ten firſt, numerical, and does not admit of the leaſt doubt to any, who are tolerably verſed in the fundamental rules of Arithmetic.

It is of eſpecial uſe in attaining a knowledge of the properties of the Circle, as contained in the third Book. For which reaſon, I adviſe the young Student, not to paſs it over ſlightly, but to digeſt it carefully, and be ſure he underſtands it perfectly; the ten firſt particularly, before he advances further; elſe, he will frequently be obliged to turn back to them. Therefore, I adviſe him to get them by heart, ſo, as to be clearly certain of the power of a Line, ſo and ſo divided, whenever it occurs in the courſe of the Work; it will greatly facilitate his ſtudy of the remainder, and ſave him ſome trouble.

This ſecond Book treats alſo, more fully, of the Theory of Triangles, in general; ſhewing, as in the right-angled, the proportion which the Square of any Side of a Triangle, of any ſpecies, has to the Squares of the other two; by which

means, a Perpendicular, and, consequently, the Area of any Triangle may be obtained by the measures of the Sides only, a Perpendicular being otherwise unattainable; with some other properties of Triangles and Parallelograms, which are not in Euclid.*

I have made no alterations in this Book, nor abridgements; but have added some elegant Theorems. The Demonstrations, in general, as well in the other Books as this, are as brief as is consistent with perspicuity, and the nature of Demonstration.

RECTANGLES and SQUARES are already defined. (See Def. 34, & 35, in the general Introduction to Geometry.)

A RECTANGLE, or right-angled PARALLELOGRAM, is said to be under, or contained under two Lines, which are the measures of its Sides.

If a Right line be any how divided, as A B, in C, the Rectangle A C B or B C A is the same; and signifies a Rectangle under the two Segments, AC & CB; i.e. AC × CB; as X.

But the Rectangles ABC and BAC are different from that, and from each other, if the Line be divided unequally.

ABC▭ signifies a Rectangle under the whole line, AB, and the Segment, BC, or AB × BC; as Y.

And, BAC means one under the whole line, AB, and the other Segment, AC, or BA × AC; as Z.

The middle letter is always twice meant.

AXIOM. RECTANGLES, or SQUARES, contained under equal Right Lines, are equal.

As V and X, under the Line a b, eq. AB, and b c, eq. BC. And the Squares Y and Z, on c d, equal CD.

THEOREM I.

The Rectangle contained under any two Right lines, is equal to all the Rectangles, under either Line, and the ſeveral Segments of the other, divided into any number of parts, at pleaſure.

Let AD and Z be the two given Lines; and let AD be divided, at pleaſure, in B and C.

I ſay, that Z multiplied by AB, added to Z multiplied by BC, added to Z multiplied by CD, are equal to AD mult. into Z.

On AD, ſuppoſe the Rectangle ADEF conſtructed, whoſe Sides AF and DE are each equal to Z. Through B and C, draw DG, and CH, parallel to AF and DE.

DEM. Becauſe AF and DE are both equal to Z; conſ. BG and CH are alſo equal Z. - - Th. 15. 1. & Ax. 3.
For, ADEF is a Rectangle, by Conſtruction;
wherefore, AG, BH, and CE are Rectangles.
But, the Rect. AE = all the Rects. AG, BH, & CE - Ax. 2.
and the Sides AF, BG, CH, and DE are each equal Z.
Th. $Z \times AB + Z \times BC + Z \times CD = Z \times AD$. - Q.E.D.

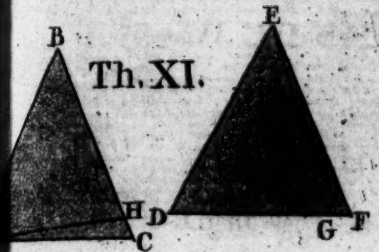
This is ſelf-evident; the whole Rectangle being divided into as many leſſer Rectangles as AD, or Z, into Parts.

Let AD repreſent, 12 Feet, and, let Z be equal 8.
Let the Segment, AB be 3, BC 4, and CD 5.

Then, the Rect. AG, i. e. $Z \times AB$, or $8 \times 3 = 24$.
\+ the Rect. BH, i. e. $Z \times BC$, or $8 \times 4 = 32$.
\+ the Rect. CE, i. e. $Z \times CD$, or $8 \times 5 = 40$.
= the Rect. AE, i. e. $Z \times AD$, or $8 \times 12 = 96$.

COR. If the two Lines are divided into Parts, at pleaſure; the Rectangle under the two whole Lines, is equal to all the Rectangles under the ſeveral ſegments of both Lines.

Th. XI.



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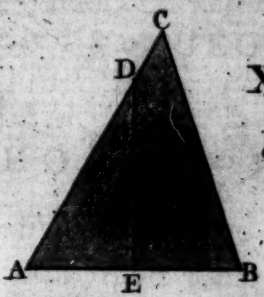

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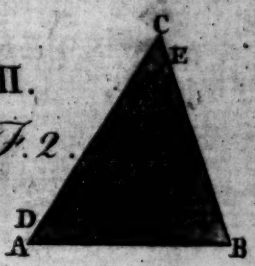

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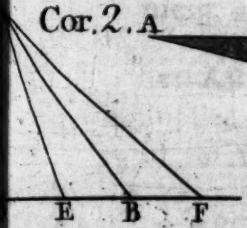

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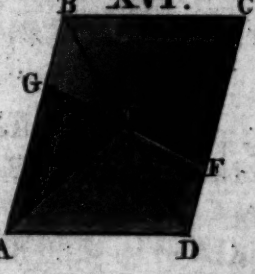

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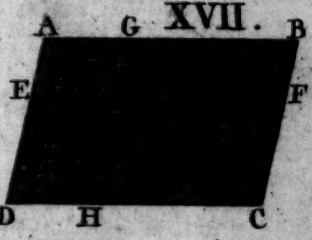

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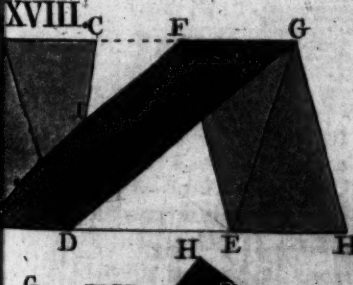

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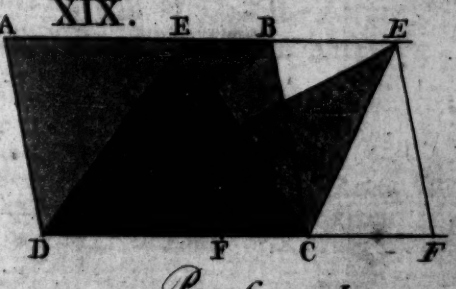

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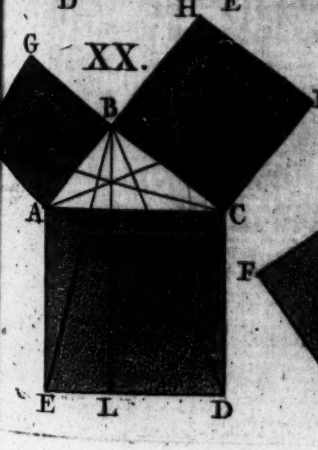

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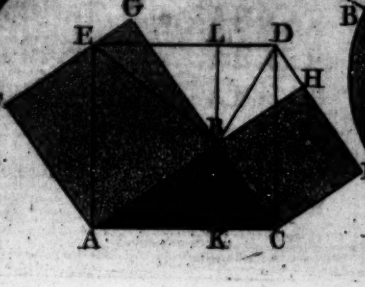

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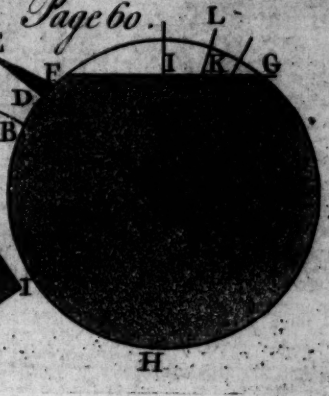
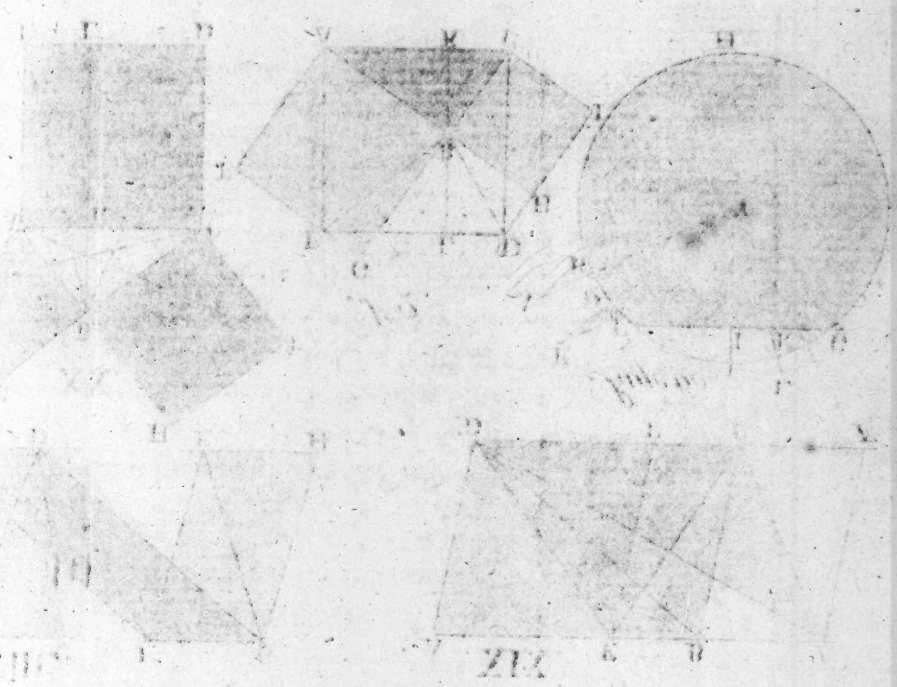




THEOREM II.

If a Right line be divided into two or more Parts; Rectangles under the whole Line and each Segment are equal to the Square of the whole Line.

Let AB be divided, at pleaſure, in E. I ſay, that AB multiplied by AE, added to AB multiplied by EB, is equal to AB multiplied by AB, i. e. the ſquare of AB.

Let ABCD be a Square, on the Line AB.
Through E, draw EF, parallel to AD.

DEM. Now, ABCD is a Square; by Conſtruction;
wherefore, AD, and BC are, each, equal to AB - Def. 35.
And, AEFD, EBCF are Rectangles - - - Con.;
wherefore, EF is equal to AD, or BC; - - - 15. 1.
But, the Square ABCD = the Rectangles AF, FB - Ax. 2.
Therefore, AB × AE + AB × EB = AB × AD, or AB□.

This Theorem is only a particular Caſe of the former, viz. when the two given Lines are equal. For another Line being aſſumed, equal to the given one; then, a Rectangle under the given and aſſumed Line is a Square; and conſequently, Rectangles under the aſſumed Line, and the ſeveral Parts of the given Line is equal to the Square of the given Line, or Rectangle under two equal Lines.

In Numbers, let the whole Line, AB, be 9;
and let it be divided in AE, 5,5, and EB, 3,5

Now, the Rect. AF, i. e. AB × AE, or 9 × 5,5, = 49,5.
\+ the Rect. FB, i. e. AB × EB, or 9 × 3,5, = 31,5.
= the Square AC, i. e. AB × AB, or 9 × 9, = 81,0.

THEOREM III.

If a Right line be divided at pleaſure; a Rectangle under the whole Line and either of the Segments is equal to a Rectangle under the two Segments, added to the Square of the Segment firſt taken.

Let AB be divided in E; unequally.

I ſay, that AB multiplied by BE is equal to AE multiplied by EB, added to EB ſquare.

Suppoſe a Rectangle ABCD, under the whole Line AB and the Segment EB. Draw EF parallel to AD.

DEM. The Rectangle AEFD added to the Square EBCF are equal to the Rectangle ABCD. - - - Ax. 2.
But, the Rect. AF is under the Segments AE, EB, (eq. EF) and, EC is the ſquare of EB, the Segment firſt taken.
Th. AB × EB (or ABE▭) = AE × EB (AEB▭) + EB□
Alſo, AB × AE = AEB▭ + AE□, of the leſſer Segment.

This Theorem is alſo a particular caſe of the firſt, viz. when the given Lines are, as there, unequal; which is the ſame thing as one, being ſo cut, that a Segment of it is equal to the leſſer Line. Or, when a Line given is divided at pleaſure; either Segment may be conſidered as another Line.

For, AB being divided, unequally, at E; take another Line, (as Z) equal to EB, or to AE.

Then, Z × AE + Z × EB = AB × Z (AB × EB) as in Th. 1.
But Z = EB, conſ. Z mult. into EB is the Square of EB.
And conſ. AB × EB, or ABE▭, = AEB▭ (AE × EB) + EB□

Let the whole Line AB be 10, 5; let AE be 4, & EB, 6, 5.
Then, the Rect. AF, i. e. AB × ED, or 4 × 6,5 = 26
+ the Square FB, i. e. EB × EB, or 6,5 × 6,5 = 42,25
= the Rect. ABCD, i. e. AB × EB, or 10,5 × 6,5 = 68,25

I have given this Line fractional, and alſo divided the former fractional-wiſe; for 'tis the ſame in Numbers, as in Lines, any how divided; which the young practitioner may prove by various fractional diviſions, if he be converſant in Decimals.

THEOREM IV.

If a Right line be divided at pleaſure; the Squares of the two Segments, added to two Rectangles under the Segments, are equal to the Square of the whole Line.

This is nothing more, or leſs, than the laſt taken both ways. For, a Rectangle under the whole Line and one Segment is equal to the Rectangle under the Segments, added to the Square of that Segment: and, the Rectangle under the whole Line and the other Segment is equal to the Rectangle under the Segments, added to the Square of the other; - - - - - - - by the 3rd.

But, Rectangles under the whole Line and each Segment are equal to the Square of the Line; - - by the 2nd.

Conſ. the Square of the Line is equal to two Rectangles under the Segments, added to the Squares of the two Segments.

Let AB, the given Line, be divided at E.

Then will the two Squares, of AE & EB, added to two Rectangles, under AE & EB, be equal to the Square of AB.

On AB, ſuppoſe the Square ABCD conſtructed. Draw EF parallel to AD; draw AC; and, through I, draw GH parallel, to AB.

DEM. Then, becauſe ABCD is a Square (Con.) the Sides, AD, DC, & BC, are each equal to AB; and the Angles A, B, C, and D are all Right ones. - - - Def 35.

But, AEIG and IHCF are Squares, of AE & EB, 17. 1.

and GF EH are Rectangles, under the Segments AE & EB.

And they are all equal to ABCD, the Square of AB. Ax. 2.

Therefore $AE\square + EB\square + 2\ AE \times EB = AB\square$. Q.E.D.

Let the Line AB be 8, divided at E, in 5 and 3.

Then, the Square GE, i. e. AE □, or $5\times5,=25$
+ the Square FH, i. e. EB □, or $3\times3,=9$
+ two Rect. GF & EH. 2 AEB ▭, 2. $5\times3,=30$
= the Square ABCD, i. e. AB □, or $8\times8,=64$

COR. If a Line be equally divided, a Rectangle under the Segments is the same as the Square of either.

Hence, the Square of a whole Line is equal to four times the Square of half the Line. $4\times4\times4=8\times8,=64$.

THEOREM. V.

If a Right line be biſected, and alſo cut unequally, at pleaſure; a Rectangle under the two unequal Parts, added to the Square of the intermediate part, is equal to the Square of half the Line.

Let AB be cut equally in C, and unequally in D. Then, the Rectangle under AD and DB, added to the Square of CD, are equal to the Square of AC, or CB.

The Rectangle under the unequal Parts is equal to a Rectangle under half the Line and leſſer Segment, + ▭ under the intermediate Part and leſſer Segment, by the firſt; and, the Rectangle under the half Line and leſſer Segment = ▭ under the intermediate Part and leſſer Segment + □ of the leſſer Segment; by the 3rd. Conſequently, the Rectangle under the unequal Segments = 2 ▭s under the intermediate Part and leſſer Segment, + □ of the leſſer. Ax. 2.

Add, on both Sides, the Square of the intermediate Part; then, ▭ under the unequal Parts, + □ of the intermediate Part, = 2 ▭s under the intermediate Part and leſſer Segment, + Square of the leſſer Segment, and of the intermediate Part; - - - - - - - - - Ax. 6.

But, two ▭s under the leſſer Segment and intermediate Part, + 2 □s, of the leſſer Segment, and of the intermediate Part, = □ of half the Line, by the laſt. Therefore, the Rectangle under the unequal Parts + ſquare of the intermediate Part, is equal to the Square of half the Line.

On CB, half AB, ſuppoſe a Square CFEB.

Draw DG parallel to CF; make DH equal DB; draw KH parallel to AB, and AK parallel to CF.

DEM. Now CFEB is a Square (Con.) and AC = CB - Hyp.
wh. BE = AC (Ax. 3.) & AK, CI are, each = DH - Def. 7.
But DH = DB (Con.) wh. AK & CI are each = DB - Ax. 3.
Th. the Rect. AKIC is equal to the Rect. DGEB. - Ax.
Add, to both, the Rect. CG; AI + CG = CG + GB - 6.
but AI + ID (eq. AH) = ADB▭, & FGHI is the □ of CD;
(CI = BD; conſ. I F and GH are each equal CD. - Ax. 7.)
Th. the Rect. AH + the Square FH = the Square CFEB, 2.
i. e. AD × DB + CD □ = AC, or CB, □. Q. E. D.

Let the whole Line, AB, be 16; AC = CB, each = 8; let CD be 3, and DB 5; then AC + CD, eq. AD, = 11.

Then, the Rect. AH, i. e. ADB ▭, or $11 \times 5 = 55$
\+ the Square FH, i. e. CD □, or $3 \times 3 = 9$
= the Square CFEB, i. e. CB □, or $8 \times 8 = 64$

THEOREM VI.

If a Right line be divided in two equal Parts, and then produced, at pleaſure, or another Line be added; the Rectangle under the whole compounded Line and the Part added, together with the Square of half the given Line, is equal to the Square of half the Line and Part added, in one Line.

Let the given Line AB be biſected in C, and let BD be added to it. I ſay, the Rectangle ADB, added to CB ſquare, is equal to the Square of CD.

A Rectangle under the whole compounded Line, and the Part added, = 2 ▭s under the half Line +, □ of the Part added; by the firſt. And, the ſquare of half the Line and part added = 2 ▭s under the half Line and the Part added, + □ of the Part added, + □ of half the Line - - 4.

Therefore, the Square of half the Line is equal to the Rectangle under the compounded Line and the part added, + Square of half the Line.

On CD, ſuppoſe the Square CFED conſtructed; and, on AD, the Rectangle AKLD; by drawing AK parallel to CF, and KL to AD, making DL equal to BD.

Draw BG parallel to DE, and join FD.

DEM. The Rect. AI = IB (Ax.) and IB = GL - - 17. 1.
therefore, AI is equal to GL. - - - Ax. 3.
Conſ. the Rect. AI + IB + BL = IB + BL + GL - Ax. 6.
And, if to both, be added the Square, IFGH, of CB;
the Rect. AKLD + □ IFGH = □s. CFED. - Ax. 2.
i. e. ADB▭, or AD × DB + CB□ = CD□. Q. E. D.

Or thus. Produce BA, and make A*E* equal to BD. Then is *E*D biſected, in C, and unequally cut, in B. Wherefore, the Rectangle *E*BD, under the unequal parts, (equal ADB) i. e. *E*B × BD, + CB□, = CD□, half the Line.

Let AB be 12; (AC and CB, 6; each) let BD be 4. Then, the Rect. AKLD, i. e. AD × BD, or 16 × 4 = 64
\+ the Square IFGH, i. e. CB□, or 6 × 6 = 36
= the Square CFED, i. e. CD□, or 10 × 10 = 100

THEOREM VII.

If a Right line be divided, any how, at pleaſure; the Square of the whole Line, added to the Square of either Segment, are equal to two Rectangles, under the whole Line and that Segment, together with the Square of the other Segment.

AB, the given Line, is divided in the point E.

Then, the Square of AB, added to the Square of AE, is equal to two Rectangles under AB and that Segment, added to the Square of EB, the other Segment.

The Square of the Line is equal two Rectangles under the Parts, + □s of the Parts, 4th; to which, let the Square of either part be added.

Then, ▭ under the whole Line and that Part = ▭ under the Parts, + □ of that Part, by the 3rd; which, being doubled, and + □ of the other Part, we ſhall have two ▭s under the whole Line and one Part, + □ of the other Part, = □ of the whole Line, + □ of the contrary Part.

Suppoſe ABCD the Square of AB; draw EF parallel to AD; draw AC; and GH, through I, parallel to AB.

DEM. The Rect. DI = IB (17. 1.) add GE to both, & DE = GB.
But, the Rectangle DE + GB, = DI + IB + 2GE; - Th. 3.
and, if FH, i. e. the Square of EB, be added;
they are equal to the Square ADCB + GE. - - Ax. 2.
But, the Rectangles DE, GB, are under the whole Line, AB, and the Segment AE; for, AD = AB, and AG = AE. Con.
Therefore 2 AB × AE + EB □, = AB □ + AE □.
Alſo, DH + FB + GE, = DB + FH;
i. e. 2ABE▭ + AE □, = AB □ + EB □.

Let the whole Line, AB, be 8; let AE be 3, and EB 5.

Then the Square ABCD, of AB, i. e. $8 \times 8 = 64$
+ the Square AEIG, of AE, i. e. $3 \times 3 = 9$ } $= 73$

But, the two Rects. DE, GB, i. e. $2AB \times AE$, $8 \times 6 = 48$
+ the Square IFCH, of EB, $5 \times 5 = 25$
is also $= 73$

THEOREM VIII.

If a Right line be divided, any how, in two Parts; four Rectangles under the whole Line and either Segment, added to the Square of the other Segment, are equal to the Square of a Line, which is compounded of the whole Line and the Segment first taken.

Let AB be divided, at pleasure, in C.

Then, four Rectangles under AB and CB added to the Square of the Segment AC, will be equal to the Square of AB added to CB.

CB being the Segment taken, make BD equal to BC.

Two Rects. under a Line, divided in two Parts, and either Segment, + Square of the other, are equal to the Square of that Line + □ of the first Segment. - Th. 7.

But, a Rectangle under a Line and Segment of it, = ▭ under the Segments, + □ of the same; - by 3.

Wherefore, two such Rectangles, − □ of the Segment taken + □ of the other, = □ of the Line. - - Ax. 7.

Conf. in this case, a ▭ under the whole compd. Line and Segment of it, = two ▭s under the Segments, + □ of the same Segment; and, consequently, two such ▭s = four under the Segments, + two □s of Segment taken.

And, the Square of the other Segment, doubled, is equal four Squares of one Segment. - - Cor. to the 4th.

Wherefore, four ▭s under the Segments, + two □s of the firſt, + four □s of the other, = □ of the compounded Line + □ of the firſt Segment. - - as by 7.

But, four ▭s under Segments of a Line, + four □s of either, = four ▭s under Line and Segment. - - as by 3.

Therefore, four ▭s under the whole Line and one Segment, + □ of the other, are equal to the Square of a Line, compounded of the given Line and a Segment of it.

On AD, ſuppoſe the Square AEHD conſtructed. Draw CF and BG, parallel to AE; make AI equal AC, and AL equal to AB, and draw IK, LM, parallel to AD.

DEM. CF and BG are parallel to AE; and IK, LM, to AD; wherefore, FG, GH, HM, and MK, &c. are each equal CB; conſequently FP, GM, OQ, and PK are equal Squares.

And, EO, LN, NB, and QD are equal Rectangles.

But the Rect. EO + FP, LN + OQ, &c. = AB × BC - 3.

Conſ. the four Rectangles EO, LN, &c. added to the four Squares FP, GM, &c. are equal to four times AB × BC; and, if the Square IC (of AC) be added, they are all equal to the Square AEHD, of the Line, AD, compounded of AB and BD, equal the Segment CB. - Ax. 2.

i. e. 4ABC▭ or 4 times AB × BC + AC□, = AD□.

Let AB be 9, divided at C, in 6 and 3; and if BD be equal CB, then AD will be equal 12.

Then, the Rect. ABC, or AB × BC, i. e. 9 × 3 = 27.

Conſ. 4 Rectangles, AB × BC is equal to 108

\+ the Square of AC, —— 6 × 6 = 36

= the Square of AD, —— 12 × 12 = 144

THEOREM IX.

If a Right Line be biſected, and alſo cut in two unequal Parts; the two Squares of the unequal Parts are, together, double the Square of half the Line, together with the Square of the intermediate Part.

Let AB be biſected in C, and cut unequally in D.

Then, the Square of AD, added to the Sq. of DB, is equal to twice the Sq. of AC, added to twice the Sq. of CD.

Let CE be perpendicular to AB, and equal to AC or CB. Join AE and EB; draw DG parallel to CE; and, where DG cuts EB, draw FG parallel to AB; laſtly, draw AG.

DEM. Becauſe ACE is a Right angle, and CE is equal AC, the Angles CAE and AEC are half right. - C. 3. 10. 1.
and AE □ = AC □ + CE □; i. e. equal 2AC □. - 20. 1.
And, becauſe CG is a Rectangle, FG = CD. - - 15. 1.
But, EFG is a Right angle, and FEG is equal FEA, half right; wherefore EGF = FEG; conſequently, FE is equal to FG, therefore equal CD - - C. 3. 9. 1.
and conſ. EG □, = EF □ + FG □, = 2 CD □. - - 20. 1.

Now, AE □ = 2 AC □; and EG □ = 2CD □ - above.
But AEG is a Right angle, and AG □ = AE □ + EG □; i. e. equal 2 AC □ + 2CD □.

Alſo, AG □ = AD □ + DG □; for ADG is a Right angle.
And DG = DB; for, GDB is a Right angle, and DBG = DGB, equal CEB, half a Right one. - - 2nd, 4. 1.
Th. AG □, eq. AE □ + EG □, i e. eq. 2AC □ + 2CD □, is alſo equal to AD □ + DB, equal DG, □.

Therefore AD □ + DB □, = 2AC □ + 2CD □. Q.E.D.

Let the whole Line AB be 14, divided equally, in C, and cut unequally, in D, in 9 & 5; AC 7, CD 2, & DB 5.

Then, AD □, i. e. $9 \times 9 = 81$
+ DB □ $5 \times 5 = 25$ } $= 106$

And AC □, i. e. $7 \times 7 = 49$
+ CD □ $2 \times 2 = 4$ } $= 53, \times 2 = 106$.

THEOREM X.

If a Right line be biſected, and then produced at pleaſure; the Square of the whole compounded Line, together with the Square of the additional Part, will be double the Square of the half Line and the Part added, together with twice the Square of the half Line.

Let AB be biſected in C, & let BD be added.

I ſay, that the Square of AD added to the Square of BD, is equal to twice the Square of CD, added to tw ce AC ſq.

Let CE be perpendicular to AB, and equal to AC.

Draw EF parallel to AB, and DE to CE; produce FD. Draw EB, and produce it, until FG is cut in G; and laſtly, draw AE and AG.

Dem. Then, ecauſe, ACE is a R. angle, (con.) & AC=CE the Angles CAE and AEC are half Right - C. 3. 10. 1. and, for the ſame reaſon, CEB and EBC are half right. And, becauſe CEFD is a Rectangle, EF=CD; 15. - 1 and ſince EFG is a Right angle, and FEG (eq. DBG, eq. EBC) is half right, FGE is half right. - Ax. 3. wherefore, FG=FE, eq. CD; and DG=DB - C. 3. 9. 1.

Now, AE□=AC□+CE□; equal 2 AC Square. - 20. 1.

And, EG□,=EF□+FG□; i. e. equal 2 CD□.

But, AEG is a R. angle; for AEC, CEG are half right; wh. AG□,=AE□+EG□; i. e. equal 2AC□+2CD□

But, ADG is a R. angle; conſ. AG□=AD□+DG; and, ſince BD=DG, BD□=DG□; - - - - Ax.

Therefore, AD□+BD□=2 AC□+2 CD□. Q. E. D.

This Propoſition may alſo be demonſtrated as the former.

Let the given Line AB be biſected in C and let BD be added; alſo make A*E* equal BD, in BA produced.

Then, becauſe *E*D is biſected in C, and cut unequally in B, *E*B □ + BD □, = 2 CD □ + 2 CB □.

But *E*B = AD; for A*E* was made equal to BD.

Wh. *E*B □ + BD □ (eq. AD + BD □) = 2 CD □ + 2 CB □.

Let AB be 12; divided, in C, equally; and let BD be 4,

Then, the Square of AD, i. e. 16 × 16 = 256
\+ the Square of BD, 4 × 4 = 16 } = 272

But, the Square of CD, 10 × 10 = 100
\+ the Square of AC, 6 × 6 = 36

136 × 2 = 272

THEOREM XI.

If a Right line be divided in extreme and mean Proportion; a Rectangle under the whole Line and leſſer Segment is equal to the Square of the greater.

AB is the Line given.

CONSTRUCTION.

At either extreme, as A, let AD be perpendicular and equal to AB; which being biſected in E; join EB.

Produce DA, until EF is equal to EB; and make AC equal to AF. I ſay, AB is ſo divided, in C, that AB multiplied into BC is equal to AC ſquare.

The Squares ADHB, of AB, and AFGC, of AC, being completed; (by drawing DH and BH, parallel to AB and AD, reſpectively; alſo, FG and CG, to AC and AF) produce GC until it cuts DH, at I.

DEM. Then, becauſe AD is biſected in E, and AF is added, the Rect. DFA, i. e. DF × AF, □ AE □ = EF □. -Th. 6.

But, EF = EB; wh. EF □ = EB □; i. e. eq. AB □ + AE □.

Conſequently, DF × AF + AE □ = AB □ + AE □.

Take AE □ from both, and there remains DF × AF = AB □.

But, the Rect. DIGF, is under the whole Line DF, and the Segment AF; and, ADHB is the Square of AB; wherefore, the Rect. FI = the Square AH; and if the Rect. ADIC, which is common to both, be taken away, there is left the Rect. CIHB equal to the Square ACGF.
i. e. the Rect. ABC (for BH = AB) or AB × BC = AC □.

This Propoſition cannot be exemplified in Numbers, or the proof aſcertained arithmetically; for there is no Number, can be ſo divided, that the product of the whole multiplied by one Part, ſhall be equal to the other ſquared. Therefore a Line, ſo divided, and the Parts, are incommenſurable.

The Number 144, being divided in 89 and 55, comes within one, in 8000, nearly; for, if 144 be to 89, as 89 is to 55; then, 89 being the Mean, the Square of 89 would be equal to a Rectangle under 144 and 55, i. e. AB × CB. But, the Square of 89 is 7921, and the Rectangle under the Extremes (144 by 55) gives 7920, only, deficiency one. By Decimals we may approximate much nearer, but never be equal.

THEOREM XII.

In an obtuſe angled Triangle, the Square of the Side which ſubtends the obtuſe angle exceeds the Squares of the other two Sides, by two Rectangles, unde either Side containing the obtuſe Angle, and the part intercepted between the obtuſe Angle, and a Perpendicular, from the adjacent Angle, to the ſame Side, produced.

ABC is an obtuſe angled Triangle. Produce the Sides AB & CB; and, to them, let AE & CD be Perpendiculars.

I ſay, the Square of AC exceeds the Squares of AB and BC, by two Rectangles ABD, or CBE.

DEM. First, AC□ is equal to AD□ + DC□. - - - 20. 1.
and AD□ = AB□ + BD□ + 2 ABD▭ - by the 14th.
wherefore, AC□ = AB□ + BD□ + CD□ + 2 ABD▭.
But, BC□ is equal to CD□ + BD□.
Therefore, AC□ = AB□ + BC□ + 2 ABD▭.

In the ſame manner, AC□ may be proved equal to AB□ + BC□ + 2CBE Rectangles.

Conſequently, the Rectangles ABD and CBE are equal.

THEOREM XIII.

In every Triangle, the Square of the Side ſubtending an acute Angle is exceeded by the Squares of the other two Sides, which contain the acute Angle, by two Rectangles under either of thoſe Sides, and a part of it, intercepted between the acute Angle and a Perpendicular from the oppoſite Angle.

In the Triangle ABC, let CD be perpendicular to the Side AB. Then, the Square of AC is exceeded by the two Squares, of AB and BC, by two Rectangles, ABD, under the whole Side, AB, and the Segment BD.

And, AB□ + AC□ = BC□ + 2BA × AD.

DEM. Firſt, AB□ = AD□ + DB□ + 2ADB. - - Th. 4.
And, BC□ = CD□ + DB□. - - - - - - 20. 1.
Wh. AB□ + BC□ = AD□ + DC□ + 2DB□ + 2ADB.
But, ADB▭ + DB□ = ABD▭. - - - - Th. 3.
conſequently, 2 ADB▭ + 2 DB□ = 2 ABD;
wherefore, by ſubſtituting theſe for the other,
then, AB□ + BC□ = AD□ + DC□ + 2 ABD.
But, AC□ = AD□ + DC□. - - - - - - 20. 1.
Therefore AB□ + BC□ = AC□ + 2 ABD▭. Q. E. D.

Secondly, thus, more briefly, when an Angle is obtuſe. Produce either Side, AC, or BC; and, let BE be perpendicular to AC produced.

Now, AB□ = BC□ + AC□ + 2 ACE (by 12th) Add, on both Sides, the Square of AC. Then AB□ + AC□ = BC□ + 2 AC□ + 2 ACE▭. - - - - Ax. 6.

But, 2ACE▭ + 2AC□ = 2EAC; - - - 3.

Therefore, AB□ + AC□ = BC□ + 2EAC, or AE × AC.

From which it is obvious, that the Rect. BAD = EAC.

By the firſt Method it may be demonſtrated; that the ſquare of BC, is exceeded by the ſquares of AB and AC; by two Rectangles BAD; and by the ſecond, the ſquares of AB and AC, exceed the ſquare of BC, by two Rectangles EAC; therefore the Rectangle BAD is equal to EAC.

SCHOL. *From theſe two Theorems it is evident, that, if the ſquare of one Side of a Triangle exceeds the ſquares of the other two, the Angle ſubtended by that Side, and contained by the other two, is obtuſe; and, if the ſquare of one Side, be exceeded by the ſquares of the other two, it ſubtends an acute Angle; conſequently, when the ſquare of one Side is equal to the ſquares of the other two, the Angle it ſubtends is a Right one.*

COR. Hence, the Perpendicular, and conſequently the Area of any Triangle may be found, or obtained by the meaſure of the Sides only.

In the △ABC, AC□ + BC□ = AB□ + 2AC × CD. Wherefore, AC□ + BC□ — AB□ = 2AC × CD; by Th. i. e. if AB□ be ſubtracted from the ſum of AC□ + BC□, the remainder will be two Rectangles, under AC & DC. Wherefore, if half that Product be divided by the Side AC, the Quotient will be the Segment DC.

But BC□ = BD□ + DC□; or, AB□ = AD□ + BD□. wh. BC□ — DC□, or AB□ — AD□ = BD□. - 20. 1. Conſ. having ſubtracted the ſquare of DC from BC□; or of AD from AB□, the remainder will be the ſquare of the Perpendicular BD; the ſquare Root of which, gives the Perpendicular required.

By the ſame means, either of the other Perpendiculars, AE or CF, may be found.

2ndly. When the Perpendicular falls without the Triangle, from the acute Angles of an obtuſe angled Triangle, it is thus found.

In the obtuſe-angled Triangle ABC, let BD be Perpendicular to the Side AC, produced.

Then, $AB\square = AC\square + CB\square + 2AC \times CD$. - - - Th. 12.
Wherefore $AB\square, -AC\square + CB\square, = 2AC \times CD$;
half of which Sum being divided by the Side AC gives CD.
But, $CB\square = BD\square + CD\square$ (20. 1.) wh. $CB\square - CD\square = BD\square$;
the ſquare Root of which gives the Perpendicular BD.

THEOREM XIV.

In every Parallelogram, the Sum of the Squares of the two Diagonals is equal to the Sum of the Squares of all the Sides.

Let ABCD be a Parallelogram.

Draw the Diagonals, AC and BD; produce the Baſe, AD, and let BE and CF be Perpendiculars to AF.

DEM. In the obtuſe angled Triangle ACD;

$AC\square = AD\square + DC\square + 2ADF\rightarrow$. - - - Th. 12.
And, in the Tri. ABD; $BD\square = AB\square + AD\square - 2DAE\rightarrow$
But, EBCF is a Parallelogram, wh. EF = BC. - 15. 1.
and, AD = BC; wherefore AD = EF - - - Ax. 3.
conſ. AE = DF, ED being common; wh. $ADF = DAE\rightarrow$
Therefore, as much as $AC\square$ exceeds $AD\square + DC\square$, viz. by the the Rect. ADF, twice; by ſo much is $BD\square$ exceeded by $AB\square + AD\square$, eq. $BC\square$; viz. by $2\,DAE\rightarrow = 2ADF$.
Th. $AC\square + BD\square = AB\square + BC\square + AD\square + DC\square$.

THEOREM XV.

In Iſoſceles Triangles, the Square of one of the equal Sides is equal to the Square of any Line drawn from the Vertex to the Baſe, added to a Rectangle under the Segments of the Baſe made by that Line.

In the Iſoſceles Triangle ABC, if BD be perpendicular, it biſects the Baſe, and is evident; for $AB\square = BD\square + AD\square$; i. e. $ADC\rightarrow$, or $AD \times DC$. - - - 20. 1.

Let BE be drawn, at pleaſure.

I ſay, the Square of AB is equal to BE ſquare, added to the Rectangle AEC.

DEM. AB□ is equal to BD□ + AD□. - - 20. 1.
and AD□ is equal to AEC▭ + DE□ - - 5. 2.
(for AC, is cut equally in D, and unequally in E).
wherefore, AB□ = BD□ + DE□ + AEC▭.
But, BE□ = BD□ + DE□. - - - 20. 1.
Therefore, AB□ = BE□ + AE × EC, or AEC▭.

THEOREM XVI.

If a Perpendicular be drawn from any Angle of a Triangle to the oppoſite Side; the Squares of the Sides containing that Angle, added to the Squares of the alternate Segments of the Baſe, are equal; and the difference between the Squares of the Sides is equal to the difference between the Squares of the Segments.

In Iſoſceles Triangles the thing is manifeſt.

In the Scalene Triangle ABC, let BD be a Perpendicular, to the Side AC.

Then, the Square of AB added to DC ſquare is equal to BC ſquare added to AD ſquare.

And the difference, between the Squares of AB and BC, is equal to the difference, between the Squares of AD&DC.

DEM. For AB□ = AD□ + BD□.
And, BC□ = BD□ added to DC ſquare. - 20. 1.
(for the Angles, ADB and BDC, are Right ones. - Con.)
Wh. AB□ + BD□ + DC□ = BC□ + BD□ + AD□;
that is, AB□ + BC□, is equal to BC□ + AB□. - Ax. 3.
Let there be taken, from both, BD□, which is common;
there is left AB□ + DC□ equal to BC□ + AD□.
2nd. AD□ = AB□ - BD□; & DC□ = BC□ - BD□.
Therefore, AD□ - DC□ = AB□ - BC□. Q. E. D.

THEOREM XVII.

If any Side of a Triangle is biſected, and a Right line be drawn from the oppoſite Angle to the biſecting Point; the Squares of the other two Sides of the Triangle will be equal to twice the Square of the biſecting Line, added to half the Square of the Side biſected.

If the Triangle be Iſoſceles, the thing is clear; for the Squares of the equal Sides, are equal to twice the Square of the Perpendicular, added to twice the Square of half the Baſe.

In the Scalene Triangle ABC, let the Side AC be biſected in D, and let BD be drawn.

Then, the Squares of AB and BC, together, are equal to twice BD ſquare, added to twice AD or DC ſquare.

Let BE be perpendicular to AE.

DEM. $AB\square = AE\square + BE\square$; and $BC\square = BE\square + EC\square$.

Wherefore, $AB\square + BC\square = AE\square + EC\square + 2BE\square$.

But $AE\square + EC\square = 2AD\square + 2DE\square$. - - 9. & 10.

Wh. $AB\square + BC\square = 2AD\square + 2DE\square + 2BE\square$.

But, $BD\square = BE + DE\square$; conſ. $2BD\square = 2BE\square + 2DE\square$

Th. $AB\square + BC\square = 2BD\square + 2AD\square$, equal $ACD\square$, or half $AC\square$. Q.E.D.

By this Theorem, may alſo be demonſtrated the 14th; viz. The Squares of the two Diagonals of every Parallelogram are equal to the Squares of all the four Sides.

In the Par. ABCD (having drawn the Diagonals) through the Center, E, draw FG and HI, parallel to AB and AD.

In the △AED, AE□ + ED = 2EF□ + AF□ + FD□.
conſ. AE□ + ED□ = AH□ + AF□ + FD□ + DI□.
And, BE□ + EC□ = HB□ + BG□ + GC□ CI + □
But, AE□ + EC□ = ½AC□; &, BE + ED□ = ½BD□.
Alſo, AF□ + FD□ = ½AD□; & AH + HB□ = ½AB□.
and, the □s of BG, GC, CI, & ID = ½BC + ½CD□.
Th. AC□ + BD□ = AD□ + AB□ + BC□ + CD□.

THEOREM XVIII.

In any Trapezium, if the mdidle Points of two oppoſite Sides be joined by a Right line; the Sum of the Squares of the two other Sides, together with the Squares of the Diagonals, is equal to the Sum of the Squares of the biſected Sides, added to four times the Square of the Line joining the middle Points.

Let the Sides AB and CD of the Trapezium. ABCD be biſected in E and F; and draw EF.

I ſay, the Squares of the two Sides, AB and CD, added to four times the Square of EF, are equal to the Squares of BC and AD, together with the Squares of AC and BD.

Draw AF and BF.

Dem. Now, AF□ + BF□ = 2AE□ + 2EF□. - Th. 17.
and, AD□ added to AC□ = 2AF□ added to 2FD□,
alſo, BC□ added to BD□ = 2BF□ added to 2FD□.
conſ. AD□ added to AC□ added to BC□ added to BD□
is equal to 2AF□ added to 2BF□ added to 4FD□.
But, AF□ added to BF□ = 2AE□ added to 2EF□.
conſ. 2AF□ added to 2BF□ = 4AE□ added to 4EF□.
But, 4AE Square is equal to AB Square;
and, 4FD Square is equal to CD Square. - Cor. to 4.

Therefore, AD□ + AC□ + BC□ + BD□,
= AB□, + CD□ + 4FF Square . Q. E. D.

ELEMENTS

OF

GEOMETRY.

BOOK III.

THE knowledge of the Properties of a Circle is the ſubject of the third Book of Elements.

A Circle is the ſimpleſt, and moſt perfect of Plane Figures; the Properties peculiar to the Circle are moſt extraordinary; and, ſome, very extenſive in their application.

My chief aim, throughout the whole of this Work, being to render the Study of Geometry eaſy, and agreeable to young Students; for which end, I conſider brevity, if clear, moſt conducive. I have, therefore, reduced the moſt ſelf-evident Propoſitions into Axioms, for the more expeditiouſly attaining the knowledge of other, more eſſential Properties; and am perſuaded, that, if the knowledge of the moſt ſimple properties of Figures can be attained, by any other means, rigid Demonſtration is uſeleſs.

If a perfect knowledge of all the Properties of a Circle, contained in the Axioms of this third Book, be not acquired by a bare recital, and inſpection of the Figure, I ſhall pronounce that perſon's capacity inſufficient, to purſue the ſtudy of the more ſublime Properties; and, if they are clearly evident, without, of what uſe is Demonſtration?

When the truth or evidence of an Aſſertion cannot be perceived, by contemplating the Figure, it is then abſolutely neceſſary to demonſtrate, by a regular Proceſs; but, where the evidence of Sight is ſufficient, 'tis Demonſtration; the other is, in that caſe, uſeleſs, and unneceſſary.

There are ſome critical Geometricians who will not admit of any thing without Demonſtration; although they cannot but acknowledge many Propoſitions to be ſelf-evident.

It would, perhaps, be no eaſy matter to give geometrical Demonſtration, that all Right lines drawn from the Center of a Circle to the Circumference are equal; at the ſame time, no Perſon of common ſenſe can deny it, who knows any thing of the Figure, and geneſis of a Circle. If, therefore, ſome things muſt be given, or granted, why not others, which are as clear and ſelf-evident? for my part, I ſee no reaſon againſt it, and have, therefore, purſued the readieſt method for attaining the end aimed at; viz. to acquire a knowledge of Geometry in the eaſieſt and moſt familiar manner poſſible, by diveſting it of all that is ſuperfluous and unneceſſary; I mean unneceſſary Demonſtration, of what is clear without it.

I think it, however, neceſſary to apologize for the liberty I have taken in abridging it. Inſtead of 31 Theorems in this third Book, according to Euclid, I have but fifteen, from Euclid; the reſt are diſpoſed of in the manner following.

The 2nd, the 5th and 6th, the 10th and 13th, are made Axioms. The 9th is a Corollary to the 5th.

The 18th and 19th are Corollaries to the 8th;
the 23rd and 24th may be deduced from the 3rd.

The 11th and 12th are both included in the 7th.

The 26th and 27th are the 2nd Corollary to the 9th;
the 28th and 29th, Corollary 2nd of the 3rd; and,
the 37th is the 3rd of the 16th.

The ſix Problems are in the practical Part.

DEFINITIONS.

The general Definitions of a Circle, and its Attributes, are given in the General introduction, in Def. 19, and the four following, which need not be repeated here; ſome few more (which follow) are particularly neceſſary.

DEF. 1. EQUAL CIRCLES are ſuch as have equal Diameters or equal Radii.

2. Circles are ſaid to cut one another, when their Circumferences croſs or interſect each other in two Points. A and B.

3. Circles are ſaid to touch, when their Circumferences meeting, either internally or externally, in any part, they do not cut each other.
As X and Z touching in the Point E.

A TANGENT is a Right line, which, being in the ſame Plane with a Circle, touches the Circumference, only, without cuting it. As AB in B.
And, the Point B, in which it touches the Circle, is called the POINT OF CONTACT.

5. A SECANT is a Right line, drawn from any Point without a Circle cuting the Circumference in two Points. As AD, in C and D.

6. A CHORD, or SUBTENSE, is any Right line drawn within a Circle, cuting it into two Segments. See Def. 22 & 23, general Introduction.

7. The ANGLE of a SEGMENT, is the mixed Angle which is contained under the Chord line, and a portion of the Circumference. As ABD or ABE.
The Angle of a greater Segment is obtuſe, of a leſſer Segment it is acute; of a Semicircle, the Angles are Right ones; as ADB or C.

8. An ANGLE in a SEGMENT is that which is contained by two Right lines, drawn from each extreme of its Baſe, or Subtenſe, to any Point in the Ark of the Segment. As ACB or ADB.

9. SIMILAR SEGMENTS are ſuch as contain equal Angles, or whoſe Angles are equal.

10. A SECTOR, of a Circle, is comprehended between two Radii or Semi-diameters, and an Ark, or portion of the Circumference, intercepted between them.

As AC, CB, and the Ark AB.

If the Radii AC, CD contain a Right angle, it is called a QUADRANT; as ADC.

11. An ANGLE in a CIRCLE is ſaid to inſiſt, or ſtand, upon the Ark of the Circumference which is oppoſite to the Angle.

Circumferences of Circles are convex outwardly; towards the Center (inwardly) they are concave.

AXIOMS.

The Six Axioms, or ſelf-evident Propoſitions which follow, contain Properties of a Circle which are neceſſary to be known, previous to the Demonſtration of the following Theorems. I call them Axioms, becauſe they are ſelf-evident; but ſince that will not, by ſome, be allowed, arbitrarily, I have endeavoured to illuſtrate them, and to make the truth of the Premiſes appear clear and manifeſt.

AXIOM 1st, All Lines drawn from the Center of a Circle to the Circumference are equal. As EA, EC, &c.

For, they are all Radii or Semidiameters.

2nd. Two or more Diameters, of a Circle, mutually biſect each other.

For they all paſs through the Center; conſequently, they are biſected, in the Center.

As AB and CD, in the Point E; for EA, EB, &c. being Radii, are equal, by the firſt.

3rd. Circles, in the ſame Plane, which cut, or touch each other inwardly (outwardly it is manifeſt) have not the ſame Center.

For, if they have the ſame Center, and equal Radii, they muſt agree in every part. And, if they have the ſame Center, and unequal Radii, they are parallel Circles, and can neither cut nor touch each other, in any part; for their Circumferences are, every where, equi-diſtant, As AB and CD.

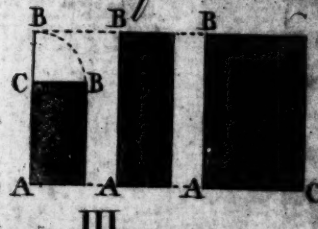
4th. Circumferences of Circles can cut each other in two points only.

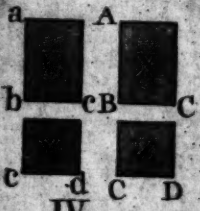
For, if it were poſſible to cut, or touch only, in three Points, A, B, & C, each of the points of Section, will be equally diſtant from the Center of each Circle; and conſequently they will have the ſame Center, which is contrary to the 3rd.

5th. A Right line joining two Points (A and B) in the Circumference of a Circle, falls entirely within the Circle. This is manifeſt.

For if not, the Curve of the Circle, ACB, muſt coincide in ſome entire part, with the Line AB; which is contrary to the 19th Definition, in the general Introduction, and the geneſis of a Circle, in N. B. Def. 20.; from which it is manifeſt, that the Curve of a Circle cannot fall in with a Right line, in any part, being uniform in every part.

Definitions Axioms





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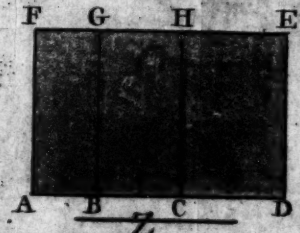

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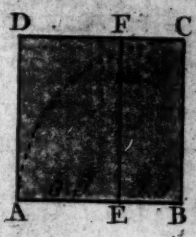

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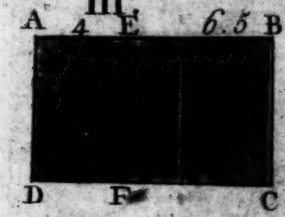

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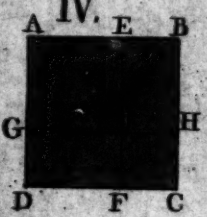

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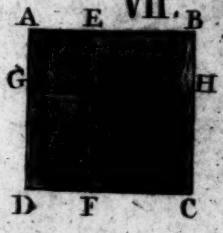

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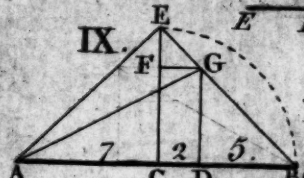

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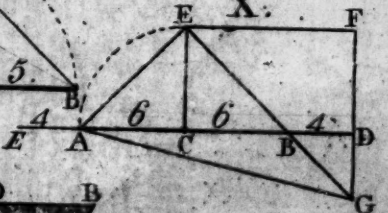

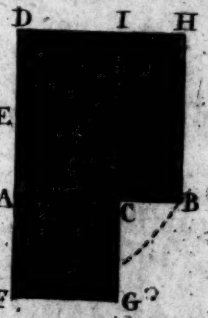

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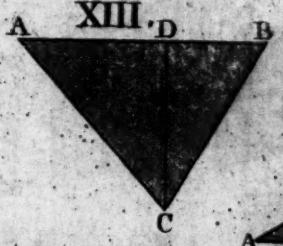

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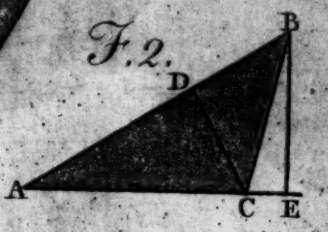

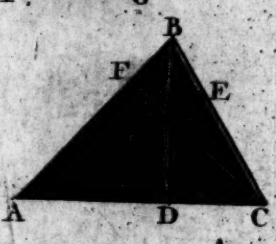

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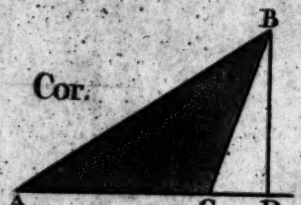

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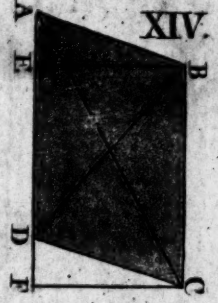

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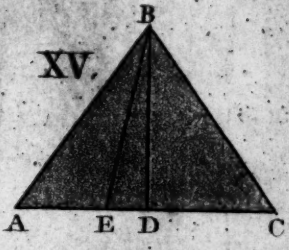

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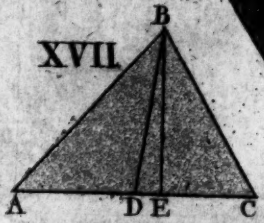

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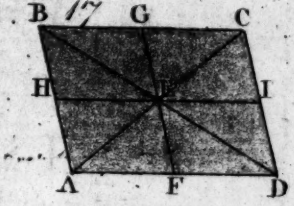

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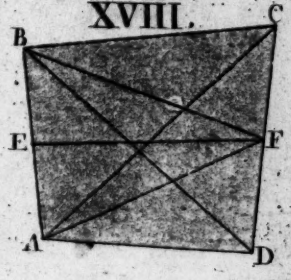
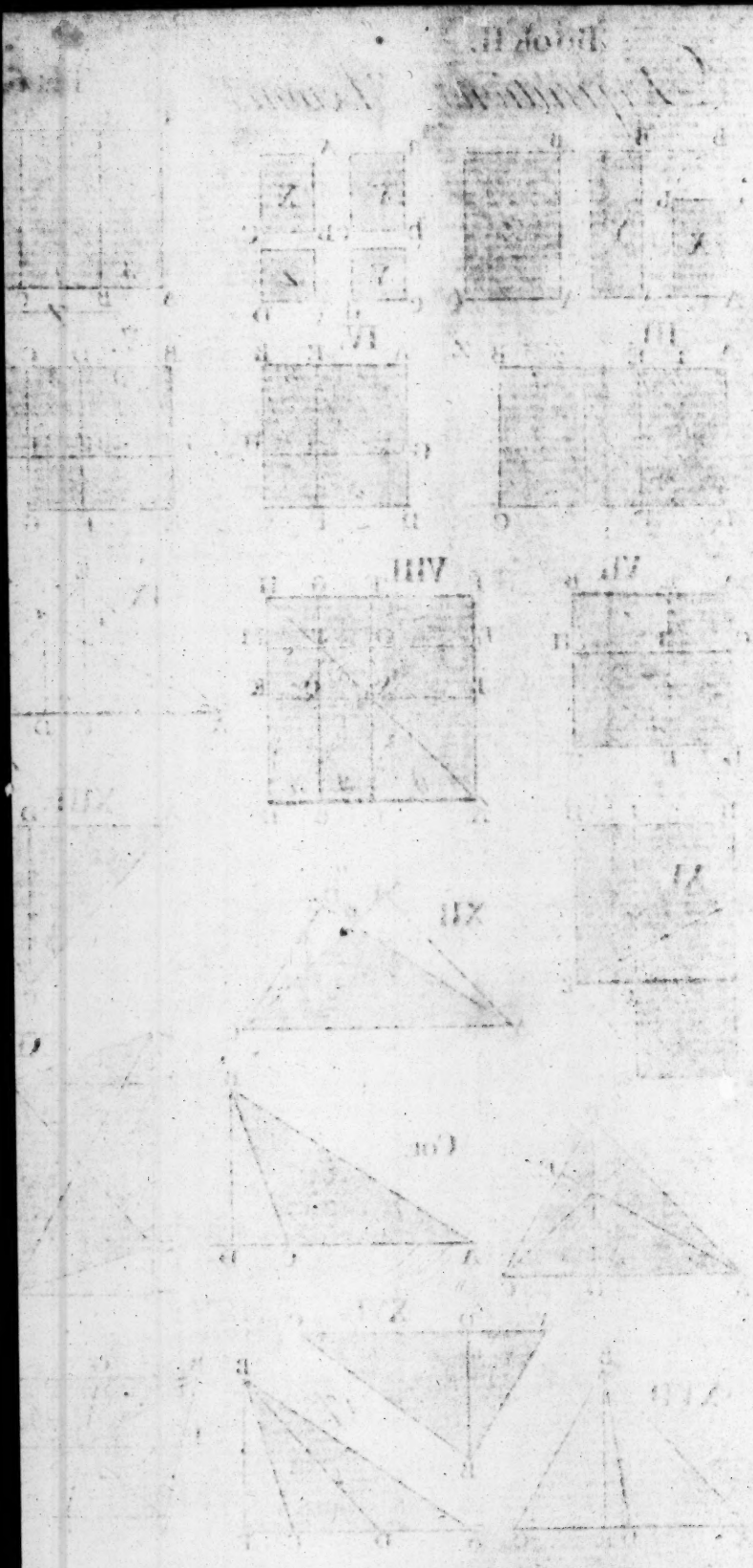




6th. Circles touch each other, or a Right line, in one Point only.

1. For, if the Circles, AD and BD, touched inwardly, in more than a Point, as at D, the Curve of the leſſer Circle, AD, muſt coincide in ſome part, entirely, with the Curve of the larger Circle, BD, which, from the geneſis of a Circle, cannot be; ſeeing, they have unequal Radii, CD and ED, which produce different Curves, according to the Radius; which cannot, in any part, fall into each other (for if they did, they muſt coincide entirely in ſome part) therefore, they touch each other but in one Point.

2. If the Circles, A and B, touched outwardly, in more than a Point, they muſt cut each other; or their Circumferences will be a Right line, in ſome part, as CD; which cannot be; for, it falls within both Circles, by the 5th.

3. The Circle, B, touches the Right line, EF, but in one Point, at G,

For, if it touched in more than a Point, it muſt coincide, in ſome entire part, with the Right line EF; which cannot be, the Radii being all equal.

If any other Line, a b, be drawn between the Line EF and the Center, it will cut the Circle in two points of its Circumference, and the part, a b, of that Line, between the two Points, falls within the Circle; by the laſt.

As this Property of Circles is purely, and entirely, ſpeculative, I preſume, that, what I have ſaid is as convictive, and ſatisfactory, as any Demonſtration whatever

THEOREM I. 3. Euclid.

If a Right line, drawn through the Center of a Circle, biſects a Chord line, not drawn through the Center, it will cut it perpendicularly.

In the Circle ADB; let the Right line DE paſs through C, the Center, dividing the Chord line AB into two equal parts, at E. I ſay, that DE will be perpendicular to AB. From the Center, C, draw CA, CB.

DEM. The Triangles ACE, ECB, are equilateral and equiangular to each other; i. e. congruous.

For, CE is common to both; AE=EB, - - Hyp.
and AC is equal to CB. - - - - - - - Ax. 1.
Wh. becauſe AC=CB, the Angle CAE=CBE. -Th. 9. 1.
Alſo, the Angle AEC=CEB. - - - - - - 8. 1.
conſequently, they are Right ones. - - C. 2. 1. 1.
Therefore, DCE is perpendicular to AB-Def. 10. & 11. 1.

COR. 1. A Chord (AB) being biſected by another Right line, at right angles with it (in E) the perpendicular Line (DE) will paſs through the Center of the Circle.

COR. 2. A Perpendicular, drawn from the Center of a Circle to a Chord line, biſects the Chord; and alſo the Ark of that Chord. As AFB, in F.

Hence, a Right line, or an Ark of a Circle, is biſected.

THEOREM II. 4. Euclid.

If, in a Circle, two Chord Lines cut each other, and are not both drawn through the Center, they cannot biſect each other.

Let AB and CD be two Chord Lines cuting each other, in E.

If one of them, CD, passeth through the Center, it is evident, that it cannot be bisected by the other, AB, which does not pass through the Center.

If neither of them passes through the Center, one of them may be bisected, but both cannot.

Dem. For, if AB (Fig. 2.) is bisected, in E, a Right line, FE, joining the Center, F, and the point of Section, E, will be perpendicular to AB. - - - Th. 1.
consequently, AEF & FEB are Right angles. - Def. 11. 1.
And, if CD be also bisected, in E;
FE is, also, perpendicular to CD. - - - Th. 1.
Wherefore, CEF and FED are Right angles. - Sup.
But, AEF, FEB are Right angles. - - proved.
consequently, CEF, FED are not Right angles; - Ax. 2. 1.
for it is evident, that one is greater and the other less.
Therefore, CD is not bisected in E. Q.E.D

THEOREM III. 14. Euclid.

Equal Chord Lines, in a Circle, are equally distant from the Center; and Chord Lines which are equi-distant, are equal.

Let AB and CD be equal Chord Lines in the Circle ADB.
I say, they are equally distant from the Center, E.
Let EF and EG be Perpendiculars, to AB and CD; join AE, EB, EC, and ED.

Dem. Now, AB is equal to CD, by the Hypothesis.
and AE, EB, EC, and ED are all equal. - - Ax. 1.
wh. the Triangles AEB, CED are congruous. - Def. 43.
consequently, they are equiangular, - - - 7. 1.
And, being Isosceles, the Angles at A and B, C and D, are all equal amongst themselves. - - - - - 9. 1.
Now, since AB & CD are bisected in F & G, AF = DG.

Wherefore, in the Triangles AFE, EGD, the Sides, AE, AF, are equal to ED, DG, reſpectively; and they contain equal Angles, FAE=EDG. - - proved. Therefore, EF is equal to EG. - - - 8. 1.

2nd. AB and CD are two Chord Lines, equally diſtant from the Center of the Circle ADB. I ſay, AB=CD.

EF, and EG, being perpendicular to AB and CD, reſpectively, will biſect them, in F and G. - Cor. 2. 1.

Draw AE and ED.

DEM. Becauſe AE=ED, EA□=ED□;&, EF□=EG□.
But, AE□=AF□+EF□;&ED□=EG□+GD□.
conſ. AF□+EF□=EG□+GD□, wh. AF□=GD□
Therefore, AF=GD (Ax. B. 2.) &conſ. AB=CD. Ax. 5. 1.

The ſecond Part of this is the converſe of the former; which, I ſhould have made a Corollary to it; but on account of the different manner of Demonſtration; by the laſt, of which, Euclid demonſtrates both.

COR. 1. Equal Chord Lines, in the ſame or equal Circles, ſubtend equal Angles at the Center.

For, ſince AB is equal to CD, and AE, EB, EC, and ED are all equal, the Triangles AEB, CED are congruous; conſ the Angle AEB=CED. - - - 7. 1.

And, ſince equal Circles have equal Radii, the Angles which equal Chords ſubtend, in equal Circles, are equal.

COR. 2. Equal Chords, in the ſame or equal Circles, ſubtend equal Arks, and cut off equal Segments. And, equal Arks have equal Chords.

AB=CD; and, ſince EF=EG, if EF and EG be produced, to H and I, FH=GI. - - Ax. 7. 1. And, ſince EF and EG are perpendicular to AB and CD, the Arks AHB, CID, are biſected, in H and I, - Cor. 2. 1. wherefore, the Triangle AHB is equal to CID; and the Arks, AH, HB, CI, and ID, are all equal, therefore, the whole Ark AHB is equal to CID. - - Ax. 5. 1.

For, if AB was applied to CD, being equal, they would perfectly agree; the Point A with D, and B with C; alſo, the Point H would coincide with the Point I, and every other Point, in the Ark AHB, with a correſponding Point, in the Ark CID; conſequently, the whole Ark, AHB, would coincide with the Ark CID; therefore the Segment AHB=CID; and, conſ. DAHC=ADIB.

The ſame may be ſaid of equal Circles; for, having equal Radii, their Circumferences will coincide (being applied to each other) in every Point.

THEOREM IV. 15. Euclid.

A Diameter is the greateſt Right line, which can be drawn in a Circle; and, of all other Chord Lines, that is the greateſt which is neareſt to the Center.

Let AB be a Diameter, and E the Center of the Circle. CD and FG are Chord Lines, at different diſtances from the Center. I ſay, that AB is the greateſt; and FG, the fartheſt off, is the leaſt of the three.

Let EH and EI be Perpendiculars, to CD and FG, from the Center.

FG is farther from the Center than CD, therefore, EI is greater than EH. (3.) Make EK equal to EH. Let KL be perpendicular to EI, and produce LK to M; and, laſtly, draw EL, EM, EF, and EG.

DEM. Now, becauſe EH is equal to EK, and are perpendicular to CD and LM; LM is equal to CD. - Th. 3.
But LE+EM (eq. AE+EB) is greater than LM. - 13. 1.
Therefore, AB is greater than LM; i. e. than CD, eq. LM.

2nd. In the Triangles LEM, FEG, the two Sides LE, EM, are equal to the two Sides FE, EG, and the Angle LEM is greater than FEG.

Therefore, FG is leſs than LM; i. e. than CD, which is equal to LM. - - - Cor. to 8. 1.

THEOREM V. 7. Euclid.

If any Point, except the Center, be taken within a Circle, and, from that point, divers Right lines be drawn to the Circumference; the greateſt of thoſe Lines is that which paſſes through the Center; and, the leaſt is the remainder of that Line, produced to the Circumference.

Of all others, drawn from that Point, that which falls neareſt to the Line, paſſing through the Center, is greater than the more remote ones; and, but two equal Right lines can be drawn, from that Point to the Circumference.

Let A be the Point aſſumed, in the Circle EFG; from which, draw AB, AD, AE, &c.

Firſt; AB, which paſſes through the Center, C, is greater than AD, or AF, or any other Line, which can be drawn from the Point A. Draw DC.

DEM. In the Triangle ADC, the two Sides, AC, CD, are greater than the remaining Side, AD - - 13. 1.
But, CD is equal to CB, - - - - - - Ax. 1.
wherefore, AC+CD=AC+CB, i. e. AB. - Ax. 6. 1.
Therefore, AB is greater than AD.

2nd. AE is the ſhorteſt Line, which can be drawn from the Point A. Draw any other Line, AF, and join FC.

Then, in the Triangle AFC, AF + AC is greater than CF
But, CF is equal to CE, - - - - - - - Ax. 1.
Wh. (taking away AC) AF is greater than AE - Ax. 8. 1.

3rd. That Line is the longeſt, which falls neareſt to AB.

In the Triangles ADC, AFC, AC, CD are equal to AC, CF, reſpectively; for AC is common, and CD=CF. Ax. 1.
But, the Angle ACD is greater than ACF. - Ax. 2. 1.
Therefore, AD is greater than AF. - - Cor. to 8. 1.

4th. No more than two equal Lines can be drawn, from the Point A, to the Circumference.

For, ſince AF is proved leſs than AD, and AB greater, every Line which can be drawn, from A, between D and F, will be leſs than AD, and between B and D greater; wherefore, no other Line, equal to CD, can be drawn on that Side EB.

Let the Angle BCG be equal BCD; and draw AG.
Then, becauſe the Angle BCG=BCD, GCA=ACD.
For, BCG + GCA=BCD + DCA. - Th. 1. 1. & Ax. 9.
And, becauſe CG=CD, and AC is common;
we have AC, CG, reſpectively equal to AC, CD;
and the Angle ACG=ACD; therefore AG=AD - 8. 1.

No other Line, equal to AG, can be drawn on that Side BE; (for the Angle ACG will be either greater or leſs than ACD) and as it cannot be on the other, conſequently, no more than two equal Lines can be drawn from the ſame Point, A, to the Circumference.

Cor. Hence it is evident, that, if from any Point in a Circle, more than two equal Lines can be drawn to the Circumference, that Point is the Center.

THEOREM. VI. 8. Euclid.

If from any Point without a Circle, Right lines are drawn through it, to the concave Circumference; that which paſſes through the Center, is the greateſt; and that Line which falls neareſt to it, is greater than that which is more remote.

Of thoſe, which fall upon the convexity of the Circumference; that is the leaſt, which, if produced would paſs through the Center; and that Line, which falls neareſt to it, is leſs than any other, farther off. And, no more than two equal Lines can be drawn from any Point, without the Circle, either to the convex or concave part of the Circumference.

Aſſume any Point, A; and, through C, draw AB; alſo, draw AD, and AE, at pleaſure.

1st. AB which paſſes through the Center, C, is greater than AD or AE, not paſſing through the Center. Draw CD.

DEM. In the Triangle A C D, the two Sides, AC added to CD, are greater than AD. - - - - Th. 13. 1.
But, CD=CB; wherefore, AC+CB, i.e. AB, =AC+CD; conſequently, AB is greater than AD.

2nd. AD is greater than AE. Draw CE.

Then, in the Triangles AEC, ADC, the Sides AC, CE, are reſpectively equal to AC, CD. - - Ax. 1.
and the Angle ACD is greater than ACE. - - Ax. 2. 1.
therefore, AD is greater than AE, - - Cor. to 8. 1.

3rd. AG is less than AH or AI. Draw CH and CI.

Then, in the Triangle ACH; AH + HC is greater than AC. But, GC is equal to HC; - - - - - - Ax. 1. wherefore, AC − GC, i. e. AG, is less than AH. Ax. 8. 1.

4th. AH is less than AI, being farther from AC.

For, in the Triangle AIC, the two Sides, AI, IC, are greater than AH, HC, drawn to any Point, H, within the Triangle. - - - - - - - 14. 1.
But IC = HC; wherefore AH is less than AI - Ax. 8. 1.

5th. No more than two equal Lines can be drawn from any Point (as A) without the Circle, either to the convex or concave part of the Circumference.

For, since all Lines drawn from A, between G and H, are less than AH, and between H and I, greater; no other Line, drawn on that Side AB, can be equal to AH.

But an equal Line, AK, may be drawn on the other Side.

So likewise; all Lines, drawn from A to the concave Periphery, between AB and AD, are greater than AD; and between AD and AE, less; consequently, no other Line can be drawn on that Side AB, equal to AD.

But, if BF be made equal BD, and AF be drawn, AF is equal to AD. Draw CF; which is equal to CD.

For, the Angle FCB = BCD, by Construction, wherefore, FCA = ACD. - - Th. 1, 1, & Ax. 7. 1.
Wh. in the Triangles ADC, AFC, the Sides AC, CD are respectively equal to AC, CF; for CF, CD are Radii; and the Angle ACD = ACF, therefore AF = AD - 8. 1.

No other Line, equal to AF (eq. AD) can be drawn on that Side of AB; wherefore, but two equal Lines can be drawn from any Point, A, without the Circle, either to the convex or concave part of the Circumference.

Cor. The greatest Right line that can be drawn, from a Point without a Circle, to the convex Circumference, is equal to the least drawn to the concave part.

For they unite in a Tangent to the Circle, at L.

THEOREM VII. 11. 12. Euclid.

If two Circles touch each other; a Right line, joining their Centers, will pass through the Point of contact of the two Circles.

First; let AB and AH be two Circles, touching each other, inwardly, in A.

C is the Center of the lesser Circle, AB.

Draw the Right line AC; it will, if produced, pass through D, the Center of the other Circle.

If not; let any other Point, as E, be the Center of AH, and draw EC, cuting the two Circles in F & G; join AE.

DEM. Then, because C is the Center of the Circle AB, CA is equal to CF. - - - - - Ax. 1.
If CE be added to both, then CE+CF=CE+CA.-6.1.
But, CE+CA is greater than AE. - - - Th.13.1.
and, EA is equal EG (by the Supposition). - Ax. 1.
wherefore, EG is less than EF (the greater than the less) which is absurd, and cannot be.
Therefore, E is not the Center of the Circle AH; consequently D is the Center; and, the Right line DC passes through the Point of Contact, A. Q. E. D.

2ndly. AB and BH are two Circles, touching each other, outwardly, at B; C is the Center of AB.

Draw the Right line CB; which, if produced, will pass through the Center, D, of the other Circle.

If it be denied, let E be supposed to be the Center of BH, and draw CE and BE.

Then, because C is the Center of the Circle AB, CF is equal to CB; and, if E be the Center of BH, EG=EB.
Wherefore, CF+GE=CB+BE. - - - Ax. 6. 1.

And, if FG be added to the former; CE, i. e. CF+FG +GE, is greater than CB+BE.

But, CB+BE is greater than CE. - - - Th. 13. 1.

wherefore, BC+BE is both greater and leſs than CE, which is abſurd, and cannot be; therefore E is not the Center of the Circle BH; conſequently D is the Center; and, the Right line, CD, joining the two Centers, C and D, paſſes through B, the Point of Contact.

Cor. Circles touch each other, either inwardly or outwardly, in a Point only.

THEOREM VIII. 16. Euclid.

If a Right line be drawn, through the extreme point of a Diameter of a Circle, at right angles with the Diameter, it will fall wholly without the Circle; and no other Right line can be drawn, from the Point of Contact, between the Tangent and Circle.

Let AB be a Diameter, and DG a Right line drawn at right angles with AB, through the extreme B.

Draw CD, cuting DG in any point, D.

I ſay, the Point D is without the Circle.

Dem. In the Tri. CDB, the Angle DBC is a Right one; conſ. BDC is acute; wh. DC is greater than BC. - 12. 1.

But, the Point B, is in the Circumference.

wherefore, the point D is beyond the Circumference - Ax. 1.

conſequently, BD falls wholly without the Circle.

Or thus, after Euclid.

If BD does not fall without, it will cut the Circle; ſuppoſe in B and E. Draw EC.

Then, in the Triangle EBC, EC=CB; - - Ax. 1.
wherefore, the Angle CEB=CBE.
But, CBE is a Right angle (Hyp.) conſ. CEB is a right one.
The two Angles, CEB, EBC, of the Triangle CEB, are Right angles, on this Suppoſition.
But all the three Angles of every Triangle=two R. angles.
Therefore, BE, i. e. BD, does not cut the Circle, but muſt neceſſarily fall without it.

2nd. Let BE be drawn (if poſſible) between DB and the Circumference of the Circle.

Now, ſince CBD is a Right angle, CBE is acute.
Let CF be drawn perpendicular to BE.
Then, CF, ſubtending an acute Angle, is leſs than CB.
But, the Point B is in the Circumference.
conſ. CF does not reach the Circumference. - Ax. 1.
wherefore, the Point F is within the Circle.
Therefore, the Right line BE cannot be drawn between the Tangent, BD, and the Circumference.

COR. 1. A Tangent touches a Circle in one Point only.
For if it touched in more, it would not be a Right line.

COR. 2. A Right line perpendicular to the Tangent, at the Point of Contact, paſſes through the Center.

COR. 3. A Right line drawn from the center of a Circle to the Point of Contact, of a Right line touching the Circle, is perpendicular to that Tangent.

COR. 4. The Angle of a Semicircle is a right one.
For, it is greater than any right-lined acute Angle.

N. B. The external Angle, made by a Tangent and the adjoining Ark of the Circle, is leſs than any right-lined Angle.

For it has been proved, that a Right line cannot be drawn, from the Point of Contact, between the Tangent and the Circumference; wherefore, the Angle GBH cannot be made leſs by a Right line, ſeeing it will cut the Circle, and conſequently make a larger Angle, at ſome other Point in the Circumference.

Yet may the Angle GBH be leſſened infinitely, by curved Lines; which is, apparently, a Paradox.

For ſince the Angle of a Semicircle is Right, and the Tangent BG is perpendicular to the Diameter, AB, there cannot be left any remainder of the Angle as a Compliment to it; becauſe, the Compliment of an Angle is its deficiency to a Right angle; or (being obtuſe) to two Right angles.

Now, ſince it has been proved that a Right line cannot touch a Circle but in one Point only, it muſt hold equally true, in reſpect of a large Circle as of a ſmall one.

Wherefore, if any other Radius, greater than BC, be taken, as BI, and an Ark, Ba, be drawn, it is evident that the Angle GBa is leſs than GBH.

If a larger Radius, BA, be taken, and, on the Center A, an Ark, Bb, be drawn, the Angle GBb is leſs than GBa; notwithſtanding, the Angles, ABH, ABa, and ABb, are the ſame (by the 4th Cor.) and it is evident, that if a ſtill larger Radius be taken, the Angle CBb may ſtill be leſſened infinitely.

SCHOL. *Hence it is manifeſt, that any Right line, as* GH, *is infinitely diviſible, by enlarging the Radius* BA *infinitely; ſeeing that, the Circumference of a Circle cannot coincide with two Points,* B *and* G, *of the Right line* BG.

THEOREM IX. 20. Euclid.

An Angle, at the Center of a Circle, is double of an Angle touching the Circumference, when both ſtand on the ſame Ark.

In the Circle ABC, let AEC be an Angle at its Center, and ABC is an Angle touching the Circumference; both ſtanding on the ſame Ark, AC.

I ſay, the Angle AEC is double the Angle ABC.

Draw the Right line BE, and produce it to D.

DEM. In the Triangle ABE, becauſe the ſide AE=EB, the Angle A is equal to ABE - - - - - - 9. 1.

But, the external Angle AED=the Angle A+ABE-10.1.

Therefore, the Angle ABD=half the Angle AED.

Alſo, DBC=½DEC; for the Triangle EBC is Iſoſceles.

Conſ. AED+DEC (i. e. AEC)=twice ABD+DBC; i. e. equal to twice ABC.

Therefore, the Angle AEC is double the Angle ABC.

CASE 2nd. When the Angle CEF, at the Center, falls without the Angle CBF, at the Circumference.

Draw BED, as before.

Now, the external Angle, DEC,=ECB+EBC; (10. 1.) and ECB=EBC (9. 1.) wh. DEC is equal twice DBC.

And, DEF=EFB+EBF; therefore, eq. twice DBF.

Conſequently, DEF-DEC, i.e. CEF,=2DBF-2DBC. i. e. 2CBF. Therefore, the Angle CEF is double CBF.

COR. 1. The Angle at the Circumference, ſtanding on any Ark, is equal to an Angle at the Center, on half that Ark.

For, if the Ark AC be=CF, the Angle AEC=CEF.

But the Angle AEC is double ABC, &, CEF double CBF conſequently the Angle AEC (eq. CEF)=ABC+CBF; i. e. the Angle ABF, on the Ark ACF, is equal to AEC, (or CEF) on AC, or CF, half that Ark.

COR. 2. In the ſame or equal Circles, Angles ſtanding on equal Arks, whether they be at the Center or at the Circumference, are equal to one another.

This is evident from the laſt; for the Ark AC being equal to CF, the Angle AEC=CEF; and, the Angle ABC (eq. half AEC)=CBF, half CEF.

And, becauſe the Radii of equal Circles are equal, the Angles on equal Arks, in equal Circles, are alſo equal. And converſely, equal Angles ſtand on equal Arks.

THEOREM X. 21. Euclid.

All Angles, which ſtand on the ſame Ark, or are in the ſame Segment of a Circle, and touching the Circumference, are equal to one another.

In the Segment ADC, I ſay the Angles ABC, ADC, and AEC, are all equal.

To the Center, F, draw AF and CF, making the Angle AFC.

Dem. Then, the Angle AFC, at the Center, is double AEC, at the Circumference.

For, they ſtand on the ſame Ark, AHC, or Chord, AC. But the Angle AFC is double ABC, or any other Angle, ADC, touching the Circumference. - - - Th. 9. Wherefore, the Angles ABC, ADC, &c. being each equal half AFC, are, therefore, equal amongſt themſelves.

If the Segment be a Semicircle, or leſs than a Semicircle, it may be demonſtrated after this manner.

AGC and AHC are Angles in a leſſer Segment, AGHC. The Angle AIC (being external, in reſpect of the Triangles GAI & ICH) = the Angle G+GAI; and alſo, to H+HCI; wherefore, the Angle G+GAI=H+HCI. But, the Angle GAI, i.e. GAH=HCG; above. Therefore, the Angle AGC=AHC. Q. E. D. 10. 1.

Cor. Right lines, BC and AH, joining the extreme Points of two equal Arks, AB and CH, in the Circumference of a Circle, ſo as not to croſs each other, are parallel.

For (having drawn AC) the Arks AB, CH being equal, the Angle BCA is equal to the Angle CAH. - C. 2. 9.
Therefore, BC is parallel to AH. - - - Th. 4. 1.

THEOREM XI. 22. Euclid.

In every Quadrilateral, inſcribed in a Circle, the oppoſite Angles are equal to two Right angles.

ABCD is a Trapezium inſcribed in a Circle.

I ſay, the Angles A and C, alſo B and D, are, together, equal to two Right ones.

Draw the Diagonals, AC and BD.

DEM. In the Triangle ABD, the Angle DAB + ABD added to BDA are equal to two Right ones. - 10. 1.
But, the Angle BCA = BDA; and ACD = ABD. - 10.
Wh. BCA + ACD (i. e. BCD) + BAD = two Right angles.

COR. 1. The external Angle, CBF, made by producing any Side, AB, of a Quadrilateral inſcribed in a Circle, is equal to the oppoſite Angle, ADC. - by Theo. & 1. 1.

2. Every Parallelogram inſcribed in a Circle is a Rectangle.

THEOREM XII. 31. Euclid.

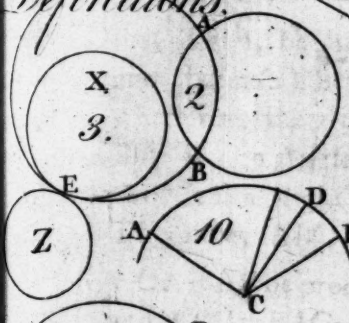
An Angle at the Circumference, in a Semicircle, is a Right angle.

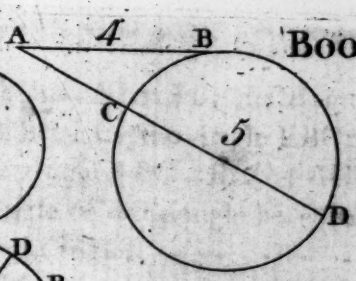
In the Circle ABCD, let AC be a Diameter, and E the Center. To any Point, as B, in the Circumference, draw AB and CB.

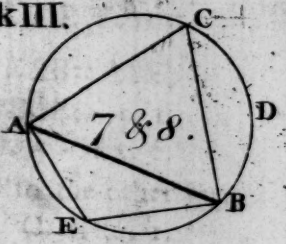
I ſay the Angle ABC is a Right angle.

Draw EB; then, the Triangles, AEB, BEC, are Iſoſceles; AE, EB, and EC being equal.

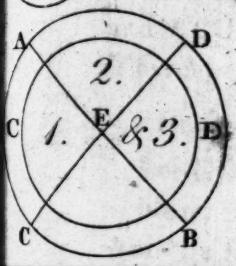
Definitions.

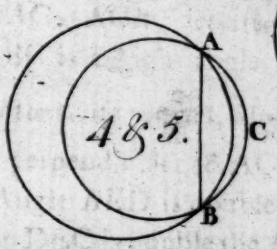


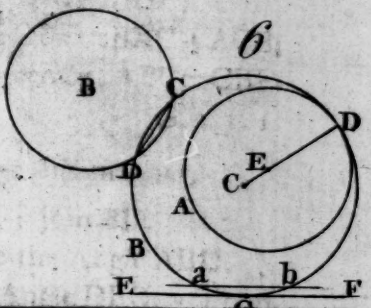




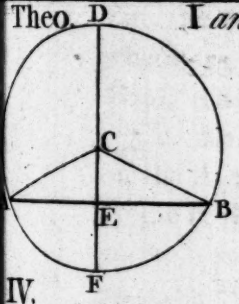
Axioms

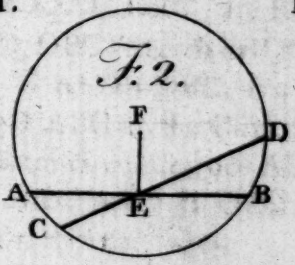






Theo. I *and* II.





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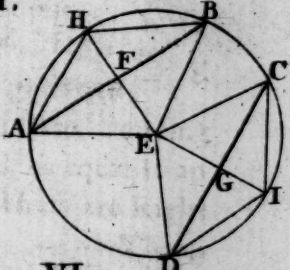

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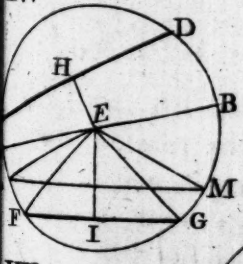

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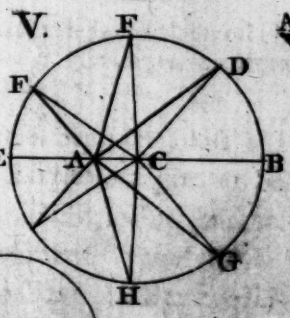

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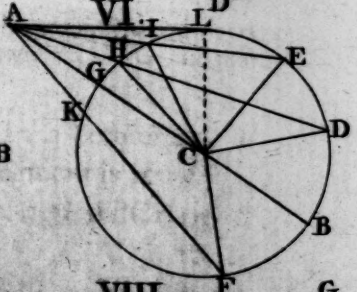

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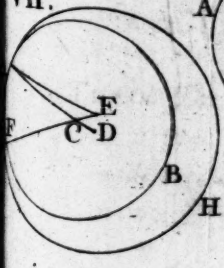

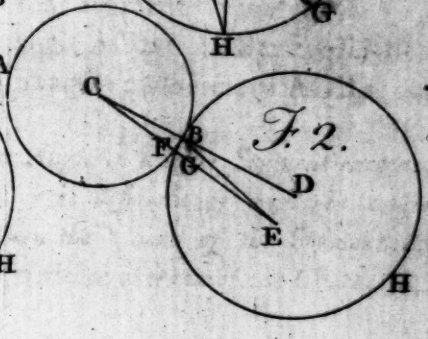

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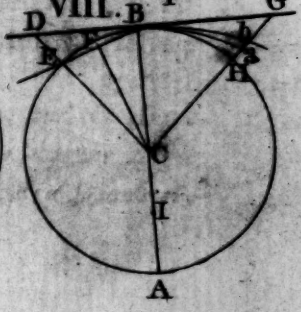
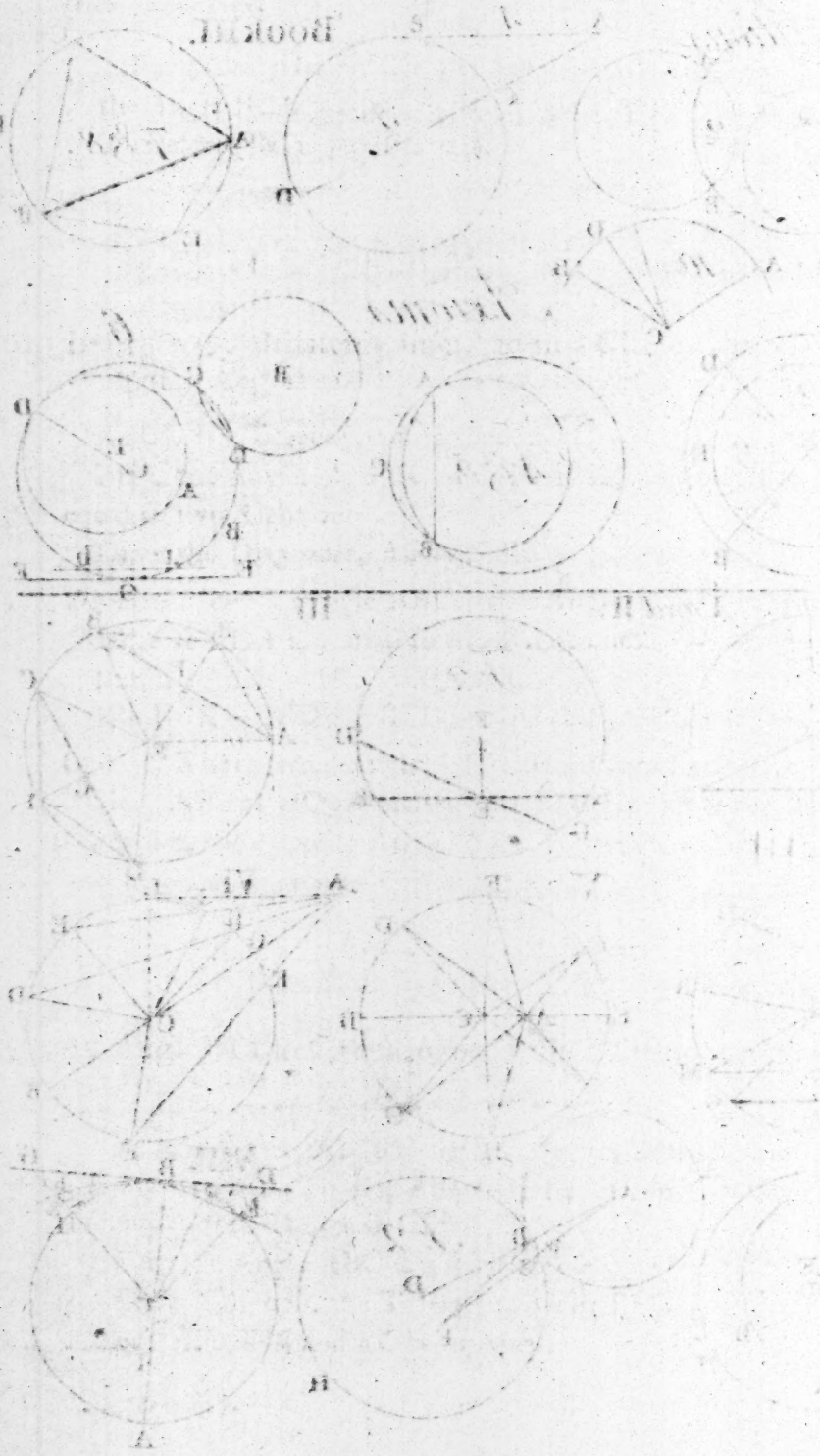




Dem. Then, becauſe AE=EB, the Angle EAB=ABE, and, becauſe EB=EC, the Angle EBC=BCE; - 9. 1. wherefore, the Angle ABC=BAC+ACB. - - Ax. 6. But, if one Angle of a Triangle be equal to the other two, it is a Right angle. - - - - - - C. 1. 10. 1. Therefore, ABC is a Right angle. Q. E. D.

Or, if AB be produced, the Angle CBF=BAC+ACB; and ABC=BAC+ACB; conſequently, ABC=CBF. Therefore, ABC is a Right angle. - - - C. 2. 1. 1.

It may be otherwiſe proved, after this manner.

Let ED be perpendicular to AC; join BD.

Now, the Angle AED is double the Angle ABD. And the Angle DEC is double the Angle DBC. - Th. 9. But, the Angles AED, DEC, are Right ones; - Con. wherefore, ABD, DBC are each half Right. - C. 1. 10. 1. Conſ. ABC, i. e. ABD+DBC, is a Right angle.

Or, the Angle ABC at the Circumference, ſtanding on the Ark or Semi-circumference ADC, is equal to an Angle at the Center, AED or DEC (which are Right angles) on half that Ark. - - - - - - - Th. 9.

From this Theorem is deduced the moſt elegant and expeditious Method for making a Right angle, or drawing a Perpendicular at the extremity of a Right line.

Cor. An Angle in a Segment, leſs than a Semicircle, is obtuſe; and an Angle in a greater Segment is acute.

In the Segment ADCB, ſeeing that the Angle ABC is right, ACB, in that Segment, is acute. - - - C. 3. 10. 1. And, the Angle AGB+ACB=two Right angles - Th. 11. But ACB is acute; wherefore AGB is obtuſe. - Def. 13.

Theo. *The Angle in a leſſer Segment exceeds, and the Angle in a greater is deficient of a Right angle, by the Angle made between the Chord of the Segment and a Diameter, drawn from either extreme of the Chord.*

Let AC be a Diameter, and ABC is a Right angle. Draw two Chords, DC and EC, and join DB and EB.

Then, in the leſſer Segment DBC, the Angle DBC is greater than the Right angle ABC, in a Semicircle; Ax. 1. and the difference is the Angle DBA.

But, the Angle DBA=DCA. - - - - Th. 10.
Therefore, the Angle DBC, in a leſſer Segment, exceeds a Right angle, by the Angle ACD.

2nd. In the greater Segment, EDC, the Angle EBC is leſs than the Right angle ABC; and, the deficiency is the Angle ABE, equal ACE.

Therefore, the Angle, EBC, in a greater Segment, is leſs than a Right angle, by the Angle ACE.

THEOREM XIII. 32. Euclid.

The Angles made by a Right line, cuting a Circle, drawn from the Point of Contact of another Line touching it, are equal to the Angles in the oppoſite alternate Segments.

Let ABC be a Right line, touching the Circle BDE, in B. Draw at pleaſure, from B, the Chord BD.

The Angle ABD is equal to DEB, or any Angle made in that Segment. And the Angle DBC is equal to DGB, in the other Segment, DBG.
Let BF be perpendicular to AC; join FE.

DEM. Now, BF is a Diameter of the Circle. - - C. 2. 8.
wherefore, FEB is a Right angle. - - - Th. 12.
But, ABF is a R. angle (Con.) conſ. FEB=ABF. - Ax. 9.
But, the Angle FED=FBD (10). Th. DEB=DBA. - 7.

2nd. BG is a Chord of the Segment GFEB.
Draw GE, making an Angle, GEB.

Then, the Angle GDB (eq. GEB) = GBA; as above.
And, the Angle G + GDB + DBG = two R. angles. - 10. 1.
But, the Angle ABD + DBC = two Right angles; - 1. 1.
and the Angle GDB (eq. GBA) + DBG = DBA. - Ax. 2. 1.
Therefore, the remaining Angle DGB = DBC. - Ax. 7. 1.

Or, in the Quadrilateral DEBG, the oppoſite Angles, DGB, DEB, are equal to two Right angles. - Th. 11.
And the Angle ABD + DBC = two Right angles. - 1. 1.
But, the Angle DEB = DBA (as above). Th. DGB = DBC.

THEOREM XIV. 36. Euclid.

In Circles, if two Chord Lines cut each other; a Rectangle, contained under the Segments of one Line, will be equal to the Rectangle under the Segments of the other.

1ſt. When both paſs through the Center, it is evident.

For, the Rectangle under the Segments of each, is the Square of the Radius; conſequently they are equal.

2nd. In the Circle ACBD; if AB, paſſing through the Center, biſects CD, which does not paſs through the Center, in E; then, the Rectangle AEB, is equal to the Square of CE, or ED.

Let F be the Center. Draw CF.

DEM. Then, becauſe AB is biſected, in F, and cut unequally, in E; AE × EB + EF □ = FB □; - - 5. 2.
i. e. = CF □, equal CE □ + EF □; - - - - - 20. 1.
for, CEF, is a Right angle (C. 2. 1.) and CF = FB. Ax. 1.
Wherefore, AE × EB + EF □ = CE □ + EF □.
Take away from both, EF □, which is common;
there remains AE × EB = CE □, i.e. ▭AEB = ▭CED.

3rd. When neither of them paſſes through the Center; and, when neither is biſected, by the other.

From the Center, F, draw FG and FH perpendicular to AB andCD; and join AF, FD, and EF.

Now AB is biſected in G, and cut unequally in E; wherefore, AE×EB+EG□=AG□. - - 5. 2.
Add, on both ſides, FG□;
then, AE×EB+EG□+FG□,=AG□+FG□-Ax.6.1.
ButEF□=EG□+FG□;&AF□=AG□+FG□.20.1.
Therefore, AE×EB+EF□=AF□.

After the ſame manner it may be proved, that the Rectangle CED+EF□ is equal to AF□.

For, CD is alſo biſected, in H, and unequally cut, in E.
Wherefore, CE×ED+HE□=HD□. Add HF□to both.
Then, CE×ED, +HE□+HF□ (equal EF□) is equal to HD□+HF□; i. e. equal FD□. - - - - 20. 1.
But FD=AF; wherefore, FD□=AF□; - Ax. B. 2.
conſequently, CE×ED+EF□=AF□ (equal FD□).
But, it was proved, that, AE×EB+EF□=AF□.
Wherefore, taking away EF□, from both,
there remains AE×EB=CE×ED. - - Ax. 7. 1.
Or, the Rectangle AEB, equal CED. Q. E. D

THEOREM XV.

If two Chord lines interſect at right angles; the four Squares, of the Segments of thoſe Chords, will be equal to the Square of the Diameter.

And, the Squares of the two oppoſite Sides of a Quadrilateral, formed by Right lines joining the extreme Points of the Chords, are alſo equal to the Square of the Diameter.

Let the Chords AB and CD cut each other perpendicularly, in E.

Then, the Squares of AE, EB, EC, and ED, are equal to the Square of AI, the Diameter of the Circle.

From the Center, F, draw FG and FH parallel to the Chords CD and AB, respectively; and join AF, and FD.

DEM. Now, since AB & CD cut each other perpendicularly, and, FG, FH, are respectively parallel to CD & AB-Con. FG and FH are perpendicular to AB and CD. -C.2.4.1. Then, AB is bisected, in G (1.3.) and cut unequally, in E; wherefore, $AE\square + EB\square = 2AG\square + 2GE\square$; also $CE\square + ED\square = 2EH\square + 2HD\square$ - - } - 9. 2.

Wh. $AE+EB+CE+ED\square = 2AG+2GE+2EH+2HD\square$

But GFHE is a Parallelogram, by Construction; wherefore, FH=GE, and FG=EH. - - - 15. 1.

Conf. the Squares of the four Segments, AE, EB, CE, & ED are eq. to the Squares of AG, GF, FH&HD, twice taken.

But, $AF\square = AG\square + GF\square$, &$FD\square = FH\square + HD\square$ 20.1.

conf. $2AF\square + 2FD\square = 2AG+2GF+2FH+2HD\square$

Wherefore, $AE+EB+CE+ED\square = 2AF\square + 2FD\square$; i. e. $=4\ AF\square$; for, AF is equal to FD.

But, four times $AF\square = AI\square$; i. e. of the Diameter- 4.2.

Th. the Squares of the four Segments, AE, EB, CE, &ED, are equal to the Square of the Diameter. - - Q.E.D.

2nd. Having joined the extremes AD, DB, AC, and CB.

Then, the Squares of AD and CB, together, are equal to the Squares of AC and DB together.

For, $AD\square = AE\square + ED\square$; and, $CB\square = CE\square + EB\square$.

But, $AC\square = AE\square + EC\square$; and $DB\square = DE\square + EB\square$; therefore, $AD\square + CB\square = AC\square + DB\square$; and, consequently, each is equal to the Square of the Diameter.

This extraordinary Property of the Circle is otherwise demonstrated by Mr. Stone; which, notwithstanding it is indisputably

true, does not carry with it that positive conviction; insomuch, that, a young Geometrician would be somewhat at a loss to perceive it.——It is as follows.

Because, the Angle AED, is a Right one, the Angles EAD, ADE, are equal to a Right angle (32. 1.) wherefore the Arks, AC, DB, on which they stand, are, together, equal to half the circumference of a Circle (26. 3.).

Therefore, because the Angle in a Semicircle is a Right angle, the Squares of AC and DB are equal to the Square of the Diameter (47. 1.). And, because the Angles at E are Right angles, the Squares of AE and EC are equal to the Square of AC; and the Squares of DE, EB, equal to the Square of DB. Therefore, the Squares of AE, EC, DE, & EB, are equal to the Squares of AC and DB; which, were proved equal to the Square of the Diameter. Therefore, those four Squares are equal to the Square of the Diameter.

The References, in this Demonstration, are to Euclid.

SCHOL. *It is worthy of observation, that, as, in a Semicircle, the Sides containing the Right angle are two Chords perpendicular to each other, and are equal to the Square of the Diameter; so the Squares of the Segments of all Chords, which cut each other at right angles, are equal to the Square of the Diameter.*

THEOREM XVI. 37. Euclid.

If, from any Point without a Circle, two Right lines be drawn, the one a Tangent, the other a Secant (i. e. one touching the Circumference, the other cuting it twice); the Rectangle, under the whole Secant and the external Segment (between the Point, assumed, and the Circle) will be equal to the Square of the Tangent.

A is the assumed Point.

Let AB touch the Circle, CBD; in the Point B; AD is a Secant, cuting the Circumference in two Points, C and D.

Then, the Rectangle under AD and AC is equal to the Square of AB.

First, when AD passes through E, the Center of the Circle.

Draw EB, from the Center to the Point of contact.

DEM. Now, because CD is bisected, in E, and AC is added to it; AD × AC + CE □ = AE □; - - - 6. 2.
and, AE □ = AB □ + BE □. - - - - - - - 20. 1.
For, ABE is a Right angle; - - - - - C. 3. 8. 3.
wherefore, AD × AC + CE □ = AB □ + BE □.
But, BE = CE; consequently BE □ = CE □. - Ax. B. 2
wherefore, AD × AC + CE □ = AB □ + CE □
Take CE □ from both, there remains ▭ DAC = AB □.

2nd. When the Secant falls on either Side of the Center.

Let EF be perpendicular to the Secant, AD, or A*D*; and join CE and EB.

Then, because EF is perpendicular to CD, CD is bisected in F. - - - - - - - C. 1. 1. 3.
And, because CD is bisected, in F, and AC is added; AD × AC + CF □ = AF □ (6. 2.) Add EF □ to both; then, AD × AC + CF □ + EF □ = AF □ + EF □ - Ax. 6. 1
But, CF □ + EF □ = CE □; and AF □ + EF □ = AE □; wherefore, AD × AC + CE □ = AE □;
i. e. equal to AB □ + BE □. - - - - - - - 20. 1.
But, BE = CE; wh. AD × AC + BE □ = AB □ + BE □.
And, if BE □ be taken from both, then (by Ax. 7.) there is left AD × AC = AB □. Q. E. D.

COR. Hence it is evident, that the Rectangles under every Right line cuting a Circle, from the same Point without the Circle, and the external Segment, are equal.

The Rectangle DAC = *DAC*; for, they are each equal the Square of the Tangent, AB

2nd. If from the ſame Point, A, without a Circle, two Right lines, AB, A*B*, are drawn, to the Circle, one on each ſide, touching the Circumference; thoſe two Tangents are equal.

For, AB□ = A*B*□, being each equal to the Rect. DAC. But, equal Squares have equal Sides; conſequently, AB is equal to A*B*. - - - Ax. B. 2

3rd. If from any Point, without a Circle, two Right lines are drawn to the Circle; one of which Lines cuts the Circle, and the other meets the Circumference, in ſome Point only; and, if the Rectangle under the whole Secant and the external Segment, be equal to the Square of the other Line, that other Line is a Tangent to the Circle, and touches the Circumference in that Point.

THEOREM XVII.

If, from any Point in the Diameter of a Circle, there be drawn two Right lines to the Circumference; one perpendicular to the Diameter, the other to the middle point of the Ark of the Semicircle; the Squares of thoſe two Lines, together, will be equal to half the Square of the Diameter.

Let AB be a Diameter, C the Center, and D the middle Point of the Semicircle, ADB. From any point, as E, let EF be perpendicular to AB; and draw ED.

I ſay, the Squares of EF and ED, together, are equal to AC ſquare, twice taken; i. e. to half the Square of the Diameter, AB. Join CD and CF.

DEM. CF□ = EF□ + EC□ - - - - - - - 20. 1.
wherefore, EF□ + EC□ = AC□; for AC = CF.

But ED□ = CD□ + EC□ ; and CD = CF, eq. AC.
Conſ. ED□ − EC□ = EF□ + EC□; i. e. = AC□,
Wherefore EF□ + ED□ = CF□ + CD□.
Therefore, EF□ + ED□ = 2AC□; i. e. = AB × AC.

THEOREM XVIII.

If from the Vertex of an equilateral Triangle, conſtructed on the Diameter of a Circle, a Right line be drawn cuting the Diameter; and, from that Point, another Line be drawn, perpendicular to the Diameter, cuting the Circumference; the Squares of thoſe two Lines, together, will be equal to the Square of the Diameter.

From C, the Vertex of the equilateral Triangle ACB (on the Diameter AB) draw CD, at pleaſure, cuting the Diameter in D; and, let DE be perpendicular to AB.

I ſay, that the Squares of CD, DE, together, are equal to the Square of AB.

If CD be drawn perpendicular to AB, as CF, it is manifeſt.
For, CF□ + FG (equal AF)□ = AC (equal AB)□.
Let CF be drawn, perpendicular to AB, and join EF.

Dem. Becauſe CF is perpendicular to AB, and AC is equal to CB, AB is biſected in F; - - C. 4. 9. 1.
wherefore, F is the Center of the Circle. - Def. 20 & 21.
Now, AC = AB (Con) and AC□ = CF□ + AF□ - 20. 1
But, AF = EF; and EF□ = ED□ + DF□; i.e. eq. AF□
wherefore, AC□ = CF□ + DF□ + DE□.
But, CD□ = CF□ + DF□; - - - - - - - 20. 1.
conſequently, AC□ = CD□ + DE□.
But AB = AC (Con.) Therefore, AB□ = CD□ + DE□.

THEOREM XIX.

If a Chord Line be parallel to a Diameter of a Circle, and, from each extreme of the Chord, Right lines are drawn to any Point in the Diameter; the Squares of thoſe Lines, together, are equal to the Squares of the Segments of the Diameter, made by that Point.

Let the Chord CD be parallel to the Diameter, AB; and, to any Point, G, in the Diameter, draw CG and DG.

I ſay, the Squares of CG, GD, together, are equal to the Squares of AG and GB. Let E be the Center.

Let CD be biſected, in F; join FG, FE, and ED.

DEM. Then, becauſe CD is biſected, in F,
$CG\square + GD\square = 2FG\square + 2FD\square$. - - - - - 17. 2.
And, becauſe CD is biſected, EF is perpend. to CD. - 1.
But, CD is parallel to AB; conſ. EF is perpend. to AB - 4. 1
wherefore, $FG\square = EF\square + EG\square$;
and, $ED\square = EF\square + FD\square$. - - 20. 1
Conſ. $CG\square + GD\square = 2GE + 2EF + 2FD$ Squares;
i. e. equal to $2GE\square + 2ED\square$. But $AE = ED$;
wherefore, $CG\square + GD\square = 2GE\square + 2AE$, equal $ED, \square$
But, $AG\square + GB\square = 2AE\square + 2GE\square$; - - - 9. 2.
therefore, $CG\square + GD\square = AG\square + GB\square$. - Ax. 3. 1.

THEOREM XX.

If two Right lines are drawn, from any two Points in the Circumference of a Circle, to the ſame Point in a Tangent to that Circle; thoſe Lines will make the greateſt Angle, when they meet in the Point of contact.

Let A and B be the Points aſſumed, in the Circumference of the Circle ABC.

Draw AC and BC to the Point C, in which a Tangent, DC, touches the Circle; and alſo, to any other Point, D, draw AD and BD.

I ſay, the Angle ACB is greater than ADB. Join AE.

DEM. The Angle AEB=ACB, - - - - Th. 10.
ſtanding on the ſame Ark, AB.
But, the Angle AEB is greater than ADB; for it is equal to ADB+DAE. - - - - - - 10. 1.
Therefore, ACB is greater than ADB.

COR. Hence it is evident, that an Angle, AFB, which falls within the Circle, is greater, and an Angle ADB, without the Circle, is leſs, than any Angle touching the Circumference, and ſtanding on the ſame, or an equal Ark, AB.

From hence may be deduced the following Problem.

A Right line being given, and two Points given, or aſſumed, without the Line; the Point in that Line may be determined, to which, if two Right lines be drawn, from the given Points, they ſhall contain a greater Angle than any other Right lines drawn, from thoſe Points, to any other Point in that Line.

ELEMENTS OF GEOMETRY.

BOOK IV.

THE fourth Book of Euclid's Elements is not elementary, but practical or problematical; it treats of the inſcription and circumſcription of right-lined Figures in and about Circles. It is of eſpecial uſe in Trigonometry, Aſtronomy, &c. and alſo, in Fortification or military Architecture, which ſeems to depend entirely on it.

I do not think proper to deviate from the order of Euclid, in his Books, and have therefore retained the fourth here (although it is wholly and entirely problematical) chiefly for the ſake of Demonſtration, which is as brief as it will admit of. To be deficient in that reſpect (though not very eſſential) would be ſufficient reaſon, with ſome Perſons, to condemn the whole. Nevertheleſs, I am of opinion, that, great part of it does not require Demonſtration, being ſufficiently evident from the Conſtruction; eſpecially, as no other part of the Elements is dependant on it.

There are, in ſome Books extant, ſundry other Propoſitions relative to the inſcription and circumſcription of right-lined Figures, in and about other right-lined Figures; particularly in Stone's Euclid; and in a curious Tract of practical Geometry by Le Clerc. But, as I ſee nothing of real utility to recommend them, I ſhall not, by uſeleſs additions, detain the Reader from matter of greater Importance, in the fifth and ſixth Books: Indeed this may be paſſed over entirely, for the preſent, there being nothing in the fourth neceſſary to be known, previous to thoſe which follow.

DEFINITIONS.

I have already, in the general Introduction, defined the Terms, to *deſcribe*, to *inſcribe*, and to *circumſcribe*, a repetition of which would be unneceſſary.

1. A Right-lined Figure is ſaid to be inſcribed, in a Circle, or to have a Circle circumſcribing it, when every Angle of the Figure touches, or is in the Circumference of the circumſcribing Circle.

2. A Right-lined Figure is, then, ſaid to circumſcribe a Circle, or to have a Circle inſcribed, when every Side of the Figure touches the Circumference of the Circle.

3. A Right-lined Figure is ſaid to be inſcribed, or to circumſcribe a right-lined Figure; when every Angle of the inſcribed one, touches ſome Side of the circumſcribing Figure.

4. So likewiſe, a Circle is ſaid to be inſcribed, in a right-lined Figure, when it touches every Side of the Figure; or to circumſcribe a right-lined Figure, when the Circumference paſſes through every Angle.

N. B. By inſcribing any Figure within a Circle, or any right-lined Figure within another, is underſtood, the deſcribing a Figure, like or ſimilar to the given one, the largeſt poſſible to be contained in the other.

To inſcribe a Right line in a Circle is the firſt Problem of Euclid's fourth Book, which I think entirely unneceſſary, as a Problem; nothing more being required, in the Operation, than to take the given Line, in the Compaſſes, and placing one point in the Circumference, at pleaſure, to cut the Circumference, with the other. As DE, or DF, in the firſt Figure.

A Right Line, greater than a Diameter, cannot be inſcribed

All the Propoſitions, of this Book, follow in the order of Euclid, omiting the Firſt. Indeed, it is immaterial in what order they are arranged, being little or nothing dependant on one another: It is moſt conſiſtent to begin with the ſimpleſt Figures, Triangles.

PROPOSITION II.

To inſcribe a Triangle in a given Circle, equiangular to a given Triangle.

Let ABC be the given Triangle.

Draw at pleaſure the Right line GH, touching the Circle in any Point of its Circumference; as D.

Make the Angle GDF equal to A, i. e. to any Angle of the Triangle; and HDE equal to another, B, and join EF.

The Triangle DEF is equiangular with ABC.

DEM. The Angle GDF (equal A) is equal to DEF, and HDE (equal B) =DFE. - - - - -. 13. 3

Wherefore, the Angle DEF, being eq. A, and DFE eq. B, the remaining Angle, EDF, muſt neceſſarily be equal to C.

Th. the Triangle DEF is equiangular to ABC. - 10. 1.

N. B. By taking the Angles in this order, the Triangle inſcribed will be poſited as the given one; but, any two being taken, the Triangle, inſcribed, will be the ſame.

OR, draw at pleaſure a d, cuting the Circumference in a and d. Make the Angle, a d c, equal to B, and join a c.

Make, c a b, equal to the Angle A, & join b c.

The Triangle a b c is equiangular to ABC.

For, the Angle a b c = a d c (10. 3.) equal B; (by Con.) and, the Angle c a b was made equal to A.

Conſ. a c b is equal to the remaining Angle C. - 10. 1.

Therefore, the Triangle, a b c, is equiangular to ABC.

PROPOSITION III.

To circumſcribe, that is, to deſcribe or draw, a Triangle, about a given Circle (touching it on every Side) equiangular to a given one.

Let ABC be the given Trian. and DFG the Circle given.

Produce any Side of the Triangle, as AC, both ways; and draw a Radius, DE.

Make the Angle DEF, equal to the external Angle A; and FEG equal C.

Through the Points D, F, and G, draw Tangents to the Circle, cuting each other in H, I, and K.

Then is HIK the Triangle required.

Dem. Becauſe HI, &c. touches the Circle, in D, F, and G; and DE, FE, and GE are drawn to the Center, the Angles, EDI, IFE, &c. are right ones. - - - 8. 3

And, becauſe the Angles, D, E, F, and I, of every Quadrilateral, are equal to four Right angles, - - 11. 3 the Angle I + DEF = two Right ones.

But, the Angle DEF is equal to A, by Conſtruction; conſequently, DIF is equal to BAC. - - 1. 1. & Ax. 7.

After the ſame Manner, the Angle K may be proved equal to ACB; and the Angle H, equal ABC.

Therefore, the Triangle HIK is equiangular to ABC.

This is obvious, from Prop. 10. B. 1; for, it follows, from it, that all the Angles of every right-lined Figure are, together, equal to twice as many Right angles as it has Sides, wanting four; and all the external Angles are equal to four. But, the three Angles of every Triangle are equal to two Right angles; and all the Angles, about a Point, are equal to four; conſequently, ſince the Angles DEF, DEG, and FEG, are reſpectively equal to the external Angles, A, B, and C, of the Triangle ABC; the Angles H, I, and K are, alſo, reſpectively equal to the internal Angles, a, b, and c.

PROPOSITION IV.

To inſcribe a Circle in a given Triangle (ABC) touching every Side of the Triangle.

Let any two Angles of the Triangle, ABC, and CAB, be biſected, by the Right lines BD, and AD, cuting each other in D; from which Point, let a Perpendicular (DE) be drawn to any Side (AB) of the Triangle.

With the radius DE, on the Center D, deſcribe the Circle EFG, which will touch every Side of the Triangle, ABC.

Let DF & DG, be perpendicular to the Sides, AC & BC.

DEM. Now, AF is equal to AE - - - - C. 2. 16. 3.
and the Angle EAD is equal to DAF. - - - Con.
Wherefore, in the Triangles AED, AFD, the two Sides AE, AD, are reſpectively equal to the two Sides AF, AD; and they contain equal Angles.
Therefore, DE is equal to DF. - - - - - - 8. 1.

After the ſame manner, DG may be proved equal to DE, and alſo to DF.

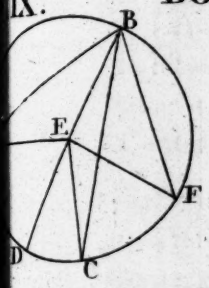
Wherefore, the three Lines DE, DF, and DG, being perpendicular to the three Sides of the Triangle, and being proved equal among themſelves, the Circle, EFG, will paſs through the three Points E, F, and G; and, conſequently, will touch every Side of the Triangle ABC.

PROPOSITION V.

To circumſcribe a given Triangle with a Circle; or, to deſcribe a Circle about a given Triangle.

ABC is the Triangle given.

IX.



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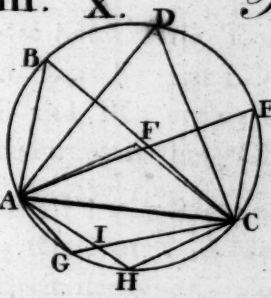

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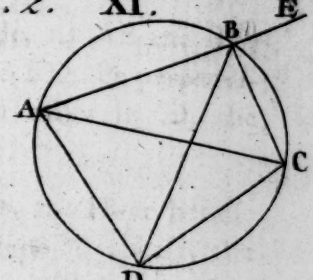

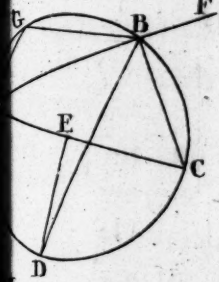

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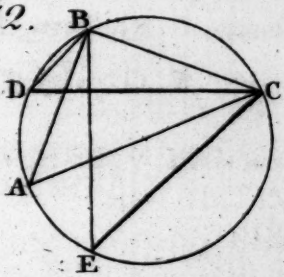

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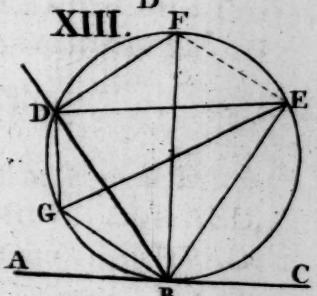


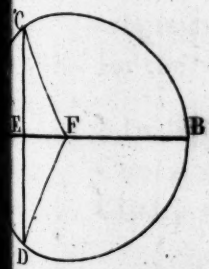
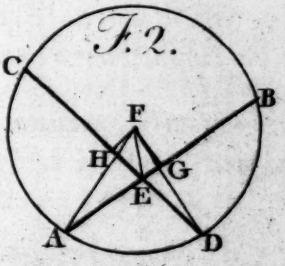
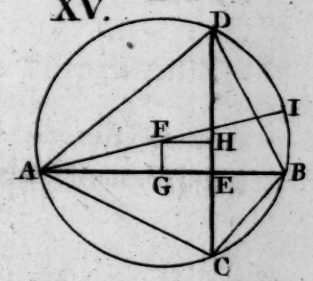


Fig. 2.



XV.



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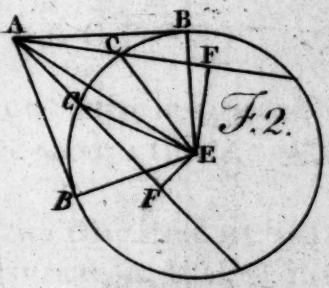

Fig. 2.

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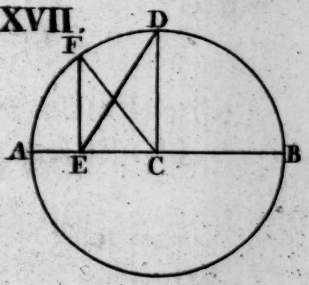

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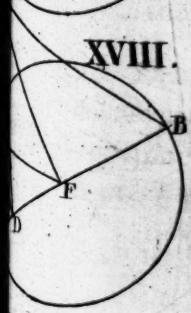

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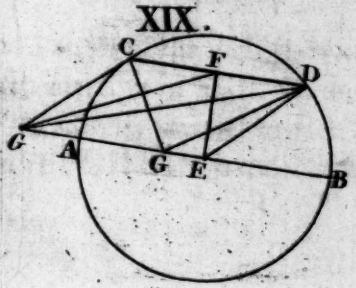

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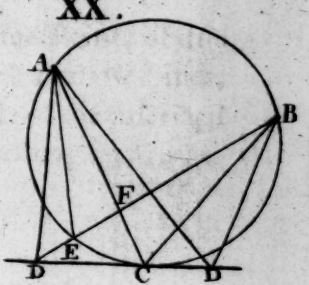
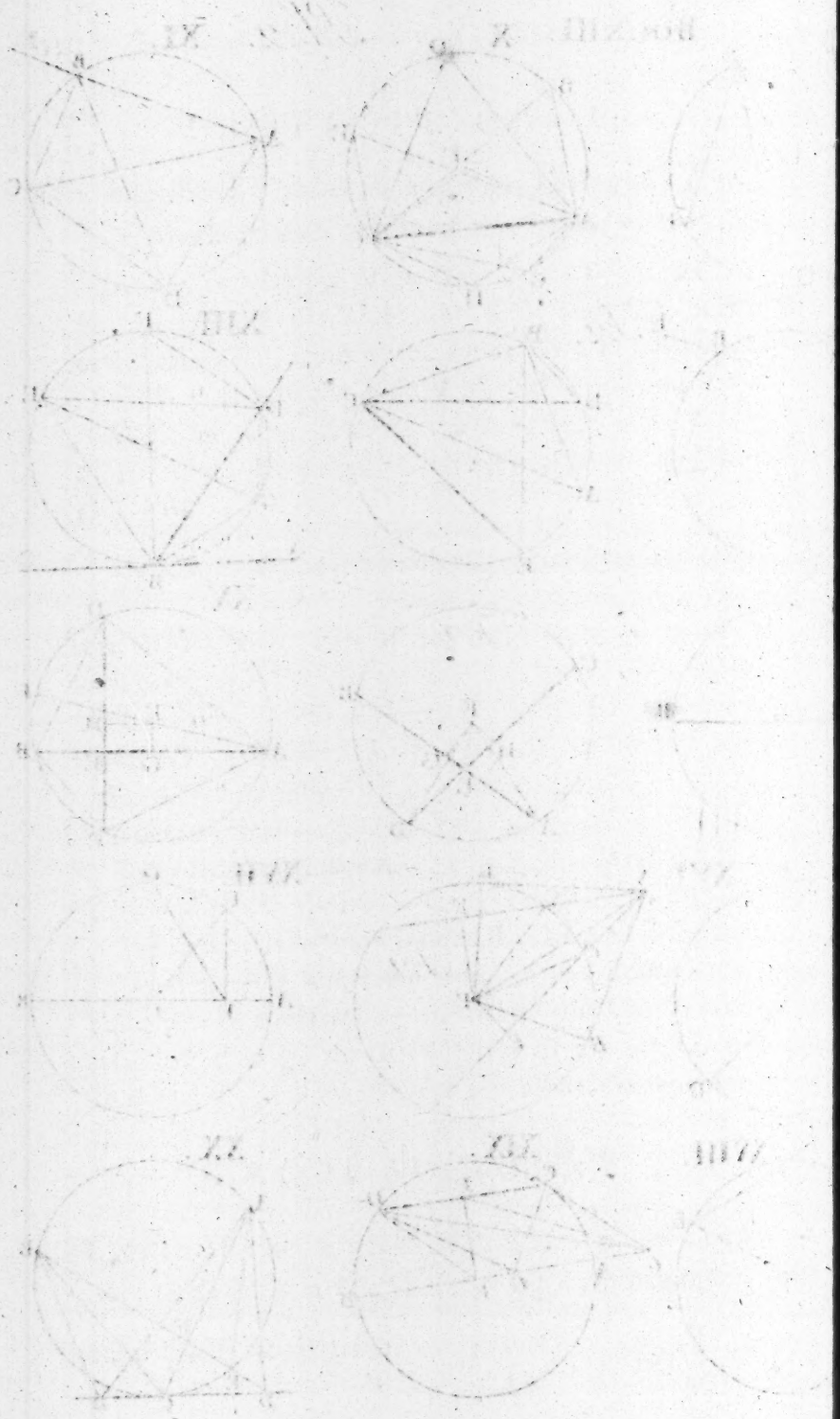




Let any two Sides of the Triangle, as AB and BC, be bisected in E and F. Let ED and FD, be perpendicular to AB and BC, cuting each other in D, the Center of the circumscribing Circle.

If the Triangle has a Right angle, the Hypothenuse bisected gives the Center. If an Angle be obtuse, the Center of a circumscribing Circle falls without the Triangle; the Operation is the same in both. (See Fig. 2.)

Join the Points AD, BD, and CD.

DEM. Because the Side AB is bisected in E, and DE is perpend. to AB, the Sides AE, ED, of the Triangle AED, are equal, respectively, to BE, ED, of the Triangle BED; and the Angle AED is equal to BED. - - Def 10. wherefore, AD is equal to BD - - - - - - 9. 1. for they subtend equal Angles, in the Triangle ABD.

In the same manner, CD may be proved eq. to AD, or BD. Consequently, D is the Center of the circumscribing Circle, ABC.

PROPOSITION VI.

To inscribe a Square in a given Circle; and to describe a Square about a Circle. ABCD, and EFGH.

Draw two Diameters, AC and BD, at right angles. Join the extremes, A, B, C, D, and a Square is inscribed.

2nd. Draw Tangents through the extreme Points of the Diameters, meeting in E, F, G, and H; or draw Lines, touching the Circle, parallel to each Side of the inscribed Square, and EFGH is a Square circumscribing the Circle.

DEM. The Diameters AC & BD biſect each other, - 16. 1.
therefore, AI, IC, IB, and ID, are equal;
and they contain equal Angles, AIB=BIC, &c. - Con.
Wh. AB, BC; CD & AD, are all equal amongſt themſelves.
And, the Angles ABC, BCD, &c. being in a Semicircle,
are Right angles. Therefore ABCD is a Square.

2ndly. Becauſe EF touches the Circle, in the Point B, it is
perpendicular to the Diameter BD. - - - - - 8. 3.
And, becauſe HG is perpend. to BD, HG is parallel to EF.
For the ſame reaſon, EH is parallel to FG.
Therefore, EFGH is a Parallelogram. - - Def. 33.
But, AC is perpendicular to BD (Con.) wherefore, EH
is perpendicular to HG, and ſo is EF to FG, and to EH;
conſequently, EFGH is a Rectangle. - - Def. 34.
But EF, FG, EH, and HG, are each equal to AC, equal BD.
Th. EFGH is equilateral; conſ. it is a Square - Def. 35.

N. B. The Square EFGH, circumſcribing a Circle, is double the
Square ABCD, inſcribed in the ſame Circle.
For the Triangle ABC=half the Rectangle AEFC - - 17. 1.
And ADC is equal to half the Rectangle AG;
conſequently, the Square EFGH is double ABCD.

PROPOSITION VII.

To inſcribe a Circle in a Square; and to circumſcribe a Square, with a Circle.

Let ABCD be a given Square.
Draw the Diameters AC & BD cuting each other in E.
From the Center E, let EF be perpendicular to AB.
With the Radius EF, on the Center E, deſcribe a Circle,
which will touch every Side of the Square ABCD.

Let EG be perpend. to BC; produce GE to I, and FE to H.

DEM. Now, becauſe ABCD is a Square, the Diagonals
AC and BD biſect each other in E; - - - - - 16. 1.
conſequently their halves are equal. - - - Ax. 1. 1.

Then, since AE, EB, and EC are equal, the Triangles AEB, BEC are Isosceles; wh. AB and BC are bisected by the Perpendiculars EF & EG; cons. FB=BG; -C.4.9.1. and consequently, BE bisects the Right angle ABC.

Now since in the Triangles EFB, BGE, the Sides, FB, BE, are respectively equal to GB, BE; and the Angle FBE is equal EBG, the remaining Side EF=EG.

Therefore the Circle FGH touches every Side of the Sq.

2nd. With the Radius EA, describe the Circle ABCD.

Then, since the Diagonals of a Square are equal, and mutually bisect each other (16. 1.) the halves EA, EB, EC, and ED are also equal; wherefore, a Circle, ABC, will pass through every Angle of the Square.

N. B. A Circle circumscribing a Sq. is double of a Circle inscribed.

For, the Square of AC=AB □ + BC □; consequently AC □ is double BC, equal FH, □.

But Circles are, to each other, as the Sqs. of their Diameters; therefore, the Circle ABCD is double FGHI. - - C. 1. 14. 6.

PROPOSITION VIII.

To make an Isosceles Triangle, having its Angles, at the Base, each double the Angle at the Vertex.

AB is a given Line, for one of the equal Sides.

AB being divided in extreme and mean Proportion, in C; with the Radius AB, on B, draw the Ark AD.

Make AD equal to CB, and draw BD. I say; in the Triangle ABD, the Angles A and D are each double the Angle B. Join CD, and describe a Circle thro' B, C, & D.

DEM. Because the Rectangle BAC=BC □; - - - 11. 2. and AD=BC (Con.) the Rectangle BAC=AD □.

Wherefore, AD touches the Circle BED, at D; - 16.3. and therefore, the Angle ABD=ADC. - - 13. 3.

But, the Angle ACD=ABD+BDC, i. e.=ADC+CDB

And, the Angle BAD=ADB (9. 1) for AB=BD; - Con.
conſequently, BAD=ACD; wh. CD=AD. C. 3. 9. 1.
But, AD=CB (Con.) wherefore, CD=CB; - Ax. 3. 1.
conſequently, the Angle CDB=CBD. - - Th. 9. 1.
But, the Angle ADC=CBD; (proved, above)
wherefore, ADC is equal to CDB; - - - Ax. 3. 1.
and therefore, ADB, BAD, are each double ABD.

The Conſtruction of this particular ſpecies of Triangles, in reſpect of itſelf, is of little or no uſe; but, the inſcribing a regular Pentagon in a Circle, according to Euclid, in the next Propoſition, depends on it entirely; which may be more readily conſtructed. Nevertheleſs, for the manner and elegance of its Demonſtration, I did not think proper to omit it.

PROPOSITION IX.

To inſcribe a regular Pentagon in a Circle.

A B C is the given Circle.

Inſcribe an Iſoſceles Triangle, ABC, whoſe Angles BAC, ACB are, each, double the Angle ABC. Prop. 1.

Let each of the Angles BAC, ACB be biſected, by the Right lines AD and CE, cuting the Circumference in D and E. Join the Points, A and E, &c. by the R. lines AE, EB, BD, & DC. AEBDC is a regular Pentagon.

DEM. For, becauſe each Angle, BAC, ACB, is double the Angle ABC (Con.) and thoſe Angles are biſected, the Angles, ABC, ACE, ECB, BAD, & DAC, are all equal.

But, equal Angles ſtand on equal Arks, - - C. 2. 9. 3
and, equal Arks have equal Chords or Subtenſes; - 2. 3 3
wherefore, AC, AE, EB, BD, and DC, are all equal;
and conſequently, the Pentagon, AEBDC, is equilateral.

Again; becauſe the Arks AE, EB, &c. are all equal, the Angles ACE, ECB, BCD, &c. are equal;

wh. the Angle EBA+ABC+CBD=EAB+BAD+DAC; conſequently the Angle EBD=EAC; and ſo, of all the reſt.

Or, becauſe each Angle of the Pentagon, AEB, EBD, &c. ſtands on equal Arks, ACDB, EACD, &c; therefore, the Pentagon AEBDC is equiangular.

Cor. The Diagonals of a Pentagon are parallel to their oppoſite Sides.

For, the Triangles AEC, ADC are equal. - 8. 1.
Therefore, ED is parallel to AC. - - Cor. 18. 1.

PROPOSITION X.

To deſcribe a regular Pentagon about a Circle.

Let ABCDE be the angular Points of a Pentagon, inſcribed in a Circle; through which, draw the Tangents FG, GH, HI, &c. cuting each other, in F, G, H, I, & K; i. e. having made the Arks AB, BC, &c. equal (viz. each equal 72 Degrees) through the Points A, B, C, &c. draw the Tangents, as before.

The Pentagon FGHIK is equilateral and equiangular.

Let L be the Center of the Circle; draw AL, BL, &c. alſo draw FL, GL, &c.

Dem. AL, BL, &c. are perpendicular to FG, GH, &c. 8. 3.
and they are all equal between themſelves. - Ax. 1. 3.
The Angle ALB is equal to BLC, by Conſtruction.
AG is equal to GB, and BH equal to HC; - C. 2. 16. 3.
wh. the Triangles ALG, GLB, BLH, &c. are equilateral and equiangular to each other; for GB=BH, &c. - 11. 1.
AF=AG, and CI=CH (ſame) conſ. FG=GH. - Ax. 1.
After the ſame manner, FK and IK, may be proved equal between themſelves, and alſo to FG, &c;
therefore, FGHIK is equilateral.

And, ſince the Triangles ALG, GLB, &c. are equiangular, the Pentagon is alſo equiangular.

For, the Angles AGL, LGB, alſo BHL and LHC are equal amongſt themſelves; conſequently, FGH=GHI.

After the ſame manner, the Angles, I and K, may alſo be proved equal between themſelves, and alſo to F, G, & H. Th. FGHIK is a regular Pentagon, circumſcribing a Circle.

Note. If the Circumference of a Circle be divided in five equal Parts, and the Points joined by Right lines; Tangents being drawn, parallel to every Chord, a regular Pentagon will be circumſcribed.

PROPOSITION XI.

To inſcribe a Circle in a regular Pentagon; and, to deſcribe a Circle about a Pentagon.

Let any two Sides, AB, & BC, of the Pentagon ABCD, be biſected by Perpendiculars, EF and FG, interſecting in F; on which, with the Radius EF or FG, a Circle, being deſcribed, will touch every Side of the Pentagon.

2nd. With the Radius AF or FB (the Point F being found as before, or by biſecting two Angles, ABC and BCD) on F, a Circle being deſcribed will paſs through every Angle of the Pentagon.

DEM. Becauſe AB=BC, EB=BG. Draw AF, FB, &c. Then, in the Triangles EBF, FBG, FB□=FE□+EB□; and, FB□=FG□+BG□; - - 20. 1. conſequently, FE□+EB□=FG□+BG□. - Ax. 3. 1. But, EB=BG (Ax. 1. 1.); wherefore, EB□=BG□; conſequently, EF□=FG□; and, therefore EF=FG.

In the ſame manner, FH may be proved equal to FG; F is, therefore, the Center of a Circle inſcribed.

2nd. Becauſe AB is biſected, in E, AE=EB, & AE□=EB□ Add EF□, to both; and AE□+EF□=EB□+EF□. But, EF is perpendicular to AB, by Conſtruction; conſequently, AEF, FEB are Right angles; - Def. 10 wh. AF□=AE□+EF□; and FB□=EB□+EF□.

Conſequently, AF□=FB□; and therefore, AF=FB.

In the ſame manner, FC, FD, &c. may be proved equal to AF and FB; and conſequently, F is the Center of a circumſcribing Circle.

PROPOSITION XII.

To inſcribe a regular Hexagon in a given Circle.

In the given Circle, AGF, draw a Diameter, AB.

With the Radius of the Circle, AC, on the Centers A and B deſcribe two Arks, DCE, FCG, cuting the Circumference in the Points D and E, F and G.

Or, having drawn the Ark DCE, only, cuting the Circle in D & E, draw DC & EC, & produce them to F & G.

Join the Points A and D, AE, EF, &c. by Right lines, which compleats the Hexagon, ADGBFE. Q. E. F.

I ſay, it is both equilateral and equiangular.

Dem. Becauſe C is the Center of the Circle AGF, the Lines DC, AC, and CE are equal; and, becauſe A is the Center of the Circle DCE, AD, AC and AE, are equal; wherefore, ADC & ACE, are equal & equilateral Triangles.

But, the Angles of an equilateral Triangle are, each, equal to one third Part of two Right Angles, - C. 10. 1. (becauſe, the Angles which are oppoſite to equal Sides are equal; and the three Angles of every Triangle are, together, equal to two Right angles; - - 9 and 10. 1.) conſequently, DCA + ACE = two thirds of two R. angles.

Now, becauſe the Right line EC falls on the Right line DF, the Angle ECF + ECD = two Right angles; - 1. 1. and, becauſe DCA + ACE (eq. DCE) = two thirds of two Right angles, ECF is equal one third part of two Right Angles; i. e. equal to ACE, equal ACD.

Again; becauſe the Right lines AB, DF, & EG cut one another, in the Point C, the Angles DCG, GCB, & BCF are, reſpectively, equal to ECF, ACE, and ACD 2. 1. Wh. thoſe ſix Angles are all equal amongſt themſelves; conſ. they ſtand on equal Arks; and conſ. on equal Chords. Therefore, AD, DG, &c. are equal to one another.

2nd. Becauſe the Triangles, ADC, CDG, &c. are equilateral, and conſequently equiangular, - - - - C. 2. 9. 1. the Angle ADC+CDG=DGC+CGB; - - Ax. 3. i. e. the Angle ADG=DGB; and ſo of all the reſt. Therefore, the Hexagon is equiangular, and equilateral.

COR. The Side of a regular Hexagon, inſcribed in a Circle, is equal to the Radius, or Semidiameter of the Circle.

For, the Triangles, ADC, &c. are proved to be equilateral.

Hence, a Duodecagon may be inſcribed in a Circle, by biſecting each Ark on the Side of a Hexagon inſcribed. i. e. The Radius being applied ſix times in the Circumference, each Ark biſected gives the Sides of a Duodecagon.

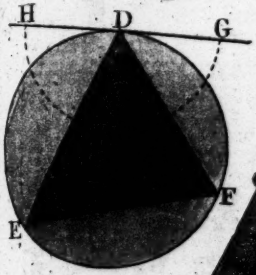
N. B. Any Poligon may be circumſcribed about a Circle, having firſt inſcribed one, by drawing Tangents through every Angle, or parallel to every Side of the inſcribed Figure.

PROPOSITION XIII.

To find the Side of a Quindecagon inſcribed in a Circle.

ABC is a given Circle, in which, to inſcribe a Quindecagon.

Draw, DG, at pleaſure, touching the Circle ABC in A. With any Radius, on A, deſcribe a Semi-circle DEFG; and make the Arks DE, &c. each equal to the Radius, AD. Draw AB and AC through E and F, and join BC; ABC is an equilateral Triangle inſcribed. Q. E. F.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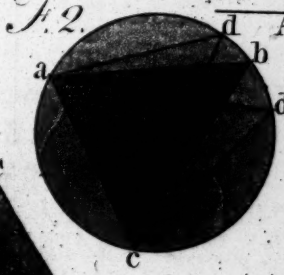
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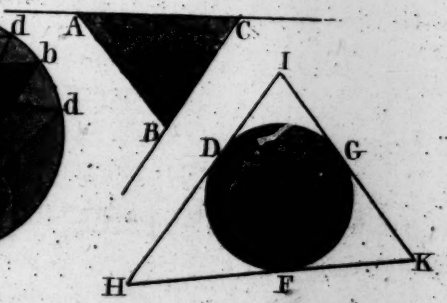
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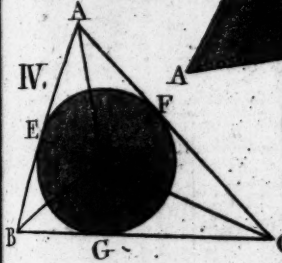
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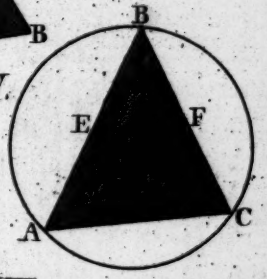
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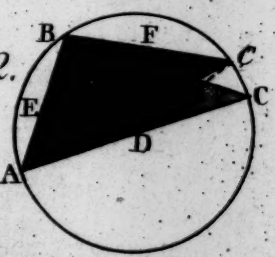
Fig. 2.
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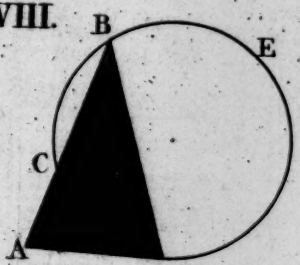
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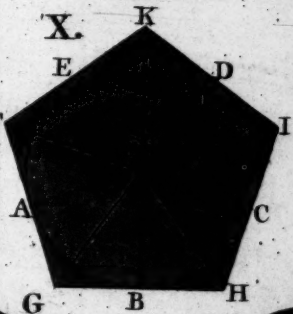
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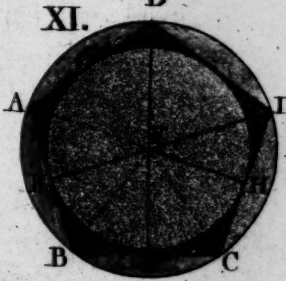
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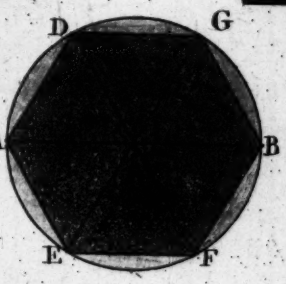
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DEM. For, the Arks, DE, EF, & FG, are equal by Conſtruction; wherefore, the Angles DAE, EAF, and FAG, are equal. But, the Angle BCA is equal to BAD, and, CBA is equal to CAG; - - - - - - - 13. 3 conſequently the Triangle ABC is equiangular; and therefore it is equilateral - - - - - - - C. 3.9.1.

Again. The Ark of the Semicircle, DEFG, being divided into five equal parts, D 1, 1 2, &c. as before into three (in E and F) A1, A2 being drawn, cuts the Circumference, at H and I.

AH and HI are Sides of a Pentagon, inſcribed.

Granting AB to be the Side of an equilateral Triangle, and AH, HI, to be the Sides of a Pentagon, inſcribed; BI is the Side of a Quindecagon, or Poligon of 15 Sides.

For, becauſe AB is the Side of an equilateral Triangle, the Ark AHB is one third part of the whole Circumference.

Alſo, AH and HI being Sides of a Pentagon, the Arks AH and HBI are each equal to one fifth part of the Circumference; i. e. the Ark AHI is equal to two fifths, equal ſix fifteenth parts; for HI, one fifth, is equal to three fifteenths. And ſince AHB is one third part of the Circumference, it contains five fifteenths; wherefore, BI is equal to one fifteenth part, and is, therefore, the Side of a Quindecagon, inſcribed in the Circle ABC.

The method of inſcribing a Quindecagon in a Circle, according to Euclid, is a matter of mere curioſity; firſt, to inſcribe an equilateral Triangle, and afterwards a Pentagon, in the ſame Circle, in order to get a fifteenth part of the Circumference; which, notwithſtanding it admits of perfect Demonſtration, is liable to greater error in Practice.

ELEMENTS
OF
GEOMETRY.

BOOK V.
OF PROPORTION.

THE fifth Book of the Elements of Euclid contains the ſublime Doctrine of Proportion; which is, perhaps, the moſt ſubtle manner of reaſoning, the moſt brief, ſolid, and convincing, that the Art of Man could deviſe. It is eſſentially neceſſary in demonſtrating all the Propoſitions of the ſixth Book; which, alone, is ſufficient to recommend it: But, when we conſider its general and extenſive uſe, throughout the Mathematics, it is impoſſible to be diſpenſed with. The Doctrine it teaches is made uſe of as the moſt general mode of Demonſtration, in almoſt every mathematical Science; without the knowledge thereof, we ſhould not be able to advance one ſtep further.

I ſhall ever be of opinion, that the eaſieſt and readieſt way of acquiring Knowledge, is the beſt. The Doctrine of Proportion is, in ſome meaſure, born with us; it is a portion of Reaſon, implanted in us by Nature. Does not every Huckſter, Pedlar, or ſmall Shopkeeper, who knows nothing of Arithmetic, nay, who probably cannot write, nor read, readily proportion the value of any quantity of their Goods, according to the Price, by the Pound, or Yard, &c. as ſoon almoſt as the Quantity is known; and that, when there are fractional Parts, either in the Price or in the Quantity, or both. Proportion is, in itſelf, very ſimple and facile, it ſcarcely requires Demonſtration, but only to be illuſtrated and familiarized.

In expreſſing and comparing Quantities, of any Species, it is neceſſary to make uſe of Symbols or Figures; which Symbols, Marks or Characters, are entirely arbitrary, and might as well have been applied to ſignify the Qualities of Things, as Quantity. But, having been accuſtomed, from our infancy, to expreſs our Ideas of Proportion by Numbers; we no ſooner become acquainted with Numbers, and the Characters applied to them, but we find it almoſt impoſſible, in our Ideas of Magnitude, or Quantity of any kind, to compare or eſtimate their Ratio, abſtracted from Numbers. In reſpect of Magnitude, as to ſolid Extenſion, or Body; we ſay, when compared, it is twice, thrice, or four times as big as another; ſo likewiſe of Extenſion, in length ſimply, we expreſs it by one, two, or three Yards, Chains, or Miles, &c. and ſo of any other Quantity whatever.

Hence, the mode of expreſſing Quantity, by Numbers, is the moſt natural, ſimple, and eaſy; and therefore, it is no wonder, when we find it expreſſed by other Characters, or Lines, as by Euclid and others, that it appears, at firſt, to thoſe who have not conſidered it, rather ſtrange, and foreign; but, in reality, the difference between Lines and Numbers, in expreſſing Quantity, conſiſts, only, in being more familiarized to one than the other.

Therefore, ſince Lines, Characters, and Numbers, are only different modes of expreſſing Quantity, I muſt needs think that the beſt, which is the moſt natural and eaſy to conceive; beſides, it is much more convincing; for, unleſs Lines are divided into the ſame equal Parts, whereby we may form a judgment by how much one Magnitude, or Quantity of any kind, exceeds another, we have no Idea of their Proportion; otherwiſe, than

that one does exceed, is more or leſs than the other; whereas, Numbers aſcertain the Ratio, whatever it is, whether equal or unequal, which Lines do not.

If Characters (as A, B, C, &c.) are made uſe of, they only denote Quantity, ſimply, but in what Ratio, or Proportion, is expreſſed by Numbers. Small Characters, uſed algebraically, are not, in themſelves, expreſſive of their value; they do not convey a competent Idea, equal to that given by Numbers, neither do they aſcertain Ratio; and, if Lines are divided into equal Parts, to expreſs a Ratio, it is the ſame as Numbers; becauſe, we only know the Proportion by numbering the Parts.

It is eaſily known that there is Analogy, or ſimilarity of Proportion, when, in four Quantities, the firſt (i. e. any one) contains the ſecond (any other) or is contained, the ſame number of times, as a third contains the fourth, or is contained by it; whether it can be expreſſed in whole numbers, as once, twice, thrice, &c; or once, twice and a half, a third, a fourth, or any other fractional number whatever.

E. g. 6 has the ſame Proportion to 24 as 4 has to 16, or 3 to 12, each being contained four times in the other; their Ratio is, therefore, as 1 to 4. 9 has the ſame Proportion to 24 as 6 to 16; for each is contained in the other, twice and two thirds; the Ratio is as 3 to 8. So likewiſe, 27 is to 9 as 21 to 7, or 12 to 4; for, each contains the other thrice; therefore, the Ratio is as 3 to 1. 18 is to 12 as 12 is to 8, for they are, each, as 3 to 2. Thus may the Ratios of any commenſurable Quantities, whatever, be expreſſed by Numbers; and, indeed, for ſuch as are incommenſurable, by the application of Decimals, we continually approach nearer the truth, till the deficiency is leſs than any aſſignable Quantity whatever.

Suppoſe any four Terms or Characters, A, B, C, D, to repreſent four Quantities, that are analogous in their Ratio, two, and two; i. e. when A (any one) has the ſame Proportion to B (any other) as C, a third, has to D, the fourth; which is thus written, A : B :: C : D and is read thus; as A is to B, ſo is C to D. The four Points, dividing the two Pairs, denote an equal Ratio between them.

The whole buſineſs, of this fifth Book, is to ſhew, what various ways they may be changed, and ſo ordered amongſt themſelves, that the Proportion ariſing, on both ſides, may ſtill be equal or analogous; which, will admit of great variety; ſeeing, the Ratios of ſeveral Quantities, that are analogous, may be expreſſed by the ſame Numbers; as, 2 to 5 expreſſes, equally, the Ratio of 6 to 15, of 8 to 20, and of 14 to 35; each being contained in the other, reſpectively, twice and a half.

Therefore, either by Alternation, Inverſion, Compoſition, Diviſion, converting, or mixing the Terms, the Ratio of each may, in each change, be expreſſed by the ſame Numbers; & conſequently, there is always equality of Proportion. Nor, indeed, is it poſſible it ſhould be otherwiſe; if the Terms, on both ſides, be ordered and changed after the ſame manner.

Wherefore, I am of opinion, notwithſtanding what the learned Dr. Barrow, in his mathematical Lectures, and Dr. Keil, in his Preface, ſays to the contrary, and the high encomiums which Cunn, in his Preface, beſtows on Euclid's Demonſtrations of the fifth Book, that 'tis involving a thing, in itſelf not very intricate, if not in darkneſs and obſcurity, at leaſt in perplexity. More than half of the Propoſitions, in Euclid, may be diſpenſed with, or paſs for Axioms; and be applied with equal validity, towards attaining Demonſtration of the whole.

DEFINITIONS.

Def. I. QUANTITY is whatever may be meaſured or numbered, eſtimated or compared, as to more or leſs.

As MAGNITUDE or Body; EXTENSION, or length ſimply; WEIGHT, or GRAVITY; MEASURE, of any kind; TIME, MOTION, &c.

One Surface may be equal, in area, to twice, thrice, or four times the area of another Surface; or a Line may have twice or thrice, &c. the length of another Line. Alſo, if one Body moves, uniformly, through twice, thrice, or four times the Space through which another Body moves, in equal Time, it will move with twice, thrice, or four times the velocity. All which, come under the Denomination of Quantity, and may be expreſſed by Right lines, of twice, thrice, or four times the length, one of another; or by Numbers; as 6 contains 3, 2, or $1\frac{1}{2}$, twice, thrice, or four times.

QUANTITIES, being compared, are (in reſpect thereof) either commenſurable, or incommenſurable.

COMMENSURABLE QUANTITIES are ſuch as have a common meaſure; that is, when both may be divided into Parts of the ſame magnitude. Or, when ſome determinate Quantity may be found, which, being taken, or multiplied ſome number of times, is equal to either, without deficiency or exceſs.

INCOMMENSURABLE QUANTITIES are ſuch as no other Quantity can meaſure; i. e. there cannot be found any determinate Quantity, how ſmall ſoever, which, being multipled, will be equal to each of the other; but that there will be a deficiency or exceſs, in one or the other.

N. B. Any two Quantities, whoſe Proportion, to each other, can be expreſſed by Numbers, are commenſurable; two Quantities, whoſe Ratio is ſuch as cannot be expreſſed in Numbers, are ſaid to be incommenſurable to each other.

Def. II. An ALIQUOT, or equal Part, is that, when a leſs Quantity is contained any number of times, preciſely, in a greater; i. e. when a leſs Quantity, being taken or multiplied any number of times, is equal to a greater.

Thus; an Inch, or a Foot, is an Aliquot part of a Yard, or Fathom; or an Ounce of a Pound, &c.

3 is an Aliquot part of 9, 12, or 15, &c.

Def. III. MULTIPLE. That Quantity is called a Multiple, in reſpect of another Quantity, when it contains, exactly, or is equal to the other, being taken any number of times; then, the leſs is ſaid to meaſure the greater.

Thus; a Foot is a Multiple of an Inch, of two, three, four or ſix Inches, but it is an Aliquot part of a Yard, &c.

A Yard is a Multiple of a Foot or of an Inch, &c. but it is an Aliquot of a Fathom, or Furlong, &c.

Def. IV. RATIO* (or PROPORTION) is the mutual habitude or relation which ſubſiſts between Magnitudes, of the ſame kind, reſpecting Quantity; or their affinity to each other, in reſpect of Magnitude.

Thoſe QUANTITIES are ſaid to have Ratio to one another, which, being multiplied, can exceed each other.

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* This Definition is rejected, by ſome, as ungeometrical. Profeſſor Simſon ſays, after a long quotation from Dr. Barrow's 18th Lecture, "that he fully believes, the 3rd and 8th Definitions of the 5th Book (the 4th and 6th of this) are not Euclid's, but are added by ſome unſkilful Editor." I muſt own, that I am at a loſs to conceive why it is they cavil at them; which, to me, ſeem eſſentially neceſſary.

Dr. Barrow ſays, that, "Euclid had, perhaps, no other deſign in giving this Definition (which he calls metaphyſical) than to give a general, though a groſs and confuſed notion, or idea of Ratio, to beginers," grant it; was it not neceſſary, then, to give ſuch a general Idea?

He ſays, further, that nothing in Mathematics depends on it, (the 6th) and that, both might well be ſpared, without any loſs to

In order to which, it is neceſſary that they be of the ſame ſpecies, or kind. e. g. A Line (having no breadth) cannot, by multiplying, be compared with, or be equal to the ſmalleſt

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Geometry." I am aſtoniſhed at ſuch an Aſſertion; what! does nothing depend on having a clear Idea of Analogy? can we (as he ſays we ſhould, principally) attend to thoſe which follow, before we know what is meant by RATIO and *Analogy* of RATIOS? can we have any notion of any one, from *Alternate Ratio* to *Ratio of Equality*, before we know what is meant by Ratio, ſimply? which, in itſelf, abſtractedly conſidered, is undoubtedly metaphyſical; we can have no Idea of Ratio in abſtract. The Doctor takes great pains (in his 20th Lecture) to convince us, that Ratios are not Quantities; who ever ſuppoſed it? at the ſame time, I find it impoſſible to form an idea of Ratio, abſtracted from Quantity. Ratio neceſſarily implies Quantity; it has no exiſtence but in or between Quantities; can we conceive a Ratio without Quantity? the ſuppoſition is abſurd, and cannot be. But ſome Terms muſt be uſed, to expreſs or denote Ratio and Analogy of Ratios, which, I think, are abſolutely neceſſary in Geometry; that they were given by Euclid, is indiſputable, in my opinion; the Elements of Proportion would be imperfect, how could we reaſon on the ſubject without them?

Some Geometers make a diſtinction between Ratio and Proportion; for, Dr. Barrow, in his 4th, and Mr. Stone, in his 8th Definitions ſay, "PROPORTION is a ſimilitude of RATIOS," which, if Ratio and Proportion be the ſame, (as the Doctor aſſerts, in a Note annexed) is an abſurd Definition; viz. Proportion is a ſimilitude *of* Proportions. Now, in my opinion, the diſtinction between them is ſo very nice, that it requires very acute diſcernment to diſcover the difference. What is, by ſome, called Proportion is, properly, Analogy;* i. e. when the Ratios are equal or ſimilar. Dr. Keil, in his 8th Definition, ſays, "ANALOGY, is a ſimilitude of Proportions;" but, Prof. Simſon ſays, "ANALOGY, or PROPORTION, is the ſimilitude of RATIOS." Again; Dr. Keil, in his 4th Definition, ſays, "Magnitudes are ſaid to have Proportion to one another." Mr. Stone ſays, "Magnitudes are ſaid to have a Ratio to one another." Dr. Barrow, in his fifth, changes Magnitudes for Numbers, but makes uſe of the Term Ratio. So, in the ſixth Definition, Keil ſays, "Magnitudes that have the ſame Proportion;" Simſon and Stone ſay, in the 6th, "Magnitudes which have the ſame Ratio, are called PROPORTIONALS;" conſequently, none but Keil means the ſame by Ratio and Proportion. Profeſſor Simſon, expreſsly, makes Proportion ſynonimous with Analogy; and 'tis evident, that both Dr. Barrow and Mr. Stone make uſe of the Term Proportion to ſignify Analogy.

* See Def. 6.

ſurface. Alſo, a Surface (having no thickneſs) cannot, by multiplying, be equal to, or compared with a Solid. So neither can Weight be compared with Time, or Motion, &c. Wherefore, ſuch Quantities as are not of the ſame kind, are heterogeneous, and can have no Ratio to one another.

Def. V. ANTECEDENT and CONSEQUENT are the Terms of a Ratio.

Ratio neceſſarily ſuppoſes two Terms, denoting two Quantities. The ANTECEDENT is that which is firſt named, and being refered to, or compared with another Quantity, the latter is called the CONSEQUENT.

Thus, A : B, or B : A, ſignifying, that A has ſome Ratio to B, or B to A (which it neceſſarily muſt if they be ſuppoſed to repreſent two Quantities of the ſame kind). A or B is the Antecedent, B or A is the Conſequent, according as they are taken, firſt or laſt.

N. B. The Ratio of two Quantities is not expreſſed by the deficiency or exceſs of the Antecedent to the Conſequent; although they may be compared by them, it is not their Ratio.

RATIO, is the Value of the Antecedent Term in reſpect of the Conſequent, not the Difference.

Thus, If A is to B as 3 to 5; the Ratio is $\frac{3}{5}$, 3 fifths of the Conſequent. But, if B be taken for the Antecedent, the Ratio is greater, viz. as 5 to 3; $\frac{5}{3}$, i. e. 5 thirds; which is one, and two thirds of the Conſequent; whereas, in the former Ratio, the Antecedent is two fifths deficient of its Conſequent.

Def. VI. ANALOGY,* of RATIOS, is a ſimilitude or equality of RATIOS. When A (any Quantity what-

* IT was not the deſign, of thoſe Geometers who made uſe of this Term, to define Analogy, in abſtract, but to explain what is meant by analogous Ratios, which I think a very proper and

ever) has the ſame Proportion to B (any other of the ſame kind) as C (a third) has to D, another of the ſame kind; i. e. when, in four Quantities, the Antecedent, of any two, contains its Conſequent, equally, as the Antecedent of the other two contains its Conſequent; or, when the Antecedents are contained equally in their reſpective Conſequents, the Ratios being equal, they are then ſaid to be analogous.

Thus; 9 has the ſame Ratio to 3, as 6 to 2; for each is as 3 to 1, i. e. each Antecedent contains its Conſequent thrice; alſo, 6 is to 9 as 2 to 3, for each Antecedent is contained once and a half in its Conſequent.

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expreſſive Term, for equality of Ratios. As it was neceſſary that ſome Term ſhould be uſed, Analogy ſeems, to me, more expreſſive of the thing meant than Proportion; which is only another Term for Ratio, or Ratio for Proportion; for theſe are ſynonimous, and equally expreſſive. In all the various changes which are made, by Alternation, Inverſion, &c. the reſult of them is Analogy; the mode of reaſoning by Proportion, in general, is from ANALOGY; conſequently it is not an unneceſſary Term.

If we muſt be caviling, there are other Definitions which are more exceptionable; as the fourth of Euclid, viz. "Magnitudes are ſaid to have a Ratio to one another," &c. This is not a Definition, but meant only to acquaint us, what Quantities *can* have Ratio to each other; which Common ſenſe has amply provided for: Who would ever think of comparing Time with Gravity; or either of them with Surfaces, or Solids.

The fifth is not a Definition, but only a ſign of Analogy; (ſee the Remark on it, at the end of this fifth Book). The 9th is leſs ſo than either of the former, viz. "Proportion (or Analogous Ratio) conſiſts of three Terms, at leaſt." What does this define? nothing; nor can any Perſon be at a loſs to know, that there muſt needs be three Terms, who has had Analogy of Ratios defined, where he always finds four; but, one of the four, is ſometimes taken twice. I therefore, look on it as ſuperfluous, unneceſſary, ſeeing it is a neceſſary conſequence in Analogy; yet, 'tis not improper, by way of Nota bene, or Remark; but there could be no impediment, to the attaining of the reſt, if it was never mentioned. Therefore it may, very ſafely, be bloted out of the Vocabulary of Definitions; for, to uſe Dr. Barrow's own words, Proportion would ſuſtain no loſs by it. I can ſcarcely believe, that it was given by Euclid as a Definition.

ANALOGY of Ratios is thus ſymbolized; A:B::C:D.

The firſt and third Terms, A and C, are the Antecedents, and may be taken for any Quantities whatever; B and D, the third and fourth, are the Conſequents.

Wherefore, if A be ſuppoſed, in Numbers, equal to 9, and C to 3; and if B be equal 6, it will be thus, 9:6::3:2; conſequently, D will be equal to 2. But if B is equal to C, then will D expreſs or ſignify but 1, viz 9:3::3:1.

Or, if B repreſents a greater Quantity than A, D will alſo be greater than C; thus, 9:12::3:4; or 9:15::21:35; i. e. as 3 to 5.

The three firſt Terms being fixed or known Quantities, and in a certain order, the fourth is a neceſſary conſequence.

UNEQUAL RATIOS are ſuch as are not analogous.

Def. VII. CONTINUED RATIO is when, in three or more Quantities, the firſt is to the ſecond, as the ſecond to the third, as the third to the fourth, &c.; or, more properly, the third is to the fourth, as the ſecond to the third, and that, as the firſt is to the ſecond.

Thus. In four or more Quantities, A, B, C, D; if C has the ſame Proportion to D, as B has to C, and, if B has alſo the ſame Proportion to C, as A has to B, they are then in continual Proportion, or geometrical PROGRESSION, which begins with the two firſt taken; and are thus ſymbolized; A:B:C:D÷.

N. B. Each Term being twice taken, except the firſt and laſt, are, therefore, both Antecedent and Conſequent, i. e. the ſecond Term is a Conſequent in reſpect of the firſt, and the third of the ſecond; but they are, alſo, Antecedents, in reſpect of the third and fourth Terms.

N. B. 2. In continued Ratio, all the Terms are, neceſſarily, of the ſame ſpecies or kind. i. e. The Quantities muſt be homogeneal

If A and B, the two first Terms, or C and D, the two last (or *any* two, that are contiguous) are fixed or determinate Quantities, all the the rest are determinable; which, in Numbers, is obvious.

Let A be 4, and B 6; it will then be thus, $4:6:9:13\frac{1}{2}$ ∺&c. each consequent Term containing its Antecedent once and a half. But if A be 3, and B 6, it will be 3:6:12:24 ∺ the Ratio being as 1 to 2, or half. Or, if A represents 2, and B 6, then 2:6:18:54 ∺ &c. the Ratio being as 1 to 3. Also, 81:27:9:3 ∺ that is, as 3 to 1, or triple. So that, whatever Ratio there is between the first and second, that Ratio is continued, *ad libitum*.

Therefore, when the Antecedent is the lesser Quantity, it is determinable either by Addition or Multiplication; when the Antecedent is the greater Quantity, by Division, or Subtraction.

When the second Term has not the same Proportion to the third, as the first has to the second, or the third to the fourth, they are said to be *directly* or *discretely* proportional.

Thus, 4:6::8:12, directly or discretely; but, $4:6::9:13\frac{1}{2}$ is directly, and also continued.

Def. VIII. PROPORTIONALS are such Quantities as are in the same Ratio; discretely or continual.

PROPORTIONALS consist, at least, of three Terms, or Quantities, which are thus symbolized by Characters, as A:B:C; or by four, as A:B:C:D ∺; or A:B::C:D; the Ratio being determined by Numbers.

PROPORTIONALS cannot subsist without Analogy, or similitude of Ratios, each consisting of two Terms (Def. V.) But, since the Consequent of any two may be the Antecedent to another, three Terms may constitute Proportionals; in which case, it is necessary that they are all of the same species or kind.

But, when there are four Terms, thus, A:B::C:D, discretely, the first Pair may be of a different species from the other; for, there is the same Ratio between a Foot and an Inch, two, three, or four Inches, &c. as between a Pound, and one, two, three, or four Ounces (Troy Weight); consequently, there may be Analogy of Ratios between heterogeneous Quantities.

Def. IX. MEAN PROPORTIONAL and THIRD PROPORTIONAL.

When there are but three Terms, which are Proportionals, as A, B, C, they are in continued Proportion; thus, if A is to B, as B to C, in equal Ratio; then, the middle Term, B, is a Mean, between the other two, and is, therefore, called a MEAN PROPORTIONAL.

The firſt and laſt Terms are the Extremes; either of which (A or C) when there are but three, is, in reſpect of the other two, called a THIRD PROPORTIONAL.

Hence it is evident, that, in continued Proportion, of four or more Quantities, each Term, except the firſt and the laſt, is a Mean, between its Antecedent and Conſequent.

N. B. If two Terms or Quantities are given, and a third be required, in the ſame Ratio to either of the given Terms as that has to the other, the Quantity found is a third Proportional.

Def. X. EXTREME and MEAN PROPORTION is when any Quantity is ſo divided, into two Parts, that, the leſs, the greater, and the whole, are in continued Ratio.

A Right line is ſaid to be divided in *extreme* and *mean* Proportion; when, being cut into two unequal Parts, the Ratio of the whole Line, to the greater Segment, is the ſame as of the greater Segment to the leſs.

If the Line AB be cut in C. A C B

Then, if the whole Line, AB, is to AC as AC is to CB; AB is divided in extreme and mean Proportion.

Def. XI. A FOURTH PROPORTIONAL.

If four Terms, A, B, C, and D, are Proportionals; whether they are diſcretely analogous, only, or in continued Proportion, either of the Extremes, A or D, is a fourth Proportional, in reſpect of the other three.

N. B. If three Terms or Quantities are given, and a fourth is required, in the ſame Ratio, to any one of the given Quantities, as the remaining two have, one to the other; the Quantity determined, is a FOURTH PROPORTIONAL.

Def. XII. DUPLICATE and TRIPLICATE RATIO.

If four, or more Terms, A, B, C, D, are in continual Proportion; the firſt, A, is ſaid to have, to the third, C, a *duplicate* Ratio of A to B, the firſt to the ſecond; i. e. as the Square of A to the Square of B.

And the Proportion of A to D, the firſt to the fourth, is *triplicate* of A to B; that is, as the Cube of A to the Cube of B; and ſo on to any higher Power.

Def. XIII. ALTERNATE RATIO is the mutation of the two middle Terms, of four proportional Quantities; * i. e. when Antecedent is compared with Antecedent, and Conſequent with Conſequent; and they are called homologous, corational, or correſponding Quantities.

Thus, A:B::C:D, directly, — 27:18::6:4.
Then, A:C::B:D, alternately — So, 27:6::18:4.

N. B. When the Product, ariſing from the multiplication of the two middle Terms, is equal to that of the two Extremes, it is a certain proof of Analogy; conſequently, it will be the ſame whether they are taken directly or alternately.

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* Alternate Ratio cannot exiſt or take place, unleſs both Ratios are between Quantities of the ſame kind; ſeeing there is no Ratio, or compariſon between Quantities that are heterogeneous. (ſee Def. IV.) For, if four Quantities are in analogous Ratio, (A:B::C:D,) it is not neceſſary that the firſt pair (A and B,) ſhould be of the ſame kind as the laſt (C and D.) But, in Alternate Ratio, the Antecedent of one Ratio is compared with the Antecedent of the other, and Conſequent with Conſequent; wherefore, it is abſolutely neceſſary that the Quantities, in both, are of the ſame ſpecies.

N. B. In the ſeven following Definitions, it is the ſame whether they are of the ſame kind or not; i. e. each pair may be of a different kind, heterogeneous to the other.

Def. XIV. Inverse Ratio is when the Conſequent of any Ratio is taken for the Antecedent, and compared with the Antecedent as a Conſequent.

In this Caſe, the middle Terms become the Extremes.

If A:B::C:D, directly.— 27:18::6:4.
Then, B:A::D:C, inverſely.—So, 18:27::4:6.

Def. XV. Compound Ratio is when the Antecedent and Conſequent, being taken or added together, as one Quantity, are compared either with the Antecedent or with the Conſequent.

If A:B::C:D; then, A+B:A, or B, ::C+D:C, or D.
6;9::2:3; then, 15:6, or 9, ::5:2, or 3.
Converſely; A or B:A+B::C or D:C+D.

Def. XVI Divided Ratio is when the Difference, between the Antecedent and the Conſequent, is compared either with the Antecedent or with the Conſequent.

If A:B::C:D. Then whether A be greater than B, or B than A, the difference, being taken, is compared either with A or B; as alſo, the difference between C and D is compared with C or D.

Thus, A−B, or B−A,:A, or B, ::C−D, or D−C,:C or D
15:6::5:2, then, the difference, 9,:6, or 15,::3:2, or 5.
Converſely; A or B:A−B, or B−A::C or D:C−D, or D−C.

Converse Ratio, according to Euclid, is, only, when the Antecedent is compared with the difference between the Antecedent and the Conſequent. But as it is, in reſpect of divided Ratio, no other, in reality, than ſimple Inverſe Ratio (Def. 14.) I do not conceive it neceſſary.

Def. XVII. Mixed Ratio is when the Antecedent added to the Conſequent, as one Quantity, is compared with the Exceſs, by which the one exceeds the other;

And, *vice verſa*, when the Difference, between them, is compared with the Sum of both.

If A:B::C:D; then, A+B:A−B, or B−A, ::C+D :C−D, or D−C. 7.3::21:9; then, 10:4::30:12; the Antecedent being the greater Quantity.
Converſely; A−B or B−A:A+B::C−D or D−C:C+D.

Def. XVIII. RATIO OF EQUALITY * is when there are two Series or Ranks of Quantities, in which, being taken (two and two) in a certain order, in each Rank, there is always Analogy.

Or, when there are four Quantities, in analogous Ratio, either diſcretely or continual; and other Quantities are added, after the ſame manner, and in equal Ratio.

RATIO of EQUALITY is either Ordinate or Inordinate.

ORDINATE RATIO of EQUALITY is when there are three or more Quantities in one Rank or Order, in any Ratio whatever, and as many in another Rank, in the ſame Ratio, comparing two and two; i. e. as the firſt is to the ſecond, in one Rank, ſo is the firſt to the ſecond, in the other; and, as the ſecond is to the third, ſo is the ſecond to the third, &c.

In the two Ranks A, B, C, D, and E, F, G, H;
If A:B::E:F, and B:C::F:G; alſo, as C:D::G:H.
They are then in ordinate Ratio of Equality.

Let the firſt Rank be as 3:7:5:8; and let E be 9
Then, F will be 21, G will be 15, and H 24.
For, as 3:7::9:21; and as 7:5::21:15; alſo, as 5:8::15:24.
Then 3:5, or 8; as 9:15, or 24; and as 3:9::5:15, or 8:24.

* By RATIO of EQUALITY is not meant equality of RATIOS, ſimply; but, in two Ranks of Quantities, whoſe Ratios are ordinate or inordinate, they are equal, at equal Diſtances from the Extremes.

Or when there are, at firſt, but two in each Rank, which are analogous, and other Quantities are added on both ſides, ordinately, in the ſame Ratio.

Firſt; let the Quantities added be Antecedents. If A:B::C:D, then, let E be to A as F to C; or, let them both be Conſequents, viz. as B is to E ſo is D to F; in either Caſe, the Ratio is ſtill the ſame, being taken progreſſively.

Inordinate Ratio of Equality is, when in two Ranks, of three or more, A, B, C, and D, E, F; as A is to B, ſo is E to F, and as B is to C, ſo is D to E.

Or, when in equal Ratios of two Quantities each; as the Conſequent of one Pair is to ſome other Quantity, ſo is another Quantity to the Antecedent of the other.

If A:B::C:D; let B:E::F:C; then, A:E::F:D.
3:4::21:28; and, 4:14::6:21; then, 3:14::6:28.

Def. XIX. Compoſition of Ratios.*

In any number of Quantities, and in any Order whatever, the Ratio of the firſt to the laſt may be conſidered as a Compoſition of all the intermediate Ratios.

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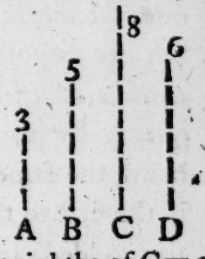
* This Definition is, in general, thought obſcure; 'tis not well underſtood, nor clearly defined. Indeed, I think it is not defined at all, as it is uſually put, but is rather a Theorem than a Definition; viz. that the Ratio of the firſt to the laſt is equal to that which is compounded of all the intermediate Ratios; of which, a Demonſtration (in Numbers) may eaſily be given.

Let A, B, C, and D be in the Proportions, as 3, 5, 8, and 6; the Ratio which is compounded of them all is as A to D, 3 to 6; i. e. as 1 to 2, or half.



Dem. For, C:D::8:6; i. e. C is equal to 8 ſixths of D, and, B:C::5:8; i. e. B is equal to 5 eighths of C. - - - - - Ax. 9.

But, an 8th part of C = a 6th part of D; conſ. 5 eighths of C = 5 ſixths of D. Again; A is 3 fifths of B. But, B is 5 eighths of C; i. e. 5 ſixths of D. Therefore, A is 3 ſixths of D; that is, half D.

If A, B, C, and D, represent as many Quantities; whatever is the Ratio of A to B, of B to C, and of C to D; the Ratio of A to D, is that which is compounded of them all. Thus, if A is to B as 3 to 5, B to C as 5 to 8, and C to D as 8 to 6.

Then, A : D :: 3 : 6; compounded of $\frac{A}{B}\frac{3}{5}$ of $\frac{B}{C}\frac{5}{8}$ of $\frac{C}{D}\frac{8}{6}$

i. e. A is 3 fifths of 5 eighths, of 8 sixths of D; that is, 3 sixths, or half D.

Compoſition of Ratios is generally given in the ſixth Book, for which I cannot conceive a reaſon. Dr. Barrow, indeed, gives nearly the ſame in both. In Def. 20. B. 5. he ſays, "any number of Magnitudes being put; the proportion of the firſt to the laſt, is compounded out of the proportion of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, &c. till the proportion ariſe."

The proportion which is between the firſt and the laſt muſt neceſſarily ariſe, by the compoſition of the intermediate Ratios; nor is it poſſible to do otherwiſe, ſeeing that, each of the intermediate Quantities is both Antecedent and Conſequent. Therefore, the Ratio, emerging at laſt, is the Ratio of the firſt to the laſt, let the intermediate Quantities be ever ſo many, and in what Proportion ſoever; which is made obvious, by multiplying all the Antecedents into one another ($3 \times 5 \times 8 = 120$) and dividing the Sum, by that of all the Conſequents ($5 \times 8 \times 6 = 240$); then is A to D as 120 to 240, or as 3 to 6; i. e. as 1 to 2, or $\frac{1}{2}$, half.

ACCORDING to Euclid, there are ſix various ways of reaſoning, from Proportion, including ordinate and inordinate Ratio in that of Equality; which, in effect, is all one. Inordinate Ratio differs from ordinate, in nothing but the diſpoſal of the laſt Quantity; for, whether it be conſidered and compared as an Antecedent or as a Conſequent, by placing it firſt or laſt, the Ratio, in either caſe, being the ſame, it will ſtill be as the firſt to the laſt, ſo is the firſt to the laſt in both Ranks; but not, one Quantity (added) to the other, as in ordinate Equality, where the Quantities, added, are both Antecedents, or both Conſequents.

But there is another way of reaſoning by Proportion diſtinct from the reſt, which is not in Euclid; viz. mixed Ratio (Def. XVII.); that is, comparing the Sum of the Antecedent and Conſequent with their Difference.

In reſpect of Converſe Ratio, it differs in nothing from Inverſe, which includes all Converſe Ratios whatever; ſince, all Ratios, that are analogous, are inverſely ſo. Fro A + or − B, or B − A, &c. may be conſidered as a ſimple Quantity, which has the ſame Ratio to A, or B, as C + or − D, or D − C, has to C, or D; by Inverſion, A is compared with the Sum or Difference between A and B, and C with the Sum or Difference between C and D, as if they were ſimple Quantities.

That the manner, and the order, of the ſeveral changes may be ſeen at one view, I have given them over again; with the Converſe of each, in the following abſtract; keeping the ſame Ratio throughout the whole. In which five Caſes, viz. alternating, inverting, compounding, dividing and mixing, with their Converſe, together with Equality, ordinate and inordinate, is contained all the variety in which the Terms, of analogous Ratios, may be changed and ſtill be analogous; at leaſt, which anſwer every purpoſe, of reaſoning from Proportion.

If A : B :: C : D, directly, as 3 to 2; thus, 21 : 14 :: 9 : 6.
Then, A : C :: B : D, alternately. - - - 21 : 9 :: 14 : 6.
Alſo, B : A :: D : C, inverſely. - - - - 14 : 21 :: 6 : 9.
And, A + B : A, or B, :: C + D : C, or D. } by Compoſition
35 : 21, or 14, :: 15 : 9, or 6. }
Alſo; A, or B, : A + B :: C, or D, : C + D. } converſely.
21, or 14, : 35 :: 9, or 6, : 15 }

And, A−B : A, or B, :: C−D : C, or D. } by Diviſion.
7 : 21, or 14, :: 3 : 9, or 6, }

Alſo; A, or B, : A−B :: C, or D : C−D. } converſely.
21, or 14 : 7 :: 9, or 6 : 3 }

Again, A+B : A−B :: C+D : C−D, mixing; 35 : 7 :: 15 : 3.
or, A−B : A+B :: C−D : C+D, converſely, 7 : 35 :: 3 : 15

Ordinate Ratio of equality.

If A : B :: C : D; and, if B : E :: D : F; then, A : E :: C : F.
21 : 14 :: 9 : 6; and, 14 : 42 :: 6 : 18; then, 21 : 42 :: 9 : 18.

Inordinate Ratio of equality.

If A : B :: C : D; and, if B : E :: F : C; then, A : E :: F : D.
21 : 14 :: 9 : 6; and, 14 : 42 :: 3 : 9; then, 21 : 42 :: 3 : 6.

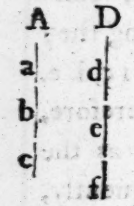
THUS, having ſhewn all the variety of changes, and, that the Ratios emerging from each, in Numbers, are equal or analogous, there remains, one might reaſonably ſuppoſe, nothing more to be done. For, after this knowledge obtained, of what uſe can it be to perplex the Student, with a tedious Demonſtration of what, in Numbers, is clear and manifeſt; and which, I am perſuaded, needs no further Demonſtration. Nor do I ſee, excepting Alternate Ratio (which is, alſo, ſufficiently evident, in Numbers) that any of the other require Demonſtration; or are made more manifeſt by it. For, the Ratios of all commenſurable Quantities may be expreſſed by Numbers; and it is ſufficiently manifeſt, that Analogy muſt hold, equally, in reſpect of Quantities which are incommenſurable.

It is evident, that, ſince A : B :: C : D, whether the Ratio of A to B be commenſurable or incommenſurable, the Ratio of C to D being the ſame, as A to B, they will ſtill be analogous in every change.

E. g. Suppoſe the Ratio of A to B and of C to D be as the Side of a Square to the Diagonal; or, as the Diameter of a Circle to its Circumference, &c. Now, it is manifeſt, that the Side of one Square is to its Diagonal, as the Side of any other Square is to *its* Diagonal; conſequently, their Ratios are equal, notwithſtanding they cannot be expreſſed in Numbers; for as 3 : 5 :: 9 : 15, i. e. 3 : 5 :: 3 : 5; conſequently, 3 : 3 :: 5 : 5, &c. Therefore, ſince the Side of one Square, is to its Diagonal, as the Side of another Square to its Diagonal; conſequently, as Side is to Side, ſo is Diagonal to Diagonal; alſo, as the Side added to the Diagonal, is to either, Side or Diagonal; ſo is any other Side added to its Diagonal, to either; likewiſe as one Diagonal is to the Side, ſo is the other Diagonal to the Side; or, as the Diagonal, leſs the Side, to either, ſo is the Diagonal, leſs the Side, to either; and thus it muſt ever be through all the variety of Changes that can be, if they are ordered and changed after the ſame manner, on both Sides.

In what I have advanced, on the ſubject of Proportion, is contained the Eſſence of the whole fifth Book of Euclid; if it be clearly underſtood, the Theory of Proportion is already acquired; and conſequently, all that can be ſaid more of it, or done, by way of Demonſtration, is, in a great meaſure, ſuperfluous. However, not to leave the whole Doctrine of Proportion without a Foundation, leſt it ſhould be cenſured by the ſcrupulous, I have demonſtrated, or briefly illuſtrated, the eſſentials, on which it depends; ſome of the Propoſitions I have made Axioms; for, being ſelf-evident, they require no Demonſtration.

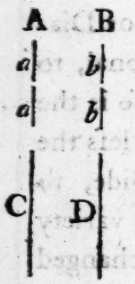
AXIOMS, or ſelf-evident PROPOSITIONS.



Ax. I. One QUANTITY is to any other Quantity, as all the PARTS of one, is to all the PARTS of the other.

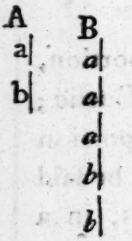
Let A and D be two Quantities, and, let A be divided, any how, into Parts, a, b, c; alſo let D be divided any how, in d, e, and f.

Then, A : D :: a+b+c : d+e+f.



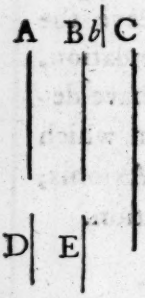
II. EQUIMULTIPLES,* or equal PARTS, of equal Quantities are equal.

Let A and B be equal Quantities, then, if C and D be equal twice or thrice, &c. of A and B, reſpectively; 2A=2B, or 3A=3B, i. e. C=D. Alſo, if A and B be divided into two or more equal Parts, *a a*, *b b*, &c; *a*=*b* or ½A=½B, &c.



III. Any MULTIPLE of the whole is equal to the ſame Multiple of all its Parts, taken together.

Let B be taken any Multiple of A; then, whatever Multiple B is of A, it is manifeſt, that it contains the ſame Multiples of a and b, Parts of A. As *a*, *a*, *a*, 3 a, and *b*, *b*, *b*, 3 b.



IV. The ſeventh Propoſition of Euclid.

Equal QUANTITIES have the ſame Ratio to any third Quantity, or to equal Quantities.

For, if A and B be equal Quantities, they have the ſame Ratio to a third Quantity, C, or to equal Quantities, D and E; ſeeing that, either may be taken for the other.

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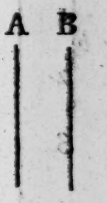
* By EQUIMULTIPLES is meant, that the Quantities are taken, or multiplied, an equal number of times.

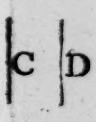
V. The ninth Propoſition of Euclid.

QUANTITIES that have an equal Ratio to the ſame Quantity, or to equal Quantities, are equal.

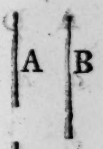
If A and B have the ſame Ratio to C, or D; A and B are equal. Alſo, if A has the ſame Ratio to C, as B to D; if C=D, A=B.



VI. The Tenth Propoſition of Euclid.

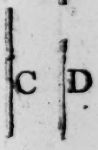


That QUANTITY which has the greater Ratio to a third Quantity, or to equal Quantities, is the greater Quantity. And, of two Quantities, that, to which a third Quantity has a greater Ratio, is the leſſer Quantity.



If C has a greater Ratio to B, than A has to B, C is greater than A.

Alſo, if A has a greater Ratio to D than to C, D is leſs than C.



VII. QUANTITIES have an equal Ratio to their Equimultiples; or to their equal Parts.

Let A and B be two Quantities, in any Ratio whatever; and let them be taken an equal number of times. Or, let them be divided into the ſame number of equal Parts; the wholes of A and B are equimultiples of thoſe Parts.

A: 2A or 3A, :: B: 2B or 3B. i. e. 1:2 or 3:: 1:2 or 3.
Or, as $\frac{1}{2}$A, or $\frac{1}{3}$A, : A :: $\frac{1}{2}$B, or $\frac{1}{3}$B, : B.
Alſo, A : A *a* ($\frac{1}{2}$A), or to A 1, :: B : B *b* ($\frac{1}{2}$B), or B 1.

VIII. The fifteenth Propoſition of Euclid.

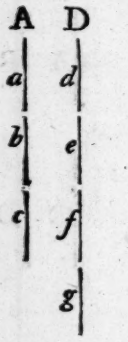
QUANTITIES are in the ſame Ratio, to each other, as their Equimultiples; or, as their equal Parts.

Let A and B be any Quantities, divided into equal Parts, as before, at *a* and *b*, or 1, 2, 3; alſo, let C and D be any equimultiples of A and B.

Then, A:B::2A:2B, or as 3A:3B; i. e. as C:D.

For C and D are Equimultiples of A and B; wherefore, as often as A contains B, or is contained in B; ſo often 2 A contains 2 B, or is contained in 2 B; i. e. as C contains D, or is contained in D; conſequently, A:B::C:D.

Alſo, as A:B::A*a* ($\frac{1}{2}$A):B*b* ($\frac{1}{2}$B)
or, as A1 to B1. ($\frac{1}{3}$ of A to $\frac{1}{3}$ of B.)



IX. Two QUANTITIES being divided into Parts of equal magnitude (conſequently equal amongſt themſelves); either Quantity is in that Ratio to the other, as the number of Parts, in one, to the number of Parts in the other.

Let A and D be two Quantities, divided into equal Parts, *a*, *b*, and *c*; *d*, *e*, *f*, and *g*;

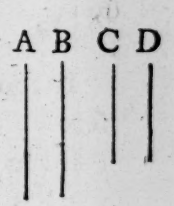
Then, A:D::3:4; or, as 6 to 8.
i. e. as $a+b+c$ is to $d+e+f+g$.

X. QUANTITIES being proportional, any one to another, as a third is to a fourth, &c; if the firſt be any Multiple or equal Part of the ſecond, the third is an equimultiple or equal part of the fourth.

For, Quantities are to each other, as the number of Parts in one, to the number of Parts in the other. - - - - - by the 9th.

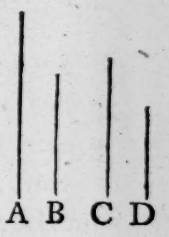
Therefore, if A:B::C:D; A contains, or is contained in B, as often as C contains, or is contained in D. - - - - - Def. 6.
Otherwiſe, the Ratios are not equal, or analogous.

AXIOM XI. In four proportional QUANTITIES, i. e. when any one is to another, as a third is to the fourth; (Def. 6.) whether the firſt be equal to, greater, or leſs than the ſecond, the third is alſo equal to, greater, or leſs than the fourth.



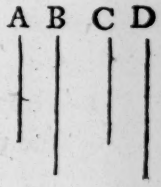
If A : B :: C : D; then, if A be equal to B, C = D; and, if A be greater, or leſs than B; C is, alſo, greater, or leſs than D.

For, A has the ſame Ratio to B, as C to D; and, a greater Quantity cannot have the ſame Ratio to a leſs, as a leſs to a greater. - Ax. 6.



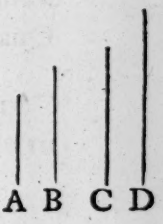
XII. The fourteenth Propoſition of Euclid.

IN four QUANTITIES which are PROPORTIONALS, if the Antecedent of one Ratio be greater than the Antecedent of the other, the Conſequent of the firſt is alſo greater than the Conſequent of the other; if the firſt are equal, the other are alſo equal; and if leſs, the other is leſs.



For, if A : B :: C : D; and, if A be equal to, greater, or leſs, than B, C is alſo equal to, greater, or leſs than D, by the foregoing. - - 11.

Wh. if A be equal to C, B is equal to D; - 5. and conſequently, if A be greater or leſs than C, B is, neceſſarily, greater or leſs than D.



XIII. The eleventh Propoſition, of Euclid.

RATIOS that are equal to the ſame Ratio, or to equal Ratios, are equal between themſelves.

This follows from the third Axiom of Book 1ſt, by ſubſtituting Ratios for Quantities.

XIV. The thirteenth Propoſition of Euclid.

IF four QUANTITIES are proportional, the firſt to the ſecond, as the third to the fourth; and if the Ratio of the third to the fourth, be greater or leſs than a fifth Quantity has to a ſixth; the Ratio of the firſt to the ſecond, is alſo greater or leſs than the fifth to the ſixth.

IF A:B::C:D; and, if the Ratio of C to D be either greater or leſs than E to F; the Ratio of A to B is alſo greater or leſs, than E to F.

For, the Ratio of A to B is equal to the Ratio of C to D; therefore, their Ratios are the ſame, in reſpect to any other.

POSTULATES.

1. Grant, that equal RATIOS may be taken, one for the other.

2. Grant, that any QUANTITY may be divided into any number of Parts, equal to one another; or as any other Quantity is divided.

3. Grant, that a QUANTITY may be taken, or aſſumed, in any Ratio to any given Quantity.

THE aſſumption of this laſt Poſtulate is not allowed by ſome Geometers; notwithſtanding, to me, it appears full as poſſible, as that equimultiples, or equal parts of Quantities may be taken; by one or other of which Suppoſitions, they demonſtrate the whole fifth Book.

THEOREM I. 1. Euclid.

IF Quantities, which are Equimultiples or equal Parts of other Quantities, be added into one Sum, or Quantity; the ſame Multiple or Part, which one Quantity is of the other, reſpectively, the whole is of the whole.

Let A, B, and C be Equimultiples, of D, E, and F, reſpectively; i. e. whatever multiple A is of D, let B be the ſame multiple of E, and C of F.

Then, A, B, and C, added together, into one Sum, is the ſame multiple of D, E, and F, added together, as A is of D, B of E, or C of F.

DEM. For, let A be equal to 2D, or 3D; then, B = 2E, or 3E, &c. conſ. A + B + C = D + E + F, taken twice, or thrice, &c. But, Quantities are to each other, as the number of Parts, in one, is to the number of equal Parts, in the other; Ax. 9. wh. as A : D, B : E, or C to F, :: A + B + C to D + E + F. Conſequently, as often as A contains D, ſo often the whole of A + B + C contains D + E + F. Therefore, &c.

If A, B, and C be Multiples of D, E, and F; the latter are, conſequently, equal Parts of the former. And therefore, as often as D is contained in A, ſo often is D + E + F contained in A + B + C.

THEOREM II. 12. Euclid.

IF any number of Quantities are Proportionals (i. e. being taken two and two, their Ratios are analogous); then, as any one of the Antecedents is to its Conſequent, ſo is the Sum of all the Antecedents to the Sum of all the Conſequents.

A——— Let A, B, C, D, E, and F be proportional Quantities; of which let A be to
B——— B, as C is to D, and as E to F.
C———
D——— I ſay, as A is to B, or C to D;
E——— ſo is A, C, and E, added together,
F——— to B, D, and F, together.

DEM. Now, A : B :: C : D, and, as E : F, therefore, their Ratios are equal. - - - - - - Def. 6.
Wherefore, if A = B, C = D, and E = F - - Ax. 11.
conſequently, A + C + E = B + D + F. - - Ax. 6. 1.
Therefore, as A : B, or as C : D, :: A + C + E : B + D + F.

Again; if A be any Multiple, or equal Part, of B; then, C is an Equimultiple, or equal Part, of D, & E of F - Ax. 10.
wherefore, A + C + E is the ſame Multiple, or equal Part, of B + D + F, as A is of B, as C of D, and as E of F - Th. 1.

But, Quantities have the ſame Ratio, to each other, as their Equimultiples, or as their equal Parts. - - Ax. 8.
Therefere, as A : B, or C : D, :: A + C + E : B + D + F.

Now, becauſe there are ſuch Quantities, or Ratios of Quantities, which cannot be Equimultiples or equal Parts, one of the other, this Demonſtration is not poſitive, in reſpect of ſuch Quantities; and, notwithſtanding they may be divided into the ſame number of equal Parts, it is no eaſy matter to give perfect Demonſtration, ſeeing that, equal Parts are in the ſame Ratio to each other as the wholes. Ax. 8th.

Suppoſe A to B, and C to D, &c. are incommenſurable, but in the ſame Ratio; ſtill, A + C + E : B + D + F, &c. :: A : B, &c. becauſe, A might either be equal to C, or E, or it may be ſome Multiple or equal Part of C, and C of E, &c.

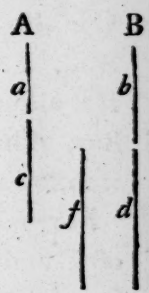
Or, ſuppoſe A be incommenſurable to C, and C to E; provided A and B, &c. be commenſurable, the ſame will hold true, as in the Theorem. But, being incommenſurable both ways, direct and alternate, equality can never be produced in either; and the Demonſtration given (or rather preſumed) from a greater or leſs Ratio, ſimply, is by no means poſitive; ſeeing, it cannot be aſcertained in what degree they are ſo.

For, A may have to B, and C to D, &c. the Ratio of the Side of a Square to its Diagonal, or as the Diagonal to the Side; and A to C may alſo be either; or, as the Diameter of a Circle to the Circumference, and C to E the ſame; or, in extreme and mean Proportion.

Yet 'tis eaſy to conceive, from what has been proved, that (let the Ratio of A to B be what it may) ſince the Ratio of C to D, and of E to F, is the ſame as A to B, conſequently, A+C:B+D, or A+C+E:B+D+F::A:B, or C to D; and although it is not eaſy to give direct proof, in ſuch Caſe, yet we may ſafely conclude that it is ſo; and, on that preſumption, all which follow will be found poſitive, let the Ratio be what it may.

THEOREM III. 19. Euclid.

IN two Quantities, if a Part taken from one, be in the ſame Ratio to a Part taken from the other, as one Quantity is to the other; the Remainders of thoſe Quantities are alſo in the ſame Ratio, as one Quantity is to the other.



Let A and B be two Quantities, in any Ratio whatever, and, ſuppoſe any Part of A be taken (as *a*) from A.

Let *b* be taken, from B, in the ſame proportion to *a*, as the whole of B is to A.

I ſay, the remaining Part, *c*, is to *d*, as A to B.

Let any other Quantity *(f)* be ſo taken, as *a*:*b* ſo is *c*:*f*. Poſt. 3.

DEM. Now, *a*:*b*::*c*:*f*; and *a*:*b*::A:B; conſ. *c*:*f*::A:B. - Ax. 13

But, A:B::*a*+*c*:*b*+*d* (Ax. 1.); and, as *a*:*b*::*c*:*f*; - Poſt.

wherefore, *a*+*c*:*b*+*f*::*a*:*b*; (i. e. as A:B.) - Th. 2.

conſ. *a*+*c*:*b*+*f*::*a*+*c*:*b*+*d*; wh. *b*+*f*=*b*+*d*. - Ax. 5.

And by taking away *b*, which is common, *f*=*d*. - Ax. 7. 1.

But, *c*:*f*::*a*:*b* (Poſt.); conſequently, *c*:*d*::*a*:*b*. - Ax. 4.

And, *a*:*b*::A:B (Hyp.); therefore, *c*:*d*::A:B. - Ax. 13.

COR. If one Quantity be to another as a third is to a fourth; and, if a fifth be to a sixth, as the third to the fourth; then, the first and fifth, added together, has the same Ratio to the second and sixth, together, as the third has to the fourth. This follows from either of the last.

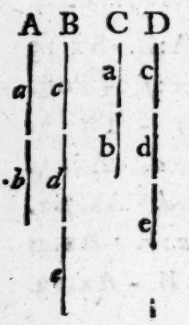
THEOREM IV. 16. Euclid.

IF four Quantities, of the same kind, are Proportionals; which, in a certain order, aredirectly proportional, and analogous in their Ratio; they are also analogous when taken alternately.

If A is to B, as C is to D, directly;
then, A is to C, as B is to D, alternately.

DEM. Now, since A is to B, as C to D; then,
whether A be equal to, greater or less than B;
C is also equal to, greater or less than D. - - Ax. 11.
And, if A be equal to, greater, or less than C;
B is also equal to, greater, or less than D. - - - 12.
But, A has the same Ratio to B as C to D; - - - Hyp.
it follows then, that A has the same to C, as B to D.

For, whether A and B be Equimultiples, or equal Parts of C and D; A is in the same Ratio to C, as B to D; seeing that, A is to B as C to D. - - - Ax. 7 and 8.



Suppose the Ratio of A to B be as 2 to 3; and, imagine them divided into equal Parts, in *a* and *b*; *c*, *d*, and *e*.

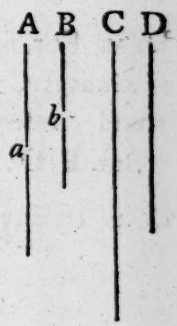
Also, suppose C and D divided into the same number of equal Parts, respectively, in a and b; c, d, and e.

Then, since A : B :: 2 : 3, & A : B :: C : D; C : D :: 2 : 3 - Ax. 13
wherefore, the Parts, a and b, (of C) and c, d, e, (of D)
are all equal amongst themselves; - - Ax. 9.
consequently, as *a* : a, or *b* : b, :: *c* : c, *d* : d, or *e* : e. - Ax. 4.

But, *a* and *b* are equal parts of A; and, a and b, of C.
Also, *c*, *d*, & *e*. are equal parts of B; and, c, d, and e, of D.
But, Quantities are, to each other, in the same Ratio as
their Equimultiples, or equal Parts. - - - - Ax. 8.
Therefore, as A is to C, so is B to D. Q. E. D.

Again. Suppose the Ratio of A to B, and of C to D, be incommensurable; and consequently, cannot be divided into Parts, which are all equal amongst themselves.

Then, let A and B be divided into the same number of equal Parts; suppose they are bisected, in *a* and *b*. - - Post. 2.



Now, A : B :: C : D; and A : B :: *a* : *b*, - - - Ax. 8.
consequently, C : D :: *a* : *b*. - - - Ax. 13.
But, A : *a* :: B : *b* (Ax. 7.); and, as *a* : *b* :: C : D. - as above.
Wherefore, by substituting one for the other, - Post. 1.
it will be as A is to C, so is B to D. Q. E. D.

THEOREM V.

Quantities which are analogous, when taken directly, are also analogous, taken inversely.

If A is to B, as C is to D; then, B is to A, as D to C.

Dem. For, A : B :: C : D; then, A : C :: B : D, - by 4.
i. e. B : D :: A : C; then, B : A :: D : C. - same.

THEOREM VI. 18. Euclid.

Quantities which are analogous, taken directly, are alſo analogous, when compounded.

If A is to B as C is to D.
Then A added to B, is to A or B; as C added to D, is to C or D.

DEM. For, ſince A : B :: C : D; then, A : C :: B : D; Th. 4.
wh. as A : C :: A + B : C + D (2); conſ. A : A + B :: C : C + D;
and conſequently, it is as A + B : A :: C + D : C. Th. 5.
But, B : D :: A : C; therefore, A + B : B :: C + D : D-Ax. 13.

COR. By inverſion, A, or B, : A + B :: C, or D, : C + D.

THEOREM VII. 17. Euclid.

Quantities which are analogous, directly, are alſo analogous when divided.

If, in four Quantities, A, B, C, and D, A is to B, as C to D;
then, according as A or B is the greater Quantity,
A leſs B (or B leſs A) is to A, or B,
as C leſs D (or D leſs C) is to C, or D.

DEM. Now, A : B :: C : D (Hyp.) wherefore, A : C :: B : D-4.
But, as A : C, (or B : D) :: A − B (or B − A) : C − D (or D − C) 3.
conſ. A (or B) : A − B :: C (or D) : C − D; - - 4.
and, conſ. A − B : A (or B) :: C − D : C (or D). - 5.
Or; - - B − A : A, or B, :: D − C : C, or D.

The Converſe of this is firſt proved, in the third ſtep.

THEOREM VIII.

If four Quantities are Proportionals, in a certain order, they are alſo proportional when mixed; i. e. their Sums and Differences are analogous.

If A is to B as C to D.

Then, A+B : A−B (or B−A) :: C+D : C−D (or D−C)

DEM. Now, A : B :: C : D (Hyp.); then A : C :: B : D - Th. 4.

Wherefore, as A : C :: A+B : C+D; - - - - - - 2.

alſo, as A : C :: A−B : C−D. - - - - - - 3.

Conſequently, A+B : C+D :: A−B : C−D; - Ax. 13.

therefore, A+B : A−B :: C+D : C−D. - Th. 4.

COR. A−B : A+B :: C−D : C+D, by inverſion. - 5.

THEOREM IX. 22 Euclid.

If there be three or more Quantities, and others equal to them in number; which, taken two and two in a certain order (ordinately, or progreſſively) are in the ſame Ratio; then will the firſt be to the laſt, as the firſt to the laſt, in each Order of Quantities.

Alſo, as the firſt is to the firſt, ſo is the laſt to the laſt.

If A : B :: C : D; and, if B : E :: D : F; A is to E as C to F.

DEM. Now, A : B :: C : D; wherefore, A : C :: B : D; } - 4.
and - B : E :: D : F; wherefore, B : D :: E : F. }

Now, ſince A : C :: B : D; and, as B : D :: E : F;

conſ. A : C :: E : F (Ax. 13). Therefore, A : E :: C : F. 4.

N. B. The laſt Part is proved firſt.

Again. If other Quantities, G and H, be added, in the ſame Ratio to the laſt Conſequents, reſpectively, as E is to G, ſo is F to H; it is manifeſt, that A is to G, as C to H; the firſt to the laſt, in equal Ratio, ſtill.

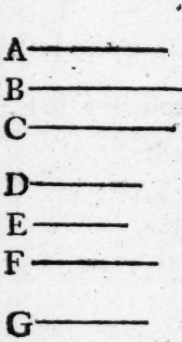
For, as A:E::C:F (Theo.) and, as E:G::F:H; by Hyp. wherefore, A:C::E:F; and, as E:F::G:H; by the 4th. Conſ. as A:C::G:H (Ax 13); th. A:G::C:H. - ſame.

And thus it will ever be, if Quantities be added (in equal Ratio) after the ſame Order, ad infinitum.

OR, if the Quantities, added are in equal Ratio to the Antecedents, reſpectively, thus;
if A:B::C:D, and, if E be to A as F to C; it is manifeſt, that as E is to B, ſo is F to D, as above.

THEOREM X. 23 Euclid.

In two Series or Ranks of Quantities, if there be three or more in each; which, taken two and two, inordinately, are in the ſame Ratio; then, as the firſt is to the laſt, in one Sett, ſo is the firſt to the laſt in the other, as by ordinate equality.



Let A, B, C, and D, E, F, be Quantities in ſuch order, that, as A is to B, ſo is E to F; and, as B is to C. ſo is D to E.

I ſay, that, as A is to C, ſo is D to F.

Let any other Quantity, G, be taken ſo, that, as D is to E, ſo is F to G; Poſt 3. conſ. as D is to F, ſo is E to G. Th. 4.

DEM. Then, as before, A:B::E:F, wh. A:E::B.F.
Alſo, as B:C::D:E, i. e. as F:G; wh. B.F::C:G. } 4
Now, A:E::B:F, & B:F::C:G; wh. A:E::C:G; Ax. 13
and conſequently, as A:C::E:G. - - - - Th. 4.
But, D:F::E:G (above). Th. as A:C::D:F. Ax. 13.

If other Quantities be added, inordinately, in equal Ratio, it will ſtill be as the firſt is to the laſt, in one Rank, ſo is the firſt to the laſt, in the other.

For, as A:C::D:F (above) and, as C is to any other (G) let H be to D; conſ. as A:G::H:F; as before.

RATIO of EQUALITY may be proved, demonſtratively (in Numbers) by Compoſition of Ratios.

Let A, B, C, and D, be in the ratios of 3, 5, 8, and 6; and let E, F, G, and H, have the ſame Ratios, inordinately; i. e. as A to B ſo is G to H; viz. as 3 to 5; and B to C as F to G (as 5 to 8); alſo C to D as E to F, as 8 to 6 (or 4 to 3); thus.

To avoid Fractions, let H be 40; then, G will be 24 (Equimultiples of A and B.) F will be 15, (the ſame multiple of B, as G of C); conſ. E will be 20; being as 4 to 3; E and F are Equimultiplies of them.

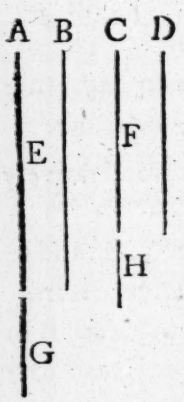
DEM. Then, E:F::20:15, F:G::15:34, and G:H::24:40 i. e. as 8 to 6, as 5 to 8, and as 3 to 5; inverſely, or inordinately, as A, B, C, and D. E is then 8 ſixths, of 5 eighths, of 3 fifths of H; equal, 3 fifths, of 5 eighths, of 8 ſixths of A to D. Conſequently, the Ratio of E to H (20 to 40) is the ſame as A to D, 3 to 6; i. e. half. Being thus proved, inordinately, it muſt hold, ordinately.

It is manifeſt, ſeeing that there are equal Ratios between the Extremes (let the intermediate Quantities be ever ſo many, and in what Ratio ſoever, provided there be an equal number in each Rank, and the Ratios equal, ordinately or inordinately) the Ratio compounded of them all is, as the firſt to the laſt, in each Rank.

But, the entreme Ratios are equal (ordinately or inordinately) conſequently, the extreme Qualities are in analogous Ratio.

THEOREM XI. 25 Euclid.

If four Quantities are Proportionals; the greateſt and the leaſt are, together, greater than the other two.



Let A, B, C, and D, be four Quantities, of which, let A be the greateſt; and let them be, as A to B, ſo is C to D. Conſequently, D is the leaſt Quantity.

I ſay, that A added to D is greater than B added to C.

For, if the firſt be greater than the ſecond, the third is greater than the fourth. Ax. 11. and, if the firſt be greater than the third, the ſecond is greater than the fourth. -12.

DEM. Becauſe A and C are greater Quantities than B and D; let E and F be taken from A and C, reſpectively, equal to B and D, reſpectively.

Now, A:B::C:D. But E=B, and F=D; - Con. wh. as A:E::C:F (Ax. 4); and, E+D=B+F. Ax. 6. 1. conſ. A−E (i. e. G):A::C−F (i. e. H):C; - Th. 7. i. e. G:A::H:C; wherefore, G:H::A:C. - Th. 4. But, A is greater than C; wh. G is greater than H; Ax. 11. conſ. G+E (i. e. A)+D, is greater than H+F (i. e. C)+B. Th. A+D is greater than C+B, the two middle Terms.

IN the laſt ten Theorems, and the Corollaries deducible from them, is contained all the various ways of reaſoning, by Proportion, which I conceive to be uſeful; two of which, the fifth and eighth, are not proved by Euclid. The eighth, indeed, is not mentioned, by him, or any of his Expounders, that I am acquainted with; but, I am ſomewhat ſurprized that

none have given a Demonſtration of Inverſe Ratio; which being proved, every Converſe Ratio is alſo proved. Profeſſor Simſon is aware of that deficiency, and has very judiciouſly introduced it, as an additional Propoſition (B); after which, I muſt think it needleſs to add the Converſe of divided Ratio (E); unleſs he had given the Converſe of all.

But what, to me, ſeems very extraordinary is, that Euclid ſhould not demonſtrate either; alſo, that he ſhould prove (if it may be ſaid to be proved) divided Ratio (the 17th) before compounded (the 18th) when the latter is, in ſome meaſure, made a Condition of the former; they are, indeed, Converſe, each of the other. In the 17th he ſays, "If Magnitudes, taken jointly or compounded, be proportional, they are alſo proportional taken ſeparately, or divided." Should it not have been proved then, previouſly, that they *are* proportional when compounded? By the tenor of theſe Premiſes, *if* one is, the other is, alſo; and conſequently, *if* one is not, the other is not; whereas, the *Condition* is preſumed, only, and therefore, there is no direct proof of the other. But, I ſee no reaſon why they ſhould be conſidered as being compounded, in order to prove that they are proportional when divided; ſeeing that, the conſequence, of their being divided, is the ſame, whether they are conſidered as ſimple Quantities, or compounded.

By Definitions 15th and 16th is to be underſtood, when Quantities are proportional (ſimply) they are alſo proportional compounded or divided; which, Profeſſor Simſon has expreſsly ſaid, in his Definitions of them; and which, I have demonſtrated in the 6th and 7th Theorems. Now, Euclid's 17th Propoſition ſays, as above; and *vice verſa* in the 18th, viz. that, if Quantities are Proportionals, when compounded or divided, they are ſo, divided or compounded; which, I cannot conceive to be the true ſenſe and meaning of it, but as I have ſtated them, in the Theorems, and have demonſtrated, on that Hypotheſis.

If what I have advanced, on the ſublime Doctrine of Proportion, be not ſufficient, for any purpoſe whatever, I will be bold to ſay, that 'tis not ſo in Euclid, or any of his Commentators. The Axioms, which I have given, are gathered from the moſt eaſy and ſimple ideas of Proportion; after the Definitions are digeſted and clearly underſtood, there is no Perſon would heſitate, one moment, to grant every Axiom.

That Quantities are in the ſame Ratio as their Equimultiples, or equal Parts (the 8th) is manifeſt to the meaneſt capacity; and this is one of the principal, on which Euclid has founded the whole Theory of Proportion; which amounts to no more than this; that one is to two, three, &c. as, two, is to four, ſix, &c; or, as three to ſix, nine, &c. i. e. as one, to two, three, &c. Or, that one Quantity is to any other Quantity, as the half, or third part, &c. of one, is to the half, or third part, &c. of the other.

The 1ſt Axiom, in the firſt Book of the Elements of Euclid (the 3rd of theſe Elements) expreſsly ſays; "*Things*, which are equal to the ſame *Thing*, are equal between themſelves." Now, the queſtion is, whether *Ratios* are *Things?* Certainly, if any thing be meant by Ratio, they muſt come within that Appellation; for, granting it to be, in itſelf, metaphyſical, yet, *ſomething* is underſtood and meant by Ratio, and conſequently, *Ratios* are *Things*. Then, the 11th Propoſition, of Euclid's fifth Book, is as much an Axiom, as the firſt Axiom of the firſt Book; and I have, accordingly, made it ſo, in the thirteenth. The remainder of the Axioms are as ſimple, obvious, and ſelf-evident; which, with the Poſtulates, being granted, all the Theorems, I am confident, will be found ſolidly, briefly, and clearly demonſtrated.

A critical Remark on Euclid's fifth Definition, of the fifth Book.

Euclid, in his fifth Definition, has given us a Criterion, whereby to determine Analogy of Ratios, rather than a ſimple Definition of what it is.

He ſays; "Magnitudes (by which is meant Quantities of any kind) are ſaid to be in the ſame Proportion, the firſt to the ſecond, and the third to the fourth, when the Equimultiples of the firſt and third, being compared with the Equimultiples of the ſecond and fourth, according to any multiplication; either together exceed the one the other, or together are equal the one to the other, or together are leſs, the one than the other." This is, literally, according to the old Tranſlation by H. Billinſley; in the Year 1570.

Dr. Barrow ſays, "Magnitudes are in the ſame Ratio, when the Equimultiples of the firſt and third, compared with the Equimultiples of the ſecond and fourth, according to any multiplication whatſoever, either both together are leſs than the ſecond and fourth both together, or equal taken together, or exceed one the other together, if thoſe be taken which anſwer one to the other."

Dr. Keil, from the Latin tranſlation of Commandine, has nearly the ſame words, viz. "the Equimultiples of the firſt and third (compared as above) are either both together, greater, equal, or leſs, than the Equimultiples of the ſecond and fourth; if thoſe be taken that anſwer each other."

Cunn did well to explain the meaning of Dr. Keil's Definition, which is literally this; that, Equimultiples of the firſt and third, taken or added together (i. e. into one Sum or Quantity) are either equal, greater, or leſs than the Equimultiples of the ſecond and fourth. I think, there is great occaſion for a Comment on Dr. Barrow's; which, to me, is quite unintelligible.

After all that has been ſaid, concerning this Definition, by the learned Doctor and others, either the original Text muſt be very obſcure, or the Expounders have render'd it ſo; for I declare, I cannot think the words bear ſenſe. But, giving the greateſt latitude poſſible to the meaning of the Words, they imply, the firſt and the third taken together, and compared with the ſecond and fourth, together (i. e. in one Sum); in which caſe, it amounts to a clear Axiom; for they muſt, certainly, be either equal, greater, or leſs, one than the other; after which, the words, "if thoſe be taken which anſwer the one to the other," mean nothing.

Mr. Stone ſays, "they are, together, either deficient, equal or exceed *each* other." They may, indeed, be equal, *each* to the other; but, it is impoſſible that they can, at the ſame time, be either greater or leſs, *each* than the other.

The word, *together*, ſeems, to me, ſuperfluous, or not expreſſive of what is meant; the tenor of it is, in this place, *at the ſame time*, or, *by the ſame multiplication*; which is rather foreign to the common acceptation of it. It is, perhaps, better to omit it, entirely, and ſupply its place with *both* (which Barrow and Keil have added) it is then plain; the firſt and the third, compared with the ſecond and fourth, will be either *both* greater, *both* equal or *both* leſs; which is the concluſion all have drawn from it; and then, the words, if thoſe be taken which anſwer each other, ſeem neceſſary.

Mr. Stone, in vindicating and illuſtrating this Definition, ſhews, that it is clearly deduced from the fourth and fourteenth Propoſitions. According to the Idea I have of the Definitions of Terms, in any Science, they ought to be defined by their ſimpleſt and moſt natural Properties; from which, the Demonſtrations of ſome Theorems are aſcertained; but it ſeems, to me, inconſiſtent, firſt to demonſtrate ſeveral preceding Propoſitions, by means of a certain Definition, previous to the demonſtration of others, from which the Definition, itſelf, is to be deduced, as a Corollary.

It is true, he ſays, "it is not ſo ſimple and plain as the Definition of Numbers, or that which might be given of commenſurable Magnitudes." And again; "Euclid could not have given any other, ſo elegant and general a Definition, that would have taken in incommenſurable Magnitudes, as well as Numbers, and commenſurable ones."

The Diagonal of a Square is not commenſurable with its Side; alſo, the Circumference of a Circle is incommenſurable by its Diameter, or Radius; that is, their Ratios cannot be expreſſed in Numbers; for, the Side of a Square cannot be divided into any number of Parts, though ever ſo ſmall, which will be an aliquot part of the Diagonal; i. e. they cannot both be divided into the ſame equal parts, or into parts of equal magnitude, without defect; ſuch Quantities are, therefore, ſaid to be incommenſurable.

The Ratio, between the Side of a Square, and its Diagonal is very different from that, between the Diameter of a Circle and its Circumference; yet, the Ratio between the Side of every Square and its Diagonal, or between the Diameter of every Circle and its Circumference, is the ſame; i. e. they are equal.

It is intimated above, and is ſtrongly ſupported by Br. Barrow and others, that this Definition comprehends or extends to Incommenſurables: I am, however, perſuaded, that 'tis not an eaſy matter to determine the analogy of ſuch Ratios by it. The Rule it preſcribes is ſimply this. Take any Equimultiples of the firſt and third Terms, in four Quantities, that are preſumed to be analogous in their Ratios; i. e. multiply the two antecedent Terms, equally, the ſame number of Times, and, either by the ſame Number or any other, multiply the two Conſequents, the third and fourth, or take Equimultiples of them; then, the firſt

and the third, being compared with the second and fourth, respectively, if the first exceeds the second, the third will also exceed the fourth, if one be equal, the other is also equal, or they will be both less, than the second and fourth; i. e. the two antecedent Terms will, at the same time, be either both equal, both greater, or both less than their respective Consequents; and, when this is the case, in all multiplications, *whatever*, we may then conclude that their Ratios are analogous.

Now, if the Ratios are of incommensurable Quantities, which cannot be expressed by Numbers, those Quantities cannot be multiplied by Numbers; and, in Lines, it would require greater accuracy than it is possible for any person to apply, to take multiples of them, till they produce Equality. It is certain, that, being multiplied, mentally, according to this Definition, if they are analogous, the Antecedents would be either greater or less than the Consequents, continually, but never equal; for if, by taking Multiples, they can be equal, they may, consequently, also be divided into equal Parts; but, it is well known they cannot, which, negative Quality, is the distinguishing characteristic of Incommensurables. And since it is manifest, that the Antecedents of Ratios, which are not analogous, may, by multiplying, be either greater or less than their respective Consequents, but never both equal, at the same time, i. e. by the same multiplication; how, then, is it possible to determine analogy of Ratios by this Criterion? seeing that, in Ratios of commensurable Quantities, which are not analogous, they may be greater or less, but never equal, as they will ever be in incommensurable Quantities that are analogous.

HENCE, it appears, to me, that to produce equality, of the Antecedent to the Consequent, by taking Multiples of them, is the only certain and infallible sign of Proportionality, or Analogy of Ratios between commensurable Quantities; but, since that Criterion fails, in Incommensurables, we cannot, by its means, positively determine Analogy, in such Quantities; unless, by trying *all multiplications, whatever*, they are, *at all times*, either *both* greater or *both* less. If the Antecedents are either both greater, or both less than their respective Consequents (and no Person would look for Proportionality otherwise) it is manifest, that, by multiplying the Antecedents any number of times, equally, and the Consequents the same, or any other, also equally, they must necessarily be, at the same time, either both greater or both less; but, if they are not analogous, they never can, at the same time, be both equal, i. e. each Antecedent to its respective Consequent; and therefore, no other Sign, but *Equality*, is necessary, or sufficient, to prove Analogy of Ratios.

Let A and B have the Ratio of the Side of a Square to the Diagonal, which is, nearly, as 12 to 17; or, nearer, as 1200 to 1697; it is not 6,10000th parts more. A being taken 126 times, B 89, produce 1512, and 1510,4, nearly. If the Decimal terminated,

it is manifeſt, they might be divided into Parts of the laſt denomination, though ever ſo ſmall; but it never will.

Suppoſe them equal; equal Quantities may be divided into equal Parts of any magnitude; but, does it not follow, that, being now equal, and therefore capable of equal Diviſion, they might alſo be equally divided in their ſimple ſtate? if they cannot, it is certain that Equality *cannot* be produced by taking Multiples of incommenſurable Quantities. For, one of them is always in whole Numbers; as here, a Side of the Square is ſuppoſed to be divided into 12 parts, which being taken any number of times is ſtill in whole Numbers; but, as the Decimal of the Diagonal does not terminate, no Multiple of it can be taken ſo as to produce the ſame, without a remainder.

I am perſuaded, that either Euclid, himſelf, or his Advocates are under ſome miſtake in reſpect of this fifth Definition; which does not, poſitively, determine the Analogy of incommenſurable Quantities. Therefore, the inveſtigation, by means of it, is vague and imperfect; and conſequently, all that is built entirely on it is ſo too; ſeeing it cannot, as it is preſumed, diſcover Analogy, in ſuch Quantities; becauſe, equality of the Antecedent to the Conſequent (the only certain Sign) cannot thereby be produced.

It is manifeſt, that if Equimultiples of incommenſurable Quantities are taken, they will ſtill be in the ſame Ratio; i. e. if the Side of a Square and its Diagonal, &c. be equally multiplied; but no multiples of the Side and of the Diagonal can be ſo taken, that they may be equal to one another; for, it is evident, if they could, they might, alſo (as obſerved above) be divided into the ſame equal Parts, which cannot be: And, for ſuch as are commenſurable, it is an unneceſſary method of determining.

Then, 'tis plain, no ſign, but equality of the Antecedent to the Conſequent, can be poſitive; and ſince that cannot ſubſiſt between Incommenſurables, although it is certain, that *if* one is, the other muſt neceſſarily be ſo too; but ſince that, *if*, can never poſſibly happen, we cannot, on that ſuppoſition, determine Analogy of Ratios, in ſuch Quantities.

THE Reverend Dr. Barrow, in his 21ſt Lecture, has made a very learned Defence of this Definition; and, ſpiritedly, encounters the Detractors of Euclid, viz. Ramus, Tacquet, and Hobbs, in the 22nd. But, having ſilenced thoſe weak Opponents, he is cloſe put to it by Borellus, in his 23rd and laſt Lecture; where, after various ſkirmiſhes (in which, the Doctor has not always the advantage) he briefly ſums up the whole Evidence, in theſe words; "that, in his judgment, there is nothing extant in the whole work of the Elements, more ſubtilly invented, more ſolidly eſtabliſhed, or more accurately handled, than the Doctrine of Proportionalities." Nevertheleſs, if his own Definition of it is to be the Touchſtone, I cannot ſubſcribe to his Atteſtation.

PROPORTION (as I have before obſerved, in the Preamble to this fifth Book) is, in a great meaſure, an innate Principle; which,

when clearly explained what is meant by it, and being explicated by Numbers, is ſo very intelligible, that, to any tolerable Capacity, it does not require other Demonſtration; and all that is built on that knowledge is as ſolid and permanent, and as ſecurely eſtabliſhed, as by all the Demonſtration which can poſſibly be given; at leaſt, on the foundation of Euclid's fifth Definition.

In reſpect of commenſurable Quantities, to all which Numbers may be applied, whether their Ratios are analogous or not is readily diſcovered; ſince, in Analogy, the Ratios being equal, the ſame Numbers will expreſs either.

E. g. 56 : 98 :: 12 : 21; which being reduced to their loweſt Denomination, the Ratio, of each, is determined. The firſt pair, 56 and 98, being divided, ſeparately, by 14, gives 4 and 7, the true Ratio of that pair. Now, if the other pair produces the ſame, they are analogous; 12 and 21, divided ſingly, by 3, produce, alſo, 4 and 7; which is, therefore, the true Ratio of both; conſequently they are analogous, ſeeing the Ratios are equal.

If there be taken Equimultiples of four proportional Quantities, i. e. being multiplied various ways, either all by one Number, (any number whatever) or the firſt pair by one Number, and the ſecond pair by another; or, if the Antecedents (the firſt of each pair) be multiplied by one Number, and the Conſequents by any other, the Products ariſing, from every ſuch multiplication, are ſtill analogous, in the Ratio of both Pairs.

Let A : B :: C : D, in the Ratio of 3 to 5.

Thus, 9 : 15 :: 3 : 5.

So, 27 : 45 :: 9 : 15, being all multiplied by 3.

Alſo, 18 : 30 :: 15 : 25; the firſt Pair by 2, and the ſecond by 5.

In both theſe Caſes, the Ratio is ſtill as A to B, or as C to D.

Again, 36 : 30 :: 12 : 10, { The firſt and third, being multiplied by 4, are both greater than the ſecond and fourth, multiplied by 2.

Alſo, 45 : 45 :: 15 : 15, { The firſt and third multiplied by 5, the ſecond and fourth by 3; in which caſe they are both equal.

And, being all multiplied by any one Number, as above, or, being multiplied by various Numbers, it is evident they will both be leſs; which illuſtrates the fifth Definition of Euclid.

Now, it is very obvious, in Numbers, from any one of theſe various operations, that there is analogy of Ratios between A and B, C and D; and it is full as obvious without. But, provided it is not known, whether the Ratios are analogous or not, not one, nor all the three firſt trials are ſufficient to determine it, merely from the firſt and third being, at the ſame time, both greater or both leſs than the ſecond and fourth; becauſe, thoſe things may, and will happen, where there is not Analogy.

E. g. Let the Ratio of A to B be as 3 to 4; and, C:D::$2\frac{3}{4}$:4.
In whole Numbers, A:B::3:4, or 12:16, and C:D::11:16.

Let there be taken Equimultiples of A and C, and other equimultiples of B and D. E. g. Let the Antecedents be taken thrice, and the Conſequents twice; it will be 3A : 2B : : 9 : 8, and 3C : 2D : : 33 : 32; by which multiplication, it is evident that the Antecedents are both greater than their Conſequents.

But, ſeeing that the Ratios are not equal, it is impoſſible to take Equimultiples of the two Antecedents, and other Equimultiples of the Conſequents, by which, the Products of the Antecedents ſhall be both equal, reſpectively, to their Conſequents; and alſo, becauſe their Ratios are not equal, ſuch Equimultiples may be taken of them, that one Antecedent ſhall be greater, and the other leſs, than its reſpective Conſequent.

E. g. Let A and C be taken four times, B and D thrice; it will then be, 4A : 3B : : 12 : 12, and 4C : 3D : : 44 : 48;

Again; let A and C be taken 7 times, alſo, B and D 5 times; it will be 7A : 5B : : 21 : 20; and, 7C : 5D : : 77 : 80.

Either of theſe operations is ſufficient to evince, that the Ratios are not analogous; i. e. A is not to B, as C to D; for, it is impoſſible that equality of Antecedent to Conſequent ſhould happen in both pairs, (i. e. of A to B, and of C to D), unleſs they are analogous; which is the only indiſputable proof of Analogy; for, one Quantity muſt neceſſarily be to itſelf, as any other Quantity is to itſelf. And ſince it is not poſſible to produce Equality, by multiplication any more than by diviſion, between incommenſurable Quantities, it is manifeſt, that Analogy of ſuch Ratios cannot be abſolutely determined by that Criterion.

Wherefore, the Demonſtration, according to Euclid, in the fifth Book, is not poſitive, in reſpect either of commenſurable or incommenſurable Quantities; but is only preſumptive, at beſt, and is very obſcure and unſatisfactory; ſince it is manifeſt, that, in unequal Ratios, Equimultiples may be ſo taken (according to Euclid) in which the Antecedents will be either both greater or both leſs than their Conſequents (as it has been ſhewn) but, it is not poſſible they can be taken, ſo, that both Antecedents ſhall be equal to their Conſequents.

I am therefore of Opinion, that, the Doctrine of Proportion is not properly inveſtigated by Equimultiples; nor do I conceive, that taking multiples of Quantities, in order to prove Analogy of Ratios, is ſo proper as to compare the Quantities by themſelves, either whole and entire, or by their equal Parts.

But it is manifeſt, that there are Ratios, which will not admit of diviſion into the ſame Parts, i. e. into Parts of the ſame magnitude; yet it is certain, that they may, with equal eaſe, and, I think, with more propriety, be divided into the ſame number of Parts, as to take Equimultiples of them; which Method I have therefore adopted, as the moſt rational and eligible.

ELEMENTS OF GEOMETRY.

BOOK VI.

THE ſixth Book of Elements is not only the moſt uſeful and valuable of the whole, but alſo the moſt entertaining and inſtructive. In it, the Doctrine of Proportion is applied to real uſe, in finding out the more extraordinary Properties of Plane Figures, by the moſt ſubtle and ſolid reaſoning that can be conceived; by which, many excellent and extenſively uſeful Problems are found out, and applied to various and moſt notable purpoſes.

In it, the Rule of Three, or Proportion, called by way of eminence, the GOLDEN RULE, has its foundation and exiſtence (Theo. 9.). Here is alſo more briefly demonſtrated, that, not only the SQUARE of the Hypothenuſe is equal to the two SQUARES of the Sides, of every right-angled Triangle, but all *ſimilar* Figures whatever; the three ſides of the Triangle being made correſponding Sides of each Figure (Th. 16.). In it, is determined the juſt proportion between two ſimilar Figures of any kind, and exhibited by two Right lines (12 & 13); by which means, any Figure, whatever, may be conſtructed, in any Ratio to a given one, or, the Side of a Square may be determined, whoſe Area ſhall be equal to that of any given right lined Figure. A Figure may alſo be conſtructed ſimilar to any given Figure, and equal to any other, of different Form and Denomination, being right-lined.

Several valuable properties of the Circle are, here, more briefly demonſtrated; in ſhort, I recommend a careful and attentive peruſal of it, as the readieſt and only means, by which its beauties and excellencies can poſſibly be communicated.

DEFINITIONS.

Def. I. SIMILAR FIGURES are ſuch as have all their Angles equal, each to the other, reſpectively; and alſo, the Sides which are oppoſite to equal Angles, or which lie between equal Angles, proportional.

The Quadrilateral a b c d, having all its Angles equal, reſpectively, to the Angles of the Quadrilateral ABCD, viz. a equal A, b equal B, &c; and, if, the Side a b has the ſame Ratio to AB, as b c has to BC, &c. then, a b c d is ſimilar to, or like, ABCD.

If EF be drawn parallel to AB, the Angles at E and F, are equal to A and B, reſpectively (4.1.) conſequently to a and b; wherefore, all the Angles of the Quadrilateral a b c d, are equal to thoſe of EFCD; but, ſeeing the correſponding Sides, a b and EF, b c and FC, a d and ED, are not proportional with c d and CD, a b c d is not ſimilar to EFCD.

Triangles ABC, a b c, having all the Angles of one, equal, reſpectively, to the Angles of the other, have their Sides neceſſarily proportional, and are, conſequently, ſimilar Figures.

The Sides a b, AB; b c, BC, and a c, AC, being oppoſite to equal Angles, or which lie between equal Angles, are HOMOLOGOUS, or correſponding Sides.

N. B. All ordinate Figures, whatever, i. e. ſuch as are equilateral and equiangular, one with another, are ſimilar Figures; as equilateral Triangles, Squares, &c. Alſo all Circles are ſimilar Figures.

Def. II. In Triangles, and Parallelograms, if there be two Sides, in one, which, with two Sides in another, of the ſame kind, and about the ſame Angle, are Proportionals; in ſuch wiſe, that the Antecedent of one, and the Conſequent of the other pair, are in the ſame Figure, thoſe Figures are ſaid to be reciprocal: but, more properly,

the Proportionals are reciprocal in ſuch Figures; reciprocal Figures has no meaning, in my Ideas.

In the Triangles ABC, DEF; if the Sides AB and BC of the one, and DE, DF, of the other, are Proportionals; and, if AB is to DE, as DF, of the ſame Triangle, is to BC of the other; then (as it is evident) the Proportionals are reciprocal in thoſe Figures; and they are uſually, though very improperly, called RECIPROCAL FIGURES.

Underſtand the ſame of Parallelograms.

N. B. Equiangular, or ſimilar, Triangles are not reciprocal; becauſe their Sides are all, reſpectively, proportional; which is not the Caſe, in the Figures annexed. For, AC and EF have not neceſſarily the ſame Ratio as the other Sides, AB to DE, and DF to BC. Neither can there, in ſimilar Figures, be found the Antecedent of one Pair, and the Conſequent of the other, in the ſame Figure, unleſs they are congruous (Def. 43). The Sides of ſimilar Figures are, therefore, diſcretely proportional, not reciprocally.

For the Definitions of BASE and ALTITUDE, ſee Def. 45 and 46, in the general Introduction.

AXIOM. If two, or more, Figures are ſimilar to the ſame Figure, they are ſimilar between themſelves.

THEOREM I. 1. Euclid.

Parallelograms, or Triangles, having the ſame Altitude, are, to each other, in the ſame Ratio as their Baſes. And, converſely; having the ſame or equal Baſes, they are as their Altitudes.

Let ADEB and BEFC be two Parallelograms, between the Parallels AC and DF.

I ſay, the Ratio of the Parallelogram AE to BF (i. e. their Areas) is, as AB to BC, their Baſes. Alſo, the Triangle AMB is to BNC, as the Baſe, AB is to BC.

Let AB be divided into any number of Parts, equal to each other, in G and H; and ſuppoſe BC divided into the ſame number of equal parts, in I and K (Poſt. 2. 5.).

Draw, GL, HM, &c. parallel to AD and BE.

DEM. Becauſe AG, GH, and HB are equal to each other; the Parallelograms AL, GM, and MB, are equal. } 18. 1.
And, the Pars. BN, NK, and KF, are alſo equal. }

Now, if AB=BC, the parts AG &c. are equal to BI, &c. If AB be greater than BC, the part AG is greater than BI; and, if AB be leſs than BC, AG is leſs than BI. Ax. 11, 5. Conſequently, the Parallelograms AL, GM, &c. are, alſo, either equal to, greater, or leſs, than BN, NK, &c. as, AG, &c. is equal to, greater or leſs than BI, &c. For, Quantities are in the ſame Ratio, to each other, as their Equimultiples, or equal Parts - - - Ax. 8, 5.

Now, AG : AB :: BI : BC }
And, Par. AL : AE :: BN : BF } in equal Ratio, - Ax. 7. 5.
Conſ. Par. AL : BN :: AG : BI, i. e. :: AB : BC, - 4. 5.
But, Par. AL : BN :: AE : BF. - - - ſame
Th. Par. AE ; BF :: AB ; BC. - - Ax. 13. 5.

2nd. Becauſe Triangles are equal to half Parallelograms, having the ſame Baſe and Altitude; - - - 19. 1.
the Parallelogram AE : BF :: Δ : AMB : BNC; - Ax. 8. 5.
and, Parallelogram, AE : BF :: AB : BC; - above;
conſ. the Triangle AMB : BNC :: AB : BC. - 13. 5.

II. Parallelograms or Triangles having the ſame or equal Baſes, have the ſame Ratio to each other, as their Altitudes.

For, if their Baſes be conſidered as their Altitudes, and the Altitudes as their Baſes, it is evident from the foregoing.

Let ABCD and AEGD be two Parallelograms, on the ſame Baſe AD. BF being perpendicular, to AD; BF and EF are their Altitudes. - - Def. 46.

Then, the Parallelogram AC: AG:: BF: EF.

For, ſince all Parallelograms having the ſame Baſe and Altitude are equal, the Parallelograms, AC, AG, are equal to Rectangles, whoſe Baſes are equal to AD, and their other Sides equal to BF and EF, reſpectively. - 18. 1.
Wherefore, BF and EF being conſidered as the Baſes of the Parallelograms AC and AG, and their Altitudes, AD; the Parallelogram AC: AG:: BF: EF. - - - Theo.
But, the Triangle ABD is half the Par. ABCD;
and, the Triangle AED is half the Par. AEGD, } - 15. 1.
Conſ. the Triangle ABD: AED:: BF: EF - Ax. 13. 5.

THEOREM II. 2 Euclid.

If a Right line be drawn parallel to any Side of a Triangle, cuting the other two Sides, they will be cut proportionally.

In the Triangle ABC, let DE be drawn parallel to BC; cuting the Sides AB and AC, in D and E.

I ſay, that AB and AC, are cut proportionally, in D and E. i. e. AD is to DB, as AE is to EC. Draw BE and DC.

Dem. Becauſe DE is parallel to BC, DBE = EDC. - 18. 1.
Wh. the Triangle ADE: DBE:: ADE: EDC. - Ax. 4. 5.
But, the Triangle ADE: DBE:: AD: DB;
and, the Triangle ADE: EDC:: AE: EC. } Th. 1.
Therefore, - - AD: DB:: AE: EC. - Ax. 13. 5.

Case 2. If ADE be the given Triangle; let BC be drawn (without it) parallel to DE.

Then, if the Sides, AD, AE be produced, cuting BC, in B and C; the Sides, AD and AE, of the Triangle, ADE, are proportional to the Segments, DB and EC, of thoſe Sides, produced to B and C.

For, AD : DB :: AE : EC; conſ. AD : AE :: DB : EC.

COR. 1. If two Sides of a Triangle (AB, AC) are cut proportionally from the ſame Angle A (in D and E) ſo, that AD : DB :: AE : EC; a Right line (DE) joining the Points, D and E, will be parallel to the remaining Side of the Triangle. Converſe of the Theorem.

For, the Sides, AB, AC, being cut, as AD : DB :: AE : EC; and the Right lines, BE, DC, drawn, as before;

$\triangle$ ADE : DBE :: ADE : EDC (1.); conſ. DBE = EDC.

Therefore, DE is parallel to BC. - - Cor. to 18. 1.

CASE 3. If DE be drawn, beyond the Vertex A, parallel to BC, the Side oppoſite; and, if the Sides BA and CA be produced, till they cut DE, in D and E; then, as AB is to AD ſo is AC to AE.

Join BE and DC; as before.

The Triangle ACD is equal to ABE.

DEM. DE is parallel to BC; wh. $\triangle$ DBE = DCE; - 18.1.
and, ADE is common to both; th. ACD = ABE. Ax. 7. 1.
Conſ. the Triangle ABC : ACD :: ABC : ABE. - Ax. 4. 5.
But, - - ABC : ACD :: AB : AD;
and, - - ABC : ABE :: AC : AE. } Theo. 1ſt.
Therefore, - AC : AE :: AB : AD. - Ax. 13. 5.

COR. 2. Two Right lines (BD and CE) cuting each other (in A) between, or without, two parallel Lines (DE and BC) are cut proportionally in that Point, and the Points in which they cut the Parallels.

For, AB : AD :: AC : AE, in the 3rd Caſe, directly; and alſo, in the other two, by Compoſition. - 6. 5.

Cor. 3. A Right line drawn parallel to the Side of a Triangle (in the firſt Caſe) cuts off a Triangle (ADE) ſimilar to the whole Triangle, ABC; and, in the other two Caſes, the Triangles ADE and ABC are alſo ſimilar. (Def. 1.)

For, the Angle ADE=ABC, and AED=ACB; -4.1. and, the Angle A is either common, or vertical.

Cor. 4. If ſeveral Lines be drawn, within a Triangle, parallel to any Side; the correſponding Segments of the other two Sides, cut by the parallel Lines, are proportional; each to the other, reſpectively.

For, if DE and FG be both parallel to BC, in the Triangle ABC; then, AD : DF : FB :: AE : EG : GC.

For, the Segment AD : DF :: AE : EG; by Theorem. DI being drawn parallel to AC; then, DF : FB :: DH : HI.
But, EDHG and GHIC are Parallelograms; - by Con. wherefore, DH=EG, and HI=GC. - - - 15. 1.
Therefore, DF : FB :: EG : GC (as DH : HI). - Ax. 4.5.
But, AD : DF :: AE : EG; th. AD : FB :: AE : GC. - 9. 5.

Or thus. AD : DB :: AE : EC, and AD : DF :: AE : EG. Th. Wh. DB : DF :: EC : EG (Ax. 13.5.); conſ. DF : DB :: EG : EC; and conſ. DF : FB (DB - DF) :: EG : GC (EC - EG). 7.5.
Therefore, AD is to FB as AE to GC. - - - 9. 5.

THEOREM III. 3 Euclid.

If any Angle of a Triangle be biſected by a Right Line, cuting the oppoſite Side; the Segments of that Side, will be proportional to the two Sides of the Triangle, containing the Angle biſected.

And, converſely, if any Side of a Triangle be cut, in the proportion of the other two Sides; and if the greater Segment be contiguous to the greater Side; then will a Right line, drawn from the point of ſection to the oppoſite Angle, biſect that Angle.

Let the Angle ABC, in the Triangle ABC, be biſected by the Right line BD, cuting the Side AC in D.

I ſay, AD is to DC, as AB is to BC.

Produce AB, make BE equal BC, and draw CE.

DEM. Then, becauſe BE=BC, the Angle BEC=BCE, 9. 1.
and the Angle ABC (being external) = BEC + BCE - 10. 1.
Now, the Angle ABD=DBC, (Hyp.); and BCE=BEC;
conſequenty, DBC is equal to BCE. - - Ax. 3. 1.
But, DBC, BCE, are alternate; conſ. BD is par. to CE. - 4. 1.
Wh. in the Tri. AEC, AD : DC :: AB : BE. - Th. 2.
But, BE=BC (Con.); th. AD : DC :: AB : BC. - Ax. 4. 5.

2nd. In the Triangle ABC, let the Side AC be ſo cut, in D, that the Segments, AD, DC, are in the Ratio of AB to BC. Then, a Right line, BD, biſects the Angle ABC.

The ſame Conſtruction remaining as before.

DEM. BE is equal BC (Con.); and, AD : DC :: AB : BC; - Hyp.
wh. AD : DC :: AB : BE; conſ. BD is parallel to CE; - 2.
and conſequently, the Angle DBC=BCE. - - 4. 1.
But, becauſe (in the Triangle CBE) BC=BE, - Con.
the Angle BCE is equal to BEC. - - 9. 1.
But, the Angle DBC=BCE; and ABD=BEC. - 4. 1.
Therefore, ABD is equal to DBC. - - - Ax. 3. 1.

N. B. This Theorem and the foregoing are of extenſive utility The third Caſe of Theo. 2., and the Corollary following, contain the whole ſubſtance of practical rectilinear Perſpective.

THEOREM IV. 4. Euclid.

In equiangular Triangles, the Sides containing equal Angles, and alſo, the Sides which ſubtend equal Angles are proportional.

Let ABC, and abc, be equiangular Triangles, having the Angle a, of one, equal to the Angle A, of the other, b equal B, and c equal C.

I ſay, the Side ab, is in proportion to AB, as ac is to AC, and, as bc to BC.

Dem. Suppoſe the Triangle abc applied to ABC, at the Angle A, ſo that ab falls on AB. - - - Poſt. 5. 1.
Then, becauſe the Angle bac=BAC, ac will fall on AC, and bc will cut the Triangle ABC, in bc.
The Ang. ABC=Abc; and they are internal & oppoſite; wh. bc is parallel to BC (4.1); conſ. Ab:bB::Ac:cC. 2
Wh. Ab:Ab+bB (i.e. AB)::Ac:Ac+cC (i.e. AC)-6.5.
Therefore, as ab:AB::ac:AC. Q. E. D.

Again; ſuppoſe the Triangle abc applied to the Angle C, of the Triangle ABC (as *ab*C); then will the Side ab (i. e. *ab*) be parallel to AB. - - - Cor. 1. Th. 2.
Then, as before, *b*C:BC::*a*C:AC.
But *a*C=Ac, &c. and, Ac:AC::Ab:AB; - above.
conſequently, *b*C (i. e. bc):BC::Ab (i. e. ab):AB.

It is thus demonſtrated, according to Euclid.

Let Abc and cdC be equiangular Triangles, ſo applied, at the Angle c, that the two Sides, Ac and cC, are in a Right line. Produce Ab and Cd, meeting in B.

Then, becauſe the Angle Acb=ACd, & Ccd=CAb, AB is parallel to cd; and, CB parallel to cb; - 4. 1.

wherefore, c b B d is a Parallelogram; - - Def. 33.
conſequently, b B = c d, and B d = b c. - - - 15. 1.
But, A b : b B :: A c : c C; - - Th. 2.
wherefore, A b : c d (equal b B) :: A c : c C;
alſo, as A c : c C :: b c (equal B d) : d C; } - Ax. 4. 5.
conſequently, as A b : c d :: b c : d C. - - - 13. 5.

COR. 1. The 5th Propoſition of Euclid. Triangles, having all their Sides directly proportional, are equiangular. This, being the Converſe, needs no proof.

COR. 2. Triangles, being equiangular, or having all their Sides directly proportional, are ſimilar.-See Def. 1, of this.

Conſequently, if two Angles, of one Triangle, are equal, reſpectively, to two Angles of another Triangle, the Triangles are ſimilar; and their Sides proportional.

For, the two remaining Angles are equal. - 10. 1.

From hence ſeveral valuable Problems are deduced.

N. B. This Theorem is of univerſal application. It is the Criterion of Proportion in every branch of the Mathematics; for, wherever there are found equiangular Triangles, ſeeing that they are neceſſarily ſimilar, the Proportion of their correſponding Sides is conſequently analogous.

The next Theorem may be deduced from this, as a Corollary; for, if two Sides, in one Triangle, are proportional to two Sides in another, and contain an equal Angle, the Sides oppoſite have been proved to be alſo proportional; and conſequently, the Triangles ſimilar.

THEOREM V. 6. Euclid.

IF, in two Triangles, an Angle of one be equal to an Angle of the other, and, if the Sides containing the equal Angles are directly proportional, the Triangles are ſimilar.

In the Triangles ABC, DEF, let the Angle BAC be equal EDF; and, as AB is to DE, ſo let AC be to DF.

I ſay, the Triangle DEF is ſimilar to ABC.

Let the Triangle DEF be applied to the Triangle ABC; the Angle D to the equal Angle A. - - Poſt. 5. 1.

Or, from the Angle A (equal D) take AG, equal DE, AH equal DF, and join GH.

Dem. Now, becauſe AB : DE :: AC : DF; - (Hyp.)
conſequently, AB : AG :: AC : AH; - - Ax. 5. 5.
wh. AB − AG (GB) : AG :: AC − AH (HC) : AH; - 7. 5.
(i.e. GB : AG :: HC : AH; or, as AG : GB :: AH : HC - 5. 5.)
wherefore, GH is parallel to BC; - - Cor. 1. 2. 6.
therefore, the Triangle AGH is ſimilar to ABC. - C. 3.
But, the Triangles AGH, DEF are congruous. - 8. 1.
Therefore, the Triangle DEF is ſimilar to ABC - Axiom.

The 7th Propoſition of Euclid, is of little or no conſequence, as it ſeldom, if ever, occurs; it is rather a critical remark, than a neceſſary Propoſition.

THEOREM VI.

In ſimilar Triangles, Perpendiculars, from equal Angles, are proportional to the correſponding Sides. And, correſponding Sides are cut proportionally by the Perpendiculars.

Let, the Triangles ABD, DEG be ſimilar, whoſe correſponding Sides are AB and DE, AD and DG, BD & EG.

I ſay, the Perpendiculars, BC, EF, from the equal Angles, B and E, are proportional to the Sides.

Dem. For, the Triangles ABC, DEF, alſo CBD, FEG are ſimilar; becauſe, the Angles, at C and F, are all Right (Def. 11.) therefore equal; - - - - Ax 9. 1.
the Angle A = EDF, and BDC = G; by Hypotheſis;

consequently, ABC=DEF, and CBD=FEG. - 10. 1.

Wherefore, as AB : DE :: BC : EF. }
But, - - AB : DE :: AD : DG. } - - Th. 4.
Therefore, - BC : EF :: AD : DG. }
Also, - - BC : EF :: BD : EG. } - Ax. 13. 5.

2nd. AD and DG are cut proportionally, in C and F, by the Perpendiculars BC and EF.

Because, AC : BC :: DF : EF; }
and, - BC : CD :: EF : FG; } - - - Th. 4.
conſ. - AC : CD :: DF : FG. - - - - 9. 5.

Cor. If parallel Lines are cut by any number of Lines, proceeding from the ſame Point, they will be cut proportionally.

And, the Lines which proceed from a Point will alſo be cut proportionally, by the parallel Lines.

Let AB and CD be parallel Lines.

From any Point, E at pleaſure, if EA, EF, &c. are drawn, cuting the Parallels, in A and C, F and H, &c; they will be cut proportionally, in thoſe Points.

The Triangles AEF and CEH, FEG and HEI, alſo, GEB and IED are, reſpectively, ſimilar.

Wherefore, EA : EC :: EF : EH :: EG : EI :: EB : ED;
conſequently, AF : CH :: FG : HI, and as GB : ID.
Therefore, CH : HI : ID :: AF : FG : GB. - - - 4. 5.
Alſo, EA : AC :: EF : FH :: EG : GI, and as EB : BD.

Hence is deduced a valuable and moſt uſeful Problem.

THEOREM VII. 8. Euclid.

In a right angled Triangle, if a Perpendicular be drawn from the Right Angle to the Hypothenuſe, it will divide that Triangle into two Triangles, which are ſimilar to each other; and alſo to the whole Triangle.

In the right-angled Triangle ABC; from the Right angle, B, let BD be perpendicular to AC.

I ſay, the Triangles ABD, DBC, are ſimilar to each other, and alſo to the whole Triangle, ABC.

DEM. The Angle ADB=ABC (Con.) and BAD is common; conſequently, the Angle ABD=BCD. - - - 10. 1.
Wherefore, the Triangles, ABD, ABC, are ſimilar. - 4.

Again; the Angle ADB is equal to BDC; -Def. 10. 1.
and ABD=BCD (above); wherefore, BAD=DBC. 10. 1.
Conſequently, the Triangle ABD is ſimilar to DBC.-4.
But, the Triangle ABD is ſimilar to ABC. - proved
Therefore, the Triangle DBC is ſimilar to ABC-Axiom.

COR. In right angled Triangles, the Perpendicular (BD) is a mean Proportional between the Segments (AD&DC) of the Hypothenuſe, made by the Perpendicular.

For, the Triangles ABD, DBC, being ſimilar, the Side AD (of the Triangle ABD) is to BD (of the ſame) as the Side BD (of the Triangle DBC) is to DC (of the ſame)
i. e. AD: BD:: BD: DC. - - - - - Th. 4.

From hence and from Theorem 12 of the 3rd Book, are deduced ſeveral Problems, of moſt extenſive Utility.

COR. 2. In right angled Triangles, each Side containing the Right Angle is a mean Proportional, between the adjoining Segment, and the whole Hypothenuſe.

AB is a mean Proportional between AD and AC;
and BC is a Mean, between DC and AC.
For, AD: AB (of the Tri. ABD):: AB: AC, of the Tri. ABC.
And, AC: BC (of the Tri. ABC):: BC: DC, of the Tri. DBC. } 4.

COR. 3. A Perpendicular from any Point in the circumference of a Circle, to a Diameter, is a mean Proportional, between the two Segments of the Diameter, made by the Perpendicular.

For every Angle touching the Circumference, in a Semicircle, is a Right one (12. 3.); consequently, the Hypothenuse of a right-angled Triangle, inscribed in the Circle, is a Diameter of the Circle.

From hence, may be deduced the following Problems.

PROBLEM I. *The two Sides, or Legs, of a right angled Triangle being given, how to find the Hypothenuse, the Perpendicular, and the two Segments of the Hypothenuse, arithmetically.*

In the right-angled Triangle, A B C, is given the two Legs, AB and BC.

1st. To find the Hypothenuse. AB□ + BC□, = AC□-20.1.

Consequently, the Squares of AB and BC, being added together, the Square Root, of that Product, gives AC.

2nd. To find the Perpendicular, BD; AC : AB :: BC : BD-4.

BD is, therefore, a fourth Proportional; AC, AB, and BC, being the three Terms given, and BD is required. - Def. 11.5.
Consequently, AB × BC = AC × BD - - - - - Th. 9.
i. e. if AB be multiplied by BC, and that Product divided by AC, (as in the Rule of Three) the Quotient arising is BD.

For, the Divisor multiplied into the Quotient is equal to the Dividend; i. e. AC multiplied into BD, is equal to AB multiplied into BC; by Theorem 9th.
Therefore, the Perpendicular, BD, is a fourth Proportional.

Hence, a Rectangle, under the two Legs of a right-angled Triangle, is equal to a Rectangle under the Hypothenuse and the Perpendicular.

3rd. To find the greater Segment, AD; AC : AB :: AB : AD-4.

Wherefore, AD, the greater Segment of the Hypothenuse, is a third proportional to AC and AB. - - - - - Def. 9. 5.
Conf. AB□ = AC × AD, or, the Rectangle CAD; - Cor. Th. 9.
i. e. AB being multiplied into itself, and the Product divided by AC, gives AD.

Otherwise; BC : BD :: AB : AD.
Conf. AD is a fourth Proportional; and AB × BD ÷ BC = AD.

4th. To find the lesser Segment, DC; AC : BC :: BC : DC, or AB : BC :: BD : DC. Also, AD : BD :: BD : DC.

When either Segment, AD or DC, is found, the other is a third Proportional, between that Segment and the Perpendicular, BD; conſ. AD×DC=BD□.

But, if the Hypothenuſe, AC, and either Segment be found, then, AC—AD is equal DC; and AC—DC=AD.

PROB. II. *The Perpendicular and either Side being given, how to find the other Side.*

Let AB and BD be given.

Becauſe AB□=AD□+BD□; - - 20. 1.
conſequently, AB□—BD□=AD□.
i. e. the ſquare of BD being ſubtracted from the Square of AB, the ſquare Root of the remainder gives AD.
Then, as AD: AB:: BD: BC; wh. BC is a fourth Proportional.
Conſ. AB×BD÷AD=BC. For AB×BD=AD×BC.

PROB. III. *The Perpendicular and either Segment being given, to find the other Segment, and the two Sides.*

Let AD be the given Segment.

Then, as AD: BD:: BD: DC; wh. DC is a third Proportional.
Conſ. BD multiplied into itſelf, and that Product divided by AD, gives DC. For AD×DC=BD□.
If DC had been given, it is juſt the reverſe.

2nd. AD□+DB□=AB□. And BD□+DC□=BC□. - 26. 1.
Wherefore, if the ſquare of AD be added to the ſquare of BD, the ſquare Root of that Sum gives the Side AB.

Again; AB being found, BC is a fourth Proportional.
For, AD: BD:: AB: BC. Wherefore, AB×BD÷AD=BC;
i. e. the ſum of AB multiplied by BD, divided by AD, gives BC.

PROB. IV. *The Segments* (AD *and* DC) *being given, to find the Perpendicular* (BD); &c.

BD is a mean Proportional between the Segments AD & DC.
Wherefore, as AD: BD:: BD: DC;
conſequently, AD×DC=BD□. (Cor. Th. 9.) i. e. If AD be multiplied by DC, the ſquare Root of that Product, is BD.
Hence, the Sides, AB and BC, may be found, as in Problem 3.

Theſe Problems, deduced from this extraordinary and extenſive Theorem, are extremely uſeful in Perſpective; to find Vaniſhing Points and their Diſtances, &c.

THEOREM VIII. 14 & 15 Euclid, reverſed.

Parallelograms having equal Angles, or, Triangles, having one Angle, in each, equal to one another, and the Sides, containing the equal Angles, reciprocally proportional, are equal.

Let ABCD and DEFG be Parallelograms having equal Angles; and as the Side AD, of the one, is to DG, of the other, ſo let DE, of the ſame, be to DC of the firſt. I ſay, they are equal.

Let the equal Angles, at D, be ſo placed together, that the Side AD, of one Parallelogram, and DG, of the other, are in one Right line; then will DE and DC be alſo in one Right line. (2. 1.) Produce BC and FG, meeting in H, forming a Parallelogram CDGH.

DEM. Then, the Par. BD is to DH, as AD is to DG; and, the Parallelm. DF is to DH, as ED is to DC. } - 1.
But, AD : DG : : ED : DC; by Hypotheſis;
wherefore, the Parallelograms, BD and DF, have an equal Ratio to the ſame Parallelogram, DH.
Therefore, the Parallelogram ABCD=DEFG.-Ax.5.5.

2 nd. Having drawn the Diagonals AC and EG; the Triangle ACD is equal to DEG; - - - - - Ax. 4. 1.
for, each is equal to half the Parallelogram BD or DF-15.1.
Or (CG being drawn) the Tri. ACD:DCG::AD:DC;
and, - - - - the Triangle DEG:DCG::ED:DC.
But, AD:DG::ED:DC (Hyp.) conſ. Tri. ACD=DEG;
for, they have an equal Ratio to the Triangle DCG.

COR. 1. Equal Parallelograms having equal Angles, or Triangles having one Angle in each alſo equal, have their Sides about the equal Angles reciprocally proportional.

This, being the converſe of the Theorem is manifeſt, from it; by the ſame conſtruction, it is thus reverſed.

Becauſe, the Parallelogram BD is equal to DF, they have an equal Ratio to the Parallelogram DH. - - - - Ax. 4. 5.
But, Par. BD : DH :: AD : DG; and, Par. DF : DH :: ED : DC. - 1.
Therefore, as AD : DG :: ED : DC. - - - Ax. 13. 5.

Alſo, the Triangle ACD is equal to half the Par. BD; and DEG is equal to half DF (15. 1.); conſequently, ACD = DEG; - Ax. 4. 1. and, their Sides, AD, DC; DE, DG, are reciprocally proportional.

Cor. 2. Parallelograms, or Triangles, having their Baſes and Altitudes reciprocally proportional, are equal.

For, whether their Angles be equal or not, the Parallelograms are equal to Rectangles having the ſame Baſe and Altitude. - 18. 1. Conſequently, if their Baſes (AD and EF) and their Altitudes (CI and IK) are reciprocally proportional, the Parallelograms, BD and DF, or the Triangles, ACD, DEG, are equal. - Th. 1.

For, the Perpendiculars (i.e. the Altitudes) of Triangles and Parallelograms, having equal Angles, are in the ſame Ratio as their Sides; i. e. CI : IK :: CD : DE. - - - Th. 6, or 2.
But, their Baſes, EF and AD, are in the ſame Ratio.
Therefore, CI : IK :: EF, or KL (equal DG) : AD; - Ax. 13. 5.
and conſequently, whether the Angles, at D, are equal or not, the Parallelogram ABCD = DGKL (equal DEFG).
Alſo, the Triangle DKG = ADC (equal DEG). } - 18, 1.

THEOREM IX. 16 Euclid.

If four Right lines are Proportionals; the Rectangle under the two Extremes (the greateſt and leaſt) is equal to a Rectangle contained under the two Means.

Let the four Right lines, A, B, C, D, be proportional; and as A is to B, ſo let C be to D.

Then will the Rectangle under A and D, be equal to that under B and C.

The Rectangles HE and EG, being conſtructed, EF equal A, and FG equal D; alſo, HI equal C, and IE equal B.

DEM. Then, becauſe HE and EG are Rectangles, conſequently their Angles are all equal. - Def. 3. 4.
But, as EF : IE :: HI : FG; i. e. as A : B :: C : D; wherefore, their Sides are reciprocally proportional.
Therefore, the Rect. EG, is equal to the Rect. HK. -8.

OTHERWISE.

AB being equal to the greateſt of four proportional Lines, at either Extreme, as B, let ABC be a Right angle.

Make BC equal to the greateſt of the Means; and AE equal to the other.

Compleat the Rectangle ABCD; by drawing AD and DC, reſpectively, parallel to BC and AB. Draw AC.

Through E draw EF, parallel to BC; and through G, where it cuts the Diagonal, draw HI parallel to AB.

DEM. Now, becauſe AB, BC, and AE are equal to three of the Proportionals, EG is the fourth;
for, AB is to BC as AE is to EG. - Th. 4.
And, ſince the Complements, DG, GB, are equal - 17. 1.
if HE be added to both, the Rect. ADFE = AHIB. Ax. 6. 1.
But, AHIB is a Rect. under the two Extremes.
And, ADFE is a Rect. under the two Means. } Con.
Therefore, the Rectangle under the two Extremes, &c.

Or thus, without a Figure. See the Lines.
A is to B, :: ▭ under A and D : ▭ under B and D; and
C is to D, :: ▭ under C and B : ▭ under D and B; - Th. 1.
But, A : B :: C : D; wh. ▭A and D = ▭C and B; - Ax. 5.
for they have an equal Ratio to the Rect. under B and D.

Corollary. The 17th Propoſition of Euclid.

If three Right lines are Proportionals, the Rectangle under the two Extremes, is equal to the Square of the Mean.

If the two middle Terms, B and C, had been equal, the Rectangle HE would be a Square. (Def. 35.) See Fig. 1.
But, as EF : IE :: HI : FG; and HI is equal to IE. - Sup.
Therefore, the Rectangle under the Extremes, of three proportional Lines, is equal to the Square of the Mean.

Or, if (by the 2nd Conſtruction) AE had been equal to BC, the Rectangle ADFE would be a Square.
Conſ. it would be equal to the Rect. AHIB under the two Extremes. For, as AB : AE (eq. BC, Sup.) :: AE : EG, equal BI.

From hence, and Theorem, 8. is deduced an ingenious method for finding a fourth Proportional.
A third Proportional may alſo be found, by conſtructing a Square on either of the given Lines, which is to be the Mean; and producing two adjoining Sides.
From which Conſtructions, it is obvious, is deduced that moſt excellent Golden Rule, or Rule of Proportion, in Arithmetic.

N. B. A Right line being divided in extreme and mean Proportion, the whole Line, the greater Segment, and the leſs, are in continual Ratio; conſequently, they are three Proportionals.

THEOREM X.

Similar Triangles are proportional to the Squares of their correſponding Sides.

Let ABC and DEF be ſimilar Triangles, having the Angles at A and D equal; the Angle B equal E, and C eq. F.

Let AGHC and DIKF be Squares, on the Sides AC and DF, which are homologous; being oppoſite equal Angles, B and E.

I ſay, the Triangles, ABC and DEF, are to each other, as GC to IF, AC ſquare to DF ſquare. &c.

Draw the Diagonals AH, and DK; alſo the Perpendiculars BL, and EM, to the Sides AC, and DF.

DEM. The △ABC:AHC::BL:HC(eq.AC,Con) } C.Th.1.
also, the △DEF:DKF::EM:KF, equal DF. }
But, - as BL:AC::EM:DF; - - Th. 6.
wherefore, △ABC:AHC::DEF:DKF; - Ax. 13. 5.
conſ. - △ABC:DEF::AHC:DKF. - Th. 45.
But, △AHC:□AGHC::△DKF:□DIKF. -Ax. 7. 5.
Wh. △ABC:□AGHC::△DEF:□DIKF. - Poſt. 1.
Th. △ABC : DEF :: □AGHC:DIKF. Th. 4. 5.
That is, the △ABC:DEF::AC□:DF□. Q. E. D.

THEOREM XI. 23. Euclid.

Equiangular Parallelograms are, to one another, in a Ratio which is compounded of the Ratio of their Sides.

Let AE and DC be equiangular Parallelograms.

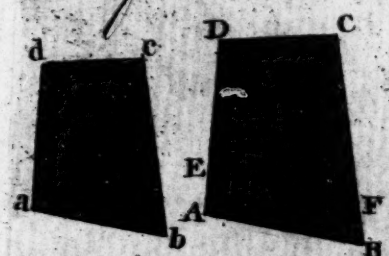
I ſay, the Ratio of the Parallelogram AE to DC is the compounded Ratio of AB, to BC, and of EB to BD.

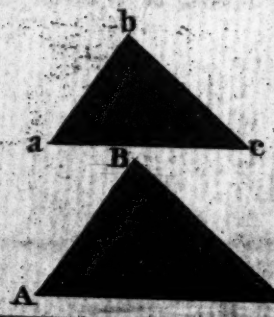
Let the equal Angles, at B, be ſo placed together, that, the Sides, AB, and BC, alſo DB and BE, are in a Right line; and produce the oppoſite Sides, meeting at F, forming a Parallelogram, BEFC.

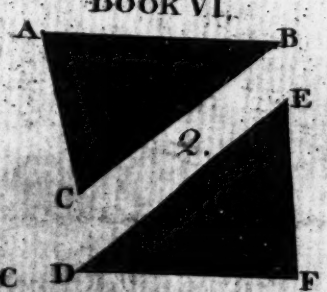
Take any Right line, G, and, as AB is to BC let G be to H; alſo, as EB is to BD, let H be to I.

DEM. Now, the Ratio of G to H is as AB to BC; and, the Ratio of H to I is as EB to BD. - by Suppoſition.
But, the Ratio of G to I, is compounded of the Ratios of G to H, and H to I (Def. 19. 5.); conſequently, the Ratio of G to I is that, compounded of the Ratios of the Sides.
But, the Par. AE:EC::AB:BC; i.e. as G to H. } -Th. 1.
And, the Par. EC:DC::EB:BD; i.e. as H to I. }
Wh. ſince the Par. AE:EC::G:H; and Par. EC:DC::H:I.

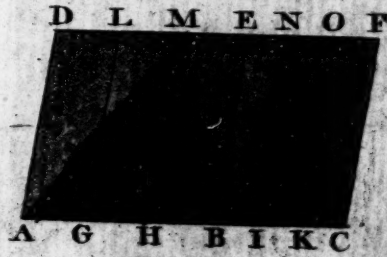
Definitions.







Theo I.



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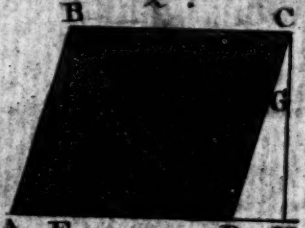

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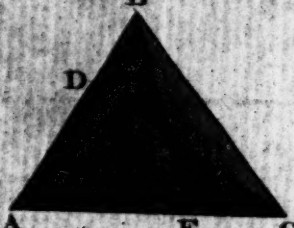
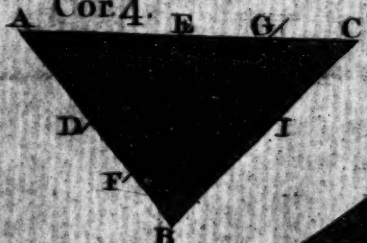


Case 3.

Cor. 4.



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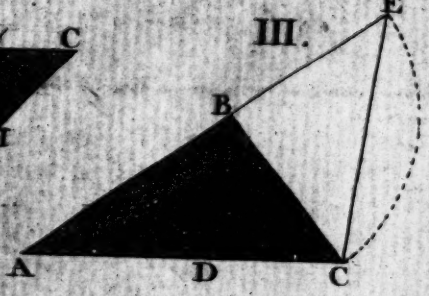

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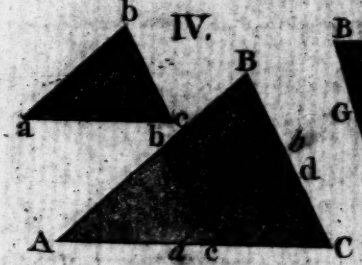

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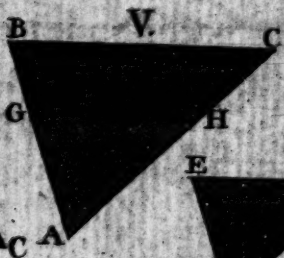

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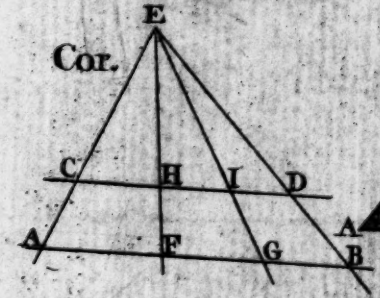

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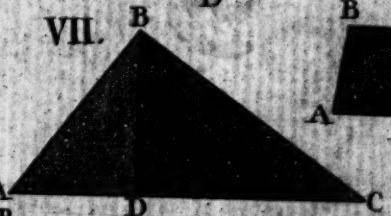

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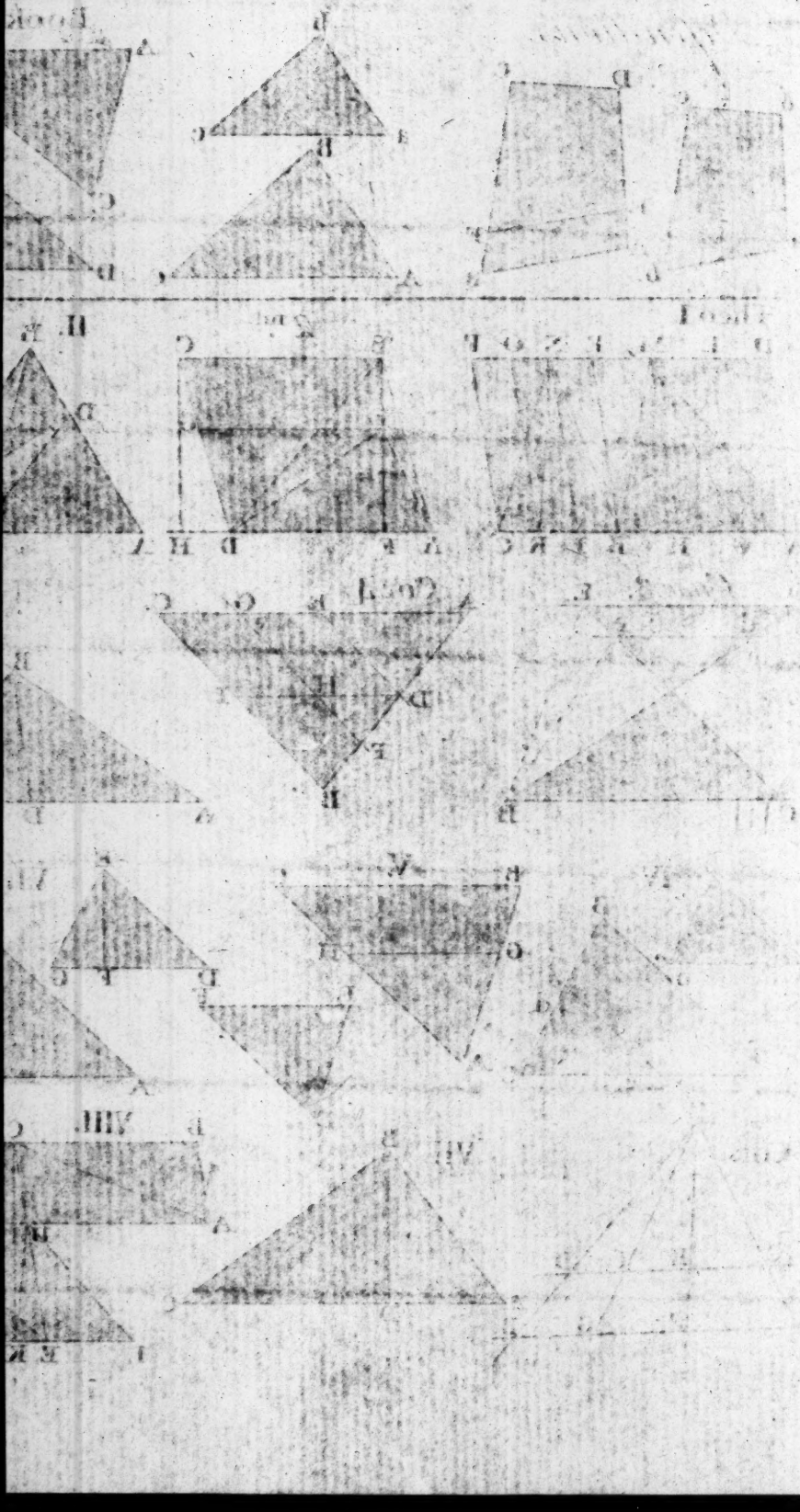


The Parallelogram, AE: Parallelm. DC::G:I. 9. 5. But, G is to I, in the compounded Ratio of AB to BC, &c. therefore, the Par. AE is to Par. DC in the Ratio, which is compounded of the Ratio of their Sides. Q.E.D.

Cor. 1. Triangles, having one Angle in each, equal to one another, have that Ratio between them, which is compounded of the Ratio of the Sides, containing equal Angles.

For, the Triangles AEB, BCD, having equal Angles, at B, EC being drawn, it is evident; as in the Theorem.

Cor. 2. Parallelograms, having equal Angles, or Triangles, which have one Angle in each, equal to one another, have that proportion to each other, as Rectangles under the Sides containing the equal Angles.

For, the Sides, containing equal Angles, have the same proportion, to each other, as the Perpendiculars, EK, KL.-Th.6. Consequently, Rectangles under the Bases and Perpendiculars, are in the same Ratio to each other, as Rectangles under the Bases and adjoining Sides, containing equal Angles, viz. in the Ratio which is compounded of their Sides; by Theorem.

THEOREM XII. 19. Euclid.

The proportion of similar Triangles, to each other, is the duplicate Ratio of their corresponding Sides.

Let the Triangles ABC, DEF, be similar and alike situated; the Angle A equal D. I say, the Triangle ABC is to DEF in the duplicate Ratio of AB to DE, &c.

Produce DF till DG be a third Proportional, to DF and AC; i. e. let DG be to AC, as AC is to DF. Draw EG.

The Triangle ABC is to DEF, as DG to DF.

Dem. Because the Triangles ABC, DEF are similar, consequently, AB: DE::AC: DF. - - - Th. 4.
But, DG: AC:: AC: DF (Con) conf. as AB: DE::DG: AC.

Wherefore, in the Triangles ABC, DEG, the Sides AB and AC, DE and DG, which are about the equal Angles (A and D) are reciprocally proportional.

conſequently, the Triangle DEG=ABC. - Th. 8.
But, the Triangle DEF: DEG :: DF : DG. - Th. 1.
Th. the Triangle DEF : ABC :: DF : DG; - Ax. 4. 5.
and, by inverſion, ABC : DEF :: DG : DF. - Th. 5. 5.
i. e. in a duplicate Ratio of their correſponding Sides.

Or, if AH be taken a third proportional to AC and DF; it will then be, as AB : DE :: DF : AH, i.e. :: AC : DF - Con.
Conſequently, the Triangle ABH = DEF - - Th. 8.
Wherefore, the Tri. DEF : ABC :: ABH : ABC. - Ax. 4. 5.
i. e. as AH is to AC (Th. 1.) viz. in a duplicate Ratio.

COR. If three Right lines are Proportionals, the Square of the firſt to the ſecond is as the firſt Line to the third.

In the Tris. DEF and ABC, the Sides DF, AC, and DG are in ∺
But, the Triangle DEF : ABC :: DF : DG; - - - Theo.
and the Triangle DEF : ABC :: DF □ : AC □. - - - 10.
Therefore, as DF : DG (the firſt to the third) :: DF □ : AC □.

THEOREM XIII. 20. Euclid.

Similar Quadrilaterals, and Poligons, are, by drawing Diagonals, divided into Triangles, equal in Number; the correſponding Triangles are ſimilar to each other, reſpectively, and proportional to the Poligons.

Alſo, the proportion of the Poligons, to each other, is in the duplicate Ratio of their correſponding Sides.

Let ABCDE and FGHIK be ſimilar Poligons.

Draw the Diagonals AC, CE, and FH, HK, &c. from any correſponding Angles, as C and H, to the oppoſite.

DEM. Every Poligon is divided into as many Triangles, as it has Sides, wanting two, by drawing all the Diagonals which

can be drawn, from any Angle; as CA, CE, or HF, HK; conſequently, the number of Triangles, in each, are equal; ſeeing the Poligons have an equal number of Sides.

2nd. Becauſe the Poligons are ſimilar, they are conſequently equiangular (Def. 1.) and, being ſimilarly poſited, the Angle A=F, B=G, and C=H, &c.

Then, becauſe AB : BC :: FG : GH, and the Angle B equal G, the Triangles ABC, FGH are ſimilar. Th. 5.
And, becauſe the Angle BAE is equal, GFK and there is taken away, from each, equal Angles BAC, GFH, the remainder, CAE is equal HFK. - Ax. 7. 1.

But, AB : AC :: FG : FH, and AB : AE :: FG : FK.
conſequently, AC : AE :: FH : FK; - - 13. 5.
therefore, the Triangles ACE FHK are ſimilar. Th. 5.
And, becauſe CD : DE :: HI : IK, and the Angles D and I are equal; the Triangles ECD, KHI are ſimilar. - ſame.

3rd. Becauſe the Triangles ABC and FGH are ſimilar, their Ratio is duplicate of their correſponding Sides; e. g. of AC to FH. - - Th. 12.
And for the ſame reaſon, the Ratio of ACE to FHK is duplicate of AC to FH; or, of EC to KH; - ſame.
alſo, the Ratio of ECD to KHI, is duplicate of EC to KH.

But, the Ratio of each Triangle, ABC, &c. to its correſponding Triangle, FGH, &c. is the ſame, viz. as AC to FH, or EC to KH; i. e. as AB or AE, to FG or FK.

Therefore, as one Antecedent, ABC, is to its Conſequent, FGH, ſo is the Sum of all the Antecedents, ABC + ACE, + ECD, to that of all the Conſequents, FGH + FHK, + KHI; (2. 5.); i.e. as Poligon to Poligon Q.E.D.

4th. The Ratio of ACE to FHK is duplicate of AE to FK - 12.
But, the Ratio of Poligon to Poligon is equal ACE to FHK.
Therefore, the Ratio of Poligon to Poligon, is duplicate, of AE to FK, of ED to KI; or, of any other Sides, which are homologous.

COR. 1. All ordinate or regular Figures whatever, Triangles, Squares, or Poligons of any kind are, reſpectively, to each other, in the duplicate Ratio of their Sides, or Diagonals.

Becauſe, all regular Figures, of the ſame Species, are ſimilar. Alſo, Circles are, to each other, in the duplicate Ratio of their Diameters, &c.

Hence, in all ſimilar Figures whatever, if the Ratio of the homologous Sides be known, the proportion of the Figures is known.

For example; If the Ratio of the Sides be as 2 to 3, their Areas are as 4 to 9. For, as $2 : 3 :: 3 : 4\frac{1}{2}$;
Therefore, as 2 is to $4\frac{1}{2}$, ſo is one Figure to the other; which, in whole numbers, is as 4 to 9.

COR. 2. The Proportion of all ſimilar Figures, whatever, to each other, is as the Squares of their correſponding Sides.

For, as the Triangle ABC or ACE, is to FGH, or FHK, ſo is Poligon to Poligon.

But, the Triangles, being ſimilar, are, as the Squares of their homologous Sides; viz. as AB to FG, or AE to FK - Th. 10.
Therefore, as the Square of AB is to the Square of FG, or of AE to FK, &c. ſo is Poligon to Poligon, - - Ax. 13. 5.

COR. 3. The Perimeters or Circuits of ſimilar Figures are, to each other, as their correſponding Sides or Diagonals.

For, ſince each Side AB, BC, &c. has the ſame Ratio to its correſponding Side FG, GH, &c; alſo, as Side is to Side, ſo is Diagonal to Diagonal, AC to FH, &c; conſequently, as any one Side, or Diagonal, is to its correſponding Side or Diagonal, ſo is the Sum of all the Sides, AB+BC+CD, &c. to the Sum of all the Sides, FG+GH+HI, &c. - - 2. 5.

COR. 4. Any ſimilar Figures whatever, deſcribed on the Mean and either Extreme of three proportional Lines, have the ſame Ratio to each other, as the two Extremes.

For, they are to each other, in the duplicate Ratio of their correſponding Sides; by the Theorem.

From hence is deduced, an excellent Problem, for finding the Side of any Figure whatever, ſimilar to another, and in any Ratio.

THEOREM XIV.

THE Perimeters of ſimilar Poligons, inſcribed in Circles, are, to each other, in the ſame Ratio as the Diameters; and their Areas as the Squares of the Diameters.

Let ABCDE and FGHIK be ſimilar Poligons, inſcribed in Circles; whoſe Diameters are AL, and FM.

Then, as AL is to FM, ſo is the Perimeter ABCDE to FGHIK. Alſo, as AL ſquare, is to FM ſquare, ſo is the Area of the Poligon ACE to FHK.

Draw AC and BL, alſo FH and GM.

DEM. The Angle BCA = BLA, and GHF = GMF. - 10 3.
But, the Angle BCA = GHF (13) th. BLA = GMF; Ax 3. 1
and, the Angle ABL = FGM (for they are R. angles) - 12. 3.
therefore, the Triangles BLA, GMF are ſimilar. - Th 4.
Wherefore, as AB : FG :: AL : FM; - - ſame.
conſequently, as AB□ : FG□ :: AL□ : FM□. - - 10.
But, as AB : FG :: Perimeter ABCDE : Per. FGHIK - 2. 5.
th. the Diameter AL : FM :: ABCDE : FGHIK - Ax. 13. 5.
Alſo, as AB□ : FG□ :: the Area of Pol. X to Z; 10. and 13.
conſ. as AL□ : FM□ :: the Area of Pol. X to Z - Ax. 13. 5.

COR. The Circumferences of Circles are proportional to their Diameters; and, their Areas are as the Squares of their Diameters.

For, the Circumference of a Circle being conſidered as the Perimeter of a Poligon, of an infinite number of Sides, and, the Diameter may alſo be conſidered as a Diagonal; wherefore, the Demonſtration evidently follows from the Theorem.

COR. 2. Circumferences of Circles are proportional, to the Perimeters of ſimilar Figures inſcribed, or circumſcribing rectilinear Figures.

For the Perimeters, of all ſimilar right-lined Figures, are in proportion to their correſponding Sides or Diagonals. - C. 3. 13. And, the Circumferences of Circles are as their Diameters.

COR. 3. The Areas of Circles are in proportion to that of any ſimilar inſcribed or circumſcribing Figures.

For, the Areas of all ſimilar Figures, are as the duplicate Ratio of their correſponding Sides or Diagonals; and, the Areas of Circles are as the duplicate Ratio of their Diameters. (Th. 13.) Wherefore, ſince the Diagonals of ſimilar Figures are in the ſame Ratio as their Sides, reſpectively; and alſo, as the Diameters of circumſcribing Circles; conſequently, the Areas of Circles are as the Areas of ſimilar Poligons, inſcribed or circumſcribed.

COR. 4. The Area of a Circle is to that of any circumſcribed right-lined Figure, as the Circumference of the Circle to the Perimeter of the circumſcribing Figure.

For, the Area of a Circle is equal to a Triangle, whoſe Baſe is equal to the Circumference, and height equal to the Radius. And, the Area of a circumſcribing Poligon is equal to a Triangle, whoſe Baſe is equal to the Perimeter of the Poligon, and its height equal to the Radius of the inſcribed Circle.* Conſequently, the Areas of all circumſcribing Figures are as their Perimeters; which may be conſidered as the Baſes of Triangles of equal height. - - - - Th. 1. Therefore, the Area of a Circle is to that of a circumſcribed Poligon, as the Circumference of the Circle to the Perimeter of the Poligon.

* See, Theory of Menſuration; Article 7, and 8.

THEOREM XV. 22. Euclid.

IF four Right lines are Proportionals; ſimilar right-lined Figures, deſcribed on each Pair, will alſo be proportional.

Let the Right lines AB, CD, EF, and GH be proportionals, and let AB be to CD, as EF to GH.

On AB and CD, let ſimilar Triangles, V and X, be conſtructed; and, on EF and GH any other ſimilar Figures, whatever, Y and Z. I ſay; as V is to X ſo is Y to Z.

Let P be a third Proportional to AB and CD; and Q to EF and GH, reſpectively.

DEM. Now, the Triangle V is to X, as AB to P; and, the Trapezium Y is to Z, as EF to Q. } Th. 12
But, as AB:CD::EF:GH; and, CD:P::GH:Q - Con.
wherefore, as AB:P::EF:Q. (9. 5.) Th. V:X::Y:Z.

It is alſo true, if ſimilar Figures are conſtructed on the firſt and third, and others on the ſecond and fourth, of four proportional Lines.

For, ſince V:X::Y:Z, conſ. V:Y::X:Z. - Th. 4. 5.

COR. Similar Figures being proportional to other ſimilar Figures, their correſponding Sides are Proportionals.

The Converſe, needs no proof; ſeeing it is but reverſing the Premiſes.

THEOREM XVI. 31. Euclid.

ANY ſimilar Figures whatever, deſcribed on the three Sides of a right-angled Triangle, and if they are made correſponding Sides of the Figures; then, that which is deſcribed on the Side ſubtending the Right angle will be equal to both the others, deſcribed on the Sides containing the Right angle.

Let ABC be a right-angled Triangle; and, let X, Y, and Z be ſimilar, irregular Pentagons, deſcribed on the three Sides, AB, BC, and AC, of the Triangle; which are correſponding Sides of the Figures, X, Y, and Z.

I ſay, that the Figure Z, deſcribed on the Hypothenuſe, AC, is equal to both X and Y, deſcribed on the other two Sides.

Let BD be perpendicular to AC.

DEM. In every right-angled Triangle (ABC,) ſince AD:AB::AB:AC; and DC:BC::BC:AC.-Cor. to 7. Conſequently, any Figure, deſcribed on AB, is to a ſimilar Figure deſcribed on AC, as AD is to AC; and, any Figure deſcribed on BC, is to a ſimilar one deſcribed on AC, as DC is to AC. - - - 12 and 13.
Therefore, ſince Z:X::AC:AD; and Z:Y::AC:DC; Z will be to X+Y::AC:AD+DC. - Cor. to 3. 5.
But, AC=AD+DC; therefore, Z=X+Y. Q. E. D.

SCHOL. *This Propoſition, by the Doctrine of Proportion, extends the famous Pythagorean Propoſition,* viz. *the 47th of the firſt Book of Euclid (the 20th, of the firſt, of theſe Elements) univerſally.*

Seeing, by Th. 10. of this 6th Book it is demonſtrated, that Triangles, and all ſimilar Figures whatever, are in proportion to the Squares of their correſponding Sides, the full and perfect demonſtration of this Propoſition neceſſarily follows from thence; which (after the 10th) was therefore unneceſſary; nevertheleſs, for the peculiar elegance of it, I did not think proper to omit it, though it includes them both, being applicable to all ſimilar Figures whatever.

This Propoſition is alſo applicable to the Diagonals of right-lined Figures, as well as to their Sides; and, conſequently, to the Diameters of Circles. Wherefore, a Circle deſcribed on the Hypothenuſe of a right-angled Triangle, for its Diameter, is equal to two Circles, whoſe Diameters are reſpectively equal to its Legs.

THEOREM XVII. 24. Euclid.

IN every Parallelogram, thoſe which are about the ſame diagonal Diameter, are ſimilar to the whole Parallelogram and alſo to each other.

And the two Complements have their Sides reciprocally proportional.

In the Parallelogram ABCD, let AEFH and FICG be about the Diameter AC. Then, the Parallelograms EH, IG, are ſimilar to the whole Parallelogram, BD, and to each other. EG is parallel to AD, and BC; and HI, to AB and DC; by Conſtruction.

Dem. Now, becauſe EF is parallel to BC, the Sides AB and AC, of the Tri. ABC, are cut proportionally.
wherefore, AE : EB :: AF : FC ; - - Th. 2.
and conſ. AE : AB :: AF : AC. - - - Conv. 6. 5.
And, becauſe HF is parallel to DC, as AF:AC :: AH:AD.
wh. as AE:AB :: AH:AD; and, as AE:AH :: AB:AD 4 5.
But, EF = AH, and BC = AD, &c. (15. 1.) wherefore, the Sides of the Parallelograms AEFH, and ABCD, have all their Sides proportional.

But, the Angle EAH is common to both; and, the oppoſite Angles of Parallelograms are equal; - - 15. 1.
wherefore, EFH = BCD. - - - - Ax. 3.
Alſo, the Angle AEF = ABC, and AHF = ADC; - 4. 1.
therefore AEFH is ſimilar to the whole Par. ABCD-Def. 1.

By the ſame Reaſoning, the Parallelogram FICG may be proved ſimilar to ABCD.

For, CI : CB :: CG : CD; viz. as CF to CA - as above; they are alſo equiangular; the Angle at C being common. Conſequently, the Par. FICG is ſimilar to AEFH; being both ſimilar to the whole Parallelogram, ABCD.-Axiom.

2nd. The Complements BF, FD are equal. - - 17. 1.
and they are equiangular (15. 1.); for EFI = HFG.-2.1.
Wherefore, as EF:FG :: HF : FI, reciprocally. - Th. 8.

Corollary. 26 Propoſition of Euclid.

Hence it is evident, that, if on any Angle of a Parallelogram, there be deſcribed, or taken away, a leſſer Parallelogram, ſimilar, and alike ſituated, they will have the ſame common Diameter.

For, the Diameter AF or FC, is common with AC.

If any other Parallelogram A e f H, ſimilar to ABCD, be deſcribed at the Angle A, not alike ſituated, they have not the ſame Diameter AC.

THEOREM XVIII. 27. Euclid.

IF a Parallelogram be deſcribed on a Right line, and from it there be taken away a Parallelogram, ſimilar, and alike ſituated, to one deſcribed on half the Line, equiangular to the firſt; that, which is deſcribed on the half Line, is greater than the remaining Parallelogram.

Let ABCD be a Parallelogram deſcribed on a given Right line, AD; and, let AEFG be an equiangular Parallelogram, deſcribed on half the Line, AG; alſo, let HICD be taken away from the Parallelogram ABCD, ſimilar, and alike ſituated, to the Parallelogram AEFG.

I ſay, the Parallelogram AEFG is greater than the remaining Parallelogram ABIH.

Draw the Diameter ID, and produce EF to K.

DEM. Then, becauſe the Parallelograms, HC and GK, are about the ſame Diameter, ID, they are ſimilar; - 17. wherefore, AEFG is ſimilar to HICD. - - Axiom.

Now, HF=FC (17.1.) and BF=FC; - - Th. 1. conſ. BF=HF (Ax 3. 1.) and HF is greater than BL; add AL, to both; and AF is alſo greater than AI. - 8. Therefore, AEFG is greater than the remaining Parallelogram, ABIH, the defect of HICD.

Again. From the leſſer Parallelogram AbcD, let there be taken away the Parallelogram hicD, ſimilar to AEFG, and ſituate alike.

Now, becauſe FD is a Diameter, in the Parallelogram GFKD, Gi is equal to iK. - - - 17. 1.

Add h i c D, on both Sides; and G c = h K. - Ax. 6. 1.
But, G b = G c (Th. 1.); wh. G b = h K. - - 3. 1.
Add G i, to both, and A i = G K − F i;
conſequently, AF (equal GK) is greater than A i.

Therefore, the Parallelogram AEFG, deſcribed on half the Line, AD, is greater than any other Parallelogram deſcribed on the whole Line, being deficient by a Parallelogram, ſimilar and alike ſituated to that which is deſcribed on half the Line. Q. E. D.

Few, if any, who have favoured the World with Treatiſes on Geometry, have taken notice of this Theorem, or the foregoing (the 27 and the 24 of Euclid) except thoſe who have trod in his path without ſteping the leaſt aſide; indeed, the laſt is ſo very obſcurely worded, that 'tis ſcarcely intelligible; which Mr. Stone has endeavoured to remedy, with little ſucceſs. He excuſes Euclid, by ſaying, that he could not have rendered it more clear, in ſo few words; and therefore, rather than appear tedious, gave it as it is. If the Propoſition, itſelf, be unintelligible, how are we to underſtand the Premiſes? Euclid has given ſome Propoſitions in more words. I am perſuaded that I have made it clearer than Mr. Stone, and in as few words as Euclid makes uſe of.

How far this Theorem may be of uſe in the Mathematics I do not pretend to ſay; but, when it is clearly underſtood, it will be found to contain ſomething extraordinary in it; inſomuch, that I could by no means diſpenſe with the omiſſion of it.

There are two Problems, following after, in Euclid, which are dependant on it; he alſo divides a Line in extreme and mean Ratio by it; in the 30th; it is certainly demonſtrable, when done, but how it is to be performed, in the operation, I cannot deviſe. See, Prop. 11th. B. 2.

As I do not conceive the Problems to be at all uſeful to Mechanics, &c. I have not inſerted them, amongſt the reſt, in the Practical Part.

The 32nd Prop of Euclid is of little conſequence, and has nothing in it worthy of notice.

THEOREM XIX. 33. Euclid.

In the ſame or equal Circles, the Angles, whether at the Center or at the Circumference, are in proportion to the Arks on which they ſtand.

First, in the Circle AED, the Angle ACB ſtanding on the Ark AB, is to the Angle BCD, on the Ark BD, as AB is to BD.

Let the Arks, AB and BD, be divided into any number of equal Parts, in F and G; and join CF, and CG.

DEM. Now, becauſe the Ark AF = FB, and BG = GD, the Angle ACF is equal FCB, and BCG = GCD; - C. 2. 9. 3.
wherefore, as the Ark AF : BG :: FB : GD } in eq
and, as the Ang. ACF : BCG :: FCB : GCD } Ratio } Ax. 4. 5
For, whether AF be equal to, greater, or leſs than BG; the Angle ACF is alſo equal to, greater, or leſs than BCG; in the ſame Ratio; conſequently,
as the Ark AF : BG :: Angle ACF : BCG, &c.
Therefore, as AF + FB : BG + GD,
ſo is the Angle ACF + FCB : BCG + GCD. - 2. 5.
i. e. as the Ark AB : BD, :: Angle ACB : BCD. Q.E.D.

But, the Angle ACB, at the Center, is double AEB, at the Circumference; and, BCD is double BED; - 9. 3
wh. as the Angle ACB : BCD :: AEB : BED. - Ax. 8. 5.
Therefore, as the Ark AB : BD, ſo is the Angle AEB : BED.

2 nd. Becauſe, in the equal Circles, AEG, a e b, the Radii are equal; and equal Arks ſubtend equal Angles - C, 2. 9. 3. conſ. as Ark AB : ab :: Angle ACB : a c b; and AEB : a e b.

If AB and a b be divided into an equal number of Parts, the Demonſtration is the ſame as above.

COR. Sectors of Circles, A C B, B C D, or a c b, are proportional to the Arks AB, BD, or a b.

For, each Sector may be conſidered as an aggregate of a number of Iſoſceles Triangles, of equal Baſes and equal Legs; as ACF, FCB, &c. conſequently, equal Triangles; and having equal Altitudes, each Sector, ACB, BCD, or a c b have, therefore, that Ratio to each other, as the Baſes of thoſe Triangles; i. e. as the Ark AB to BD, &c. - - - Th. 1.

N. B. None of the following elegant, and ſome of them elementary, Theorems are in Euclid; except the 34th; which is the 8th Propoſition of his thirteenth Book.

THEOREM XX.

If two Right lines interſect, within a Circle, and are terminated by the Circumference, the Segments of thoſe Lines are reciprocally proportional.

The two Chords, AB and CD, cut each other in E.

I ſay, that CE is to AE, as EB is to ED. Join AD and CB.

Dem. Now, the Angles AED, CEB are equal; - - 2. 1.
and ABC = ADC; alſo BAD = BCD; - - 10. 3.
wherefore, the Triangles AED, CEB, are ſimilar. Th. 4.
Therefore, CE : AE :: EB : ED, by the ſame. Q. E. D.
Conſ. CE × ED = AE × EB; by 9. 6; alſo by 14. 3.

THEOREM XXI.

If two Right lines are drawn, from the ſame Point without a Circle, to the oppoſite and concave part of the Circumference; they will have that proportion to each other, reciprocally, as their external Segments.

From the point A, draw AB and AD, at pleaſure, cuting the Circle EBD in the Points E, C, B, and D.

I ſay, that AB is to AD, as AE is to AC. Join EB & CD.

Dem. In the Tris. ABE, ADC, the Angle ABE = ADC - 10. 3
and the Angle A is common; wh. AEB = ACD; - 10. 1.
conſequently, the Triangles ABE, ADC are ſimilar.
Wherefore, AB : AD : : AE : AC. - - - Th. 4.
And, AB × AC = AD × AE, by Th. 9. and, alſo by 16. 3.

THEOREM XXII.

IF from any Point without a Circle two Right lines are drawn, one touching the Circle, and the other cuting it; that which touches the Circle is a mean Proportional, between the whole Secant and the external Segment.

From any Point, A, draw AB, touching the Circle at B; and, from the ſame Point draw AC, at pleaſure, cuting the Circle, in D and C.

I ſay, that AC is to AB, as AB to AD. Join BC, and BD.

DEM. The Triangles ACB, ABD, are ſimilar. - Th. 4.
For, the Angle ACB is equal to ABD; - - - 13. 3.
and BAC is common; wherefore, ADB=ABC. - 10 1.
Therefore, AC : AB : : AB : AD. Q. E. D.
And conſ. AC × AD = AB□. by Cor Th. 9; alſo by 16. 3.

THEOREM XXIII.

IF two Circles touch one another, either internally or externally, and if Right lines be drawn, from or through the point of Contact, cuting both Circles; thoſe Lines will be cut proportionally, by the Circumferences of the Circles.

In the Circles, ABC, ADE, and from, or through, the Point of Contact, A, draw AB and AC, cuting both Circumferences, in the Points B, C, D, and E.

I ſay, that AB is to AC, as AD to AE.

Draw AF, a Tangent to the Circles, at the Point A; and join BC, and DE, by Right lines.

DEM. Then, the Angle ABC is equal to ADE; - Ax. 3. 1.
for they are each equal to the Angle FAC. - Th. 13. 3.
And the Angle BAC (Fig. 1.) is common; but,
in the ſecond, BAC, DAE, are vertical, th. equal; 2. 1.
and therefore, the Triangles, ABC, ADE, are ſimilar;
and, conſequently, AB : AD : : AC : AE. - - Th. 4.

THEOREM XXIV.

If a Right line be drawn perpendicular to the Diameter of a Circle, whether it be within or without the Circle, or touching the Circumference, and if any Right line be drawn, from the fartheſt extreme of the Diameter, cuting the Circumference and the Perpendicular; that Line and the Diameter will be cut reciprocally proportional.

Let the Line CD be perpendicular to the Diameter AB. Draw AD, at pleaſure, cuting the Circumference in E, and the Perpendicular in the Point D.

I ſay, that AE is to AB, as AC to AD. Draw EB.

Dem. Then, becauſe AB is a Diameter, the Angle AEB (being in a Semicircle) is a Right one; - - 12. 3.
and ACD is a Right one (Con.) th. equal to AEB; Ax. 9.
and, the Angle DAC is common; conſ. ADC = ABE - 10. 1.
wherefore, the Triangles AEB, ACD are ſimilar. - 4. 6.
Therefore, as AE : AB :: AC : AD. Q. E. D.

In Fig. 3. AB and AC are the ſame; conſequently, AB is a mean Proportional between AE and AD.

Cor. The Rectangle under any Line, AD, drawn from the extreme A, of the Diameter, to the Perpendicular, DC, and the part of it, AE, within the Circle, is equal to the Rectangle under the Diameter, AB, and the whole, AC, of the Diameter produced to the Perpendicular (Fig. 1.) For, AE : AB :: AC : AD; wherefore they are Proportionals; and conſequently, AE × AD = AB × AC. - - Th. 9.

In Fig. 2, the Rectangle under the Means, AB and AC, is under the whole Diameter and a part of it.

In Fig. 3, ſince AE : AB :: AB : AD;
conſequently, the Rectangle under AD and AE, is equal to the Square of the Diameter, AB. - Cor. to 9.
For, AD × ED = BD□; - - - 16. 3.
and, AB□+BD□=AD□. - Th. 20. 1. as above.

THEOREM XXV.

IF two Circles cut each other, and a Right line be drawn cuting both Circles, it will be cut proportionally by the Circumferences, and a Right line joining the points of interſection of the Circles.

Join the Points, F and G, in which the two Circles, AFG and FBG, cut each other; and, let any Right line, AB, cut both Circles, and the Right line FG, in the Points A, C, D, E, and B.

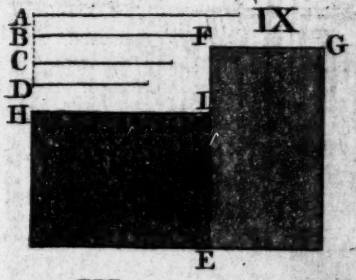
I ſay, the Line AB is cut proportionally in thoſe Points; viz. as AC:CD::EB:ED.

DEM. For, in the Circle AFG, as AD:DF::DG:DE;
and, in the Circle FBG, as DF:CD::DB:DG; } - 20.
wherefore, - - as AD:CD::DB:DE. - 10.5.
Therefore, as AD−CD : CD :: DB−DE : DE. - 7.5.
That is, as AC : CD :: EB : ED. Q. E. D.

THEOREM XXVI.

IF a Tangent to a Circle cuts any Diameter produced, and a Perpendicular be drawn, from the Point of Contact to that Diameter; the Segments of the Diameter, intercepted between the Center and the Perpendicular, the Circumference, and the Tangent, are proportionals; i. e. the Radius is a mean Proportional betwixt the part, intercepted between the Center and the Perpendicular, and the Point in which the Tangent cuts the Diameter.

IX



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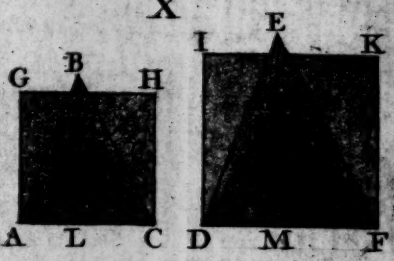

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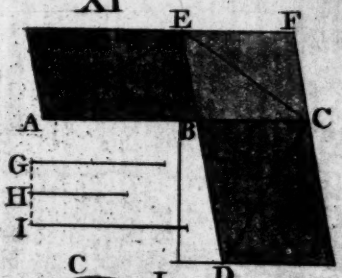

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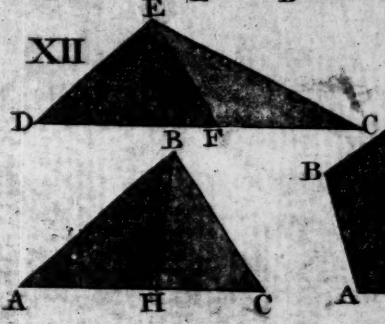

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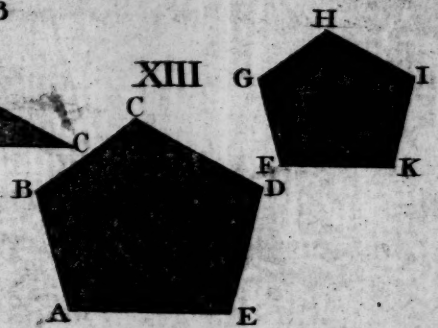

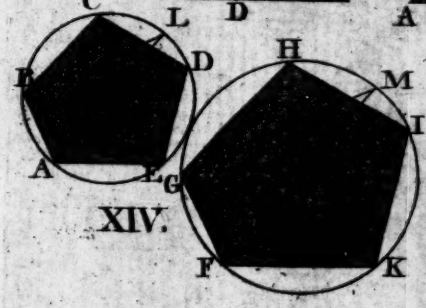

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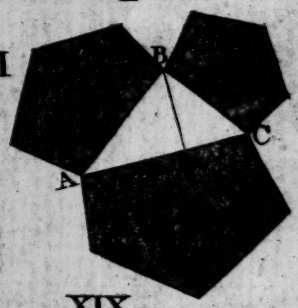

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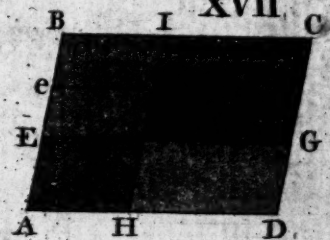

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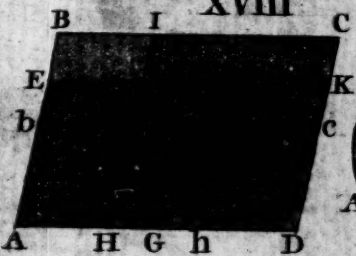

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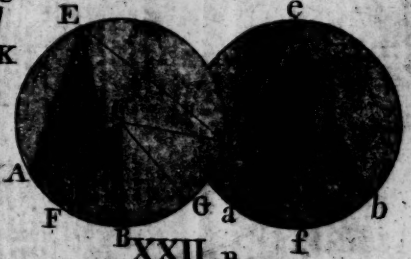

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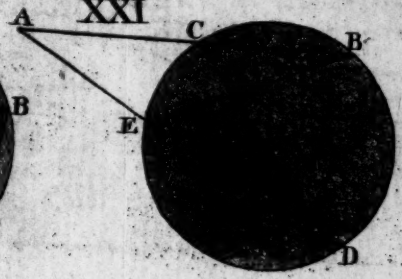

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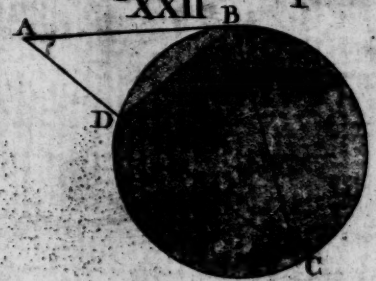

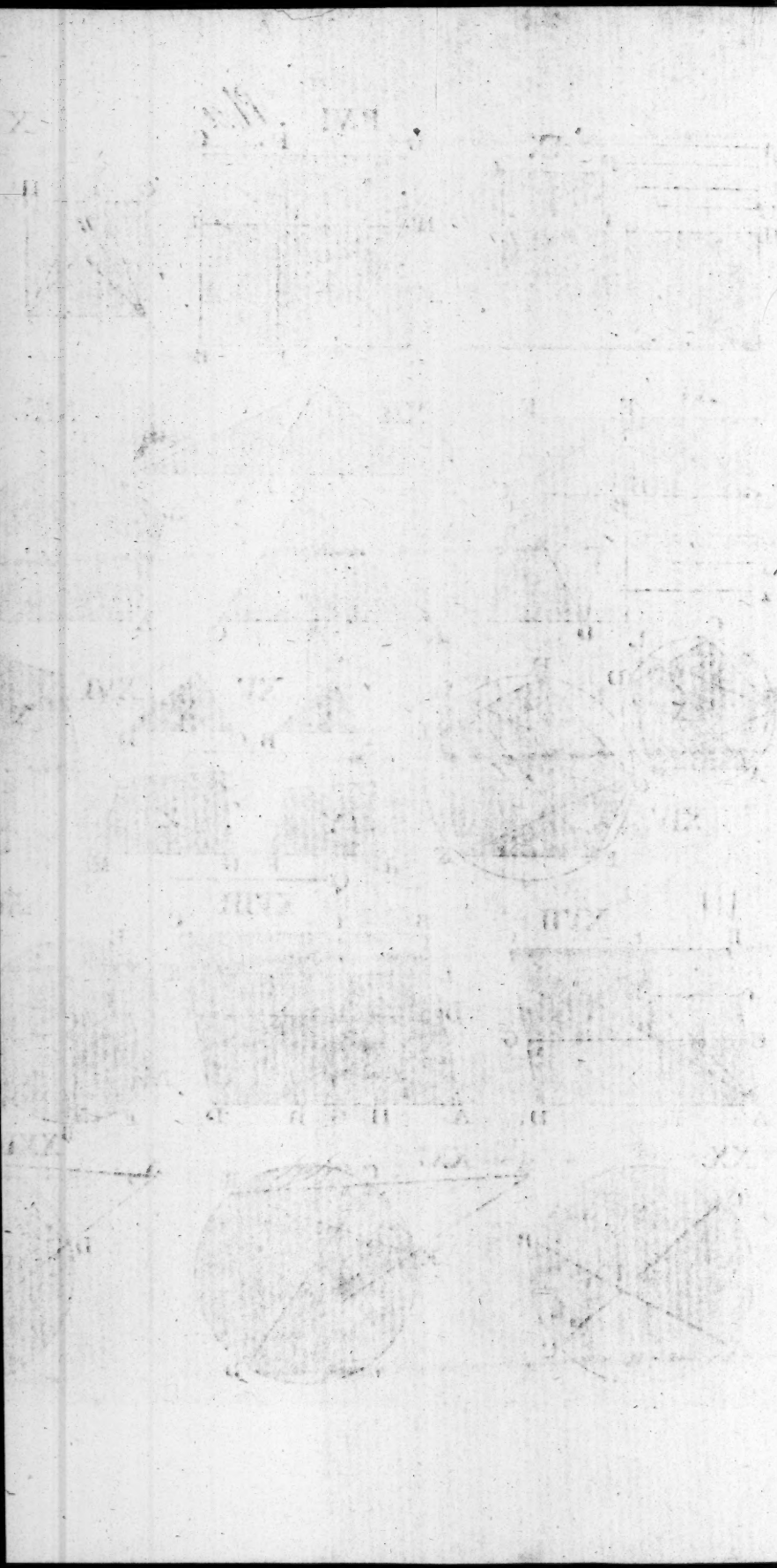

Let any Tangent, as AB, cut the Diameter CD, produced, in B; and let AF be perpendicular to CD; and E the Center. I ſay, that EF, ED, and EB are in continued Proportion. Draw AE.

DEM. For, becauſe EAB is a Right angle, - C. 3. 8. 3.
and, AF is perpendicular to EB - by Conſtruction;
the Triangles EAF, EAB are ſimilar. - - - Th. 7.
Wherefore, as EF : EA :: EA : EB. - - Cor. 2. 7.
But, EA is equal to ED;
therefore, - as EF: ED :: ED : EB. - - Ax. 4. 5.

THEOREM XXVII.

THE Squares of Chord Lines, drawn from the ſame extreme of a Diameter, are in proportion to the Parts of the Diameter, intercepted between Perpendiculars, from the other extremes of the Chords to the Diameter.

AB and AC are two Chord lines, drawn from the Extreme A, of a Diameter, AD; from the other Extremes (B and C) of the Chords, let BE and CF be perpendicular to the Diameter, cuting it in E and F. I ſay, the Square of AB is to AC ſquare, as AE to AF. Draw BD and CD.

DEM. Now, BE is a mean Proportional between AE&ED;
alſo, CF is a Mean, between AF and FD; - Cor. to 7. 6.
for ABD and ACD are Right angles. - - - 12. 3.
Wh. the Rect. AED = BE □; alſo AFD = CF □ - Cor. 9. 6.
But AB□ = BE□ + AE□; and AC□ = CF□ + AF□;
conſ. AB□ = AED + AE□; and AC□ = AFD + AF□.
But, AED, i. e. AE × ED + AE□ = AE × AD; } 3. 2.
and, AFD, i. e. AF × FD + AF□ = AF × AD; }
wh. AB□ = AE × AD; and AC□ = AF × AD - Ax. 3. 1.
But, AE × AD : AF × AD :: AE : AF. - - Th. 1.
Therefore, AB□ : AC□ :: AE : AF. Q. E. D.

THEOREM XXVIII.

IF two Points be taken in a Diameter of a Circle, produced, on the ſame ſide of the Center, ſo, that the Radius is a mean Proportional, between the diſtances of the two Points from the Center; then will two Right lines, drawn from thoſe Points to any Point in the Circumference, be in the ſame Ratio as the internal and external adjoining Segments of the Diameter.

Aſſume any Point, A, in the Diameter CD; in which Diameter, produced, let the Point B (from the Center, E) be a third Proportional to EA and EC; and, to any Point, as F, in the Circumference, let two Right lines, AF and BF be drawn. I ſay, that AF is to BF, as AC to CB.

Draw EF.

DEM. EA : EC :: EC : EB (Con.); wh. EA : EC − EA, :: EC : EB − EC (7. 5.); i. e. EA : AC :: EC : CB; conſequently, EA : EC :: AC : CB. - - 4. 5.
But, EA : EF (eq. EC) :: EF : EB; and, as Angle AEF is common, the Triangles, AFE, EBF, are ſimilar. - 5.
Th. AF : BF :: AE : EC; i. e. as AC to CB. Q.E.D.

OTHERWISE. Make EG equal EA, and draw CG.

Then, becauſe EA, EF, are equal EG, EC, reſpectively, and the Angle at E is common, the Triangles AFE, ECG are congruous; and CG is equal AF. - - 8. 1.
But, EA : EC :: EC : EB (Hyp.); wh. EG : EC :: EF : EB; conſ. EG : EC :: GF : CB (7. 5.); wh. CG is paral. to BF, 2. 6. and the Triangle ECG is ſimilar to EBF. - - - 5.
But, GF = AC (Ax. 7. 1.); wh. as GF, i. e. AC, : CB, as EG : EC, or, EF : EB :: CG, i. e. AF, : BF. Q.E.D.

THEOREM XXIX.

If any Angle of a Triangle is biſected, and the biſecting Line cuts the oppoſite Side; a Rectangle under the two Sides, containing that Angle, is equal to the Square of the biſecting Line, added to the Rectangle under the Segments of the oppoſite Side.

Alſo, the Sides of the Triangle, the biſecting Line, and that Line produced till it cuts the Circumference of a circumſcribing Circle, are Proportionals.

Let the Angle B, of the Triangle ABC, be biſected by the Right line BE, cuting AC, in the Segments AE, EC.

Then, the Rectangle under the Sides AB and BC, is equal to the Square of BE, added to the Rectangle under AE and EC.

2nd. About the Triangle ABC, deſcribe a Circle, ABCD. 5.4.

Produce BE to D, and join AD.

I ſay, that AB is to BD as BE is to BC, reciprocally.

Dem. For, the Triangles ABD, EBC, are ſimilar; Cor. 2.4.
becauſe, the Angles ABD, EBC, are equal, by Con.
and, the Angle ADB is equal to ECB; - - 10. 3.
wherefore, - BAD is equal to BEC. - C. 5. 10. 1.
Th. AB : BD :: BE : BC (4) conſ. $AB \times BC = BD \times BE$. 9. 6.
But, $\square DBE = DEB + EB\square$ (3. 2.) & $DEB = AEC$. 14. 3.
Wherefore, the Rectangle $DBE = AEC + EB\square$. Ax. 6. 1.
Conſequently, $AB \times BC = EB\square + AE \times EC$. Q. E. D.

The 2nd Part was proved in the fifth Line.

THEOREM XXX.

If a Triangle be circumſcribed by a Circle, and a Tangent to the Circle be drawn, at any Angle of the Triangle; and, if a Right line be drawn from the adjacent Angle, parallel to the Tangent, cuting

the oppoſite Side; then, the Side between the Tangent and that Angle will be a Mean-proportional, between the oppoſite Side, and that Segment which is contiguous to the Tangent.

Let AD be a Tangent to the Circle ABCE, at the Point A, in which, the Angle of a Triangle, ABC, inſcribed, cuts it; and, let BE be drawn parallel to AD, cuting, AC, in E. I ſay, that AB is a mean Proportional, between AE and AC.

DEM. BE is par. to AD (Con); wh. the Angle ABE = BAD 4.1
But, ACB = BAD (13.3.) conſ. ABE = ACB; - AX. 3. 1.
and, in the Triangles ABE, ABC, the Angle BAC is common; wherefore, thoſe Triangles are ſimilar.
Therefore, as AE : AB : : AB : AC. - - Th. 4. 6.
Conſequently, AE, AB, and AC, are three Proportionals; and therefore, AB is a Mean, to AE and AC. Q. E. D.

COR. Hence it is manifeſt, that, what is proved, in Cor. 2. of the 7th, in reſpect of a right-angled Triangle, is general, in every ſcalene Triangle; on two Sides of which, ſimilar Triangles are conſtructed. e. g.

In the Triangle ABC, if ABD be made equal to the Angle C, and CBE equal A, the Angles A and C being common; the Triangles ABD, EBC, are ſimilar to the whole Triangle ABC (C. 2. 4. 6.); conſ. to each other. - AX.

Wherefore, the ſquare of AB is equal to a Rectangle, under the Side AC, and the Segment AD; and the ſquare of BC is equal to a Rectangle under AC and EC. C. to 9.

For, becauſe the Triangle ABD is ſimilar to ABC, AD : AB : : AB : AC; &, for the ſame reaſon, EC : BC : : BC : AC. Wherefore, AB and BC are, each, a mean Proportional; between the adjoining Segments of AC, and the whole, AC.

If the Triangle be right-angled (Fig. 2.) the two Segments AD and EC are equal to the whole AC (for the

Points D and E coincide; consequently, the two Squares of AB and BC are equal to the Square of AC.

i. e. AB square is equal to the Rectangle AF, } - 20. 1.
and, BC square is equal to the Rectangle CG. - }

In Fig. 3rd, the Angle at B being acute, consequently, the two Segments AD and EC are greater than the whole AC; but they are, in each Case, subject to one general Demonstration.

When the Angles are all equal, the Square of each Side is equal to the Square of the other; for the Triangle will be equilateral.

THEOREM XXXI.

THE Rectangle, under any two Sides of a Triangle, is equal to a Rectangle contained under the Perpendicular to the other Side, from the opposite Angle, and the Diameter of a circumscribing Circle.

Describe the Circle ABCD about the Triangle ABC. (5. 4.) Draw the Diameter BD; and BF perpendicular to AC.

Then, the Rectangle under AB and BC, is equal to the Rectangle under BF and BD. Draw AD.

DEM. Now, the Angle ADB = ACB; - - - 10. 3.
and BAD is equal to BFC; - - - - - Ax. 9. 1.
(For BFC is a R. angle (Con.) and BAD is Right; 12.3)
consequently, ABD is equal to FBC;
and the Triangles ABD, FBC, are similar.
Wherefore, as FB : BC :: AB : BD; - - - 4. 6.
and therefore, FB × BD = AB × BC. Q. E. D - 9. 6.

N. B. If the Angle, ABC, was bisected by the Line BD; the Rectangle under the Sides, AB, BC, is equal to a Rectangle under the Diameter BD and the Segment BE.

For, AB is to BD, as BE is to BC; by the 29th.

THEOREM XXXII.

In every Quadrilateral inſcribed in a Circle, the Rectangle contained under the Diagonals is equal to two Rectangles under the oppoſite Sides.

In a Rectangle the thing is manifeſt. 20.1.

In the Trapezium ABCD, draw the Diagonals, AC and BD. I ſay, the Rectangle, under AC and BD, is equal to the two Rectangles, under AB and CD, BC and AD.

Make the Angle ABE equal to DBC.

DEM. The Ang. ABE = DB⌣ (Con) and BAC = BDC; 10.3
wherefore, the Triangles ABE, DBC are ſimilar; C.2.4.6.
conſequently, as AB : BD :: AE : CD. - - Th. 4. 6.
Therefore, AB × CD = BD × AE. - - - - 9. 6.
Again. Becauſe the Angle ABE = DBC by Conſtruction;
add EBD, as common to both, ABD = EBC; Ax. 7. 1.
alſo, BDA = BCA (10.3.) conſ. △ ABD is ſimilar to EBC.
Wherefore, BC : BD :: EC : AD; - - 4. 6.
conſequently, BC × AD = BD × EC.
But, AB × CD = BD × AE (above); and AE + EC = AC
wherefore, BD × AE + BD × EC = BD × AC. - 1. 2.
Th. AB × CD + BC × AD = BD × AC. Q.E.D. - Ax. 3. 1.

THEOREM XXXIII.

If all the Sides of a Trapezium are biſected, and the Points of biſection, in contiguous Sides, are joined by Right lines, the Quadrilateral formed by thoſe Lines is a Parallelogram; and, it is equal to half the Trapezium.

Alſo, the Sum of the Squares of the Diagonals, of the Trapezium, is double the Sum of the ſquares of the Diagonals of the Parallelogram.

Let the Sides of the Trapezium, A B C D, be biſected in the Points E, F, G, and H, and draw E F, F G, G H, and E H. Alſo, draw AC and BD. I ſay, firſt, that the Quadrilateral, E F G H, is a Parallelogram.

Dem. For, becauſe, in the Triangles ABC, ACD, the Sides A B, B C, and alſo A D, D C, are cut proportionally, in F and G, E and H, the Right lines F G and E H are both parallel to the common Baſe, AC; and, for the ſame reaſon, the Lines GH and FE are both parallel to BD; conſequently, they are parallel between themſelves; 5.1. therefore, E F G H is a Parallelogram. (Def. 33. 1.)

Secondly. It is equal to half the Trapezium A B C D.

For, in the Triangle ABC, becauſe FG is parallel to AC, and, becauſe AB and BC are biſected in F and G, BI is alſo biſected in K; - - - - 2. 6. and, for the ſame reaſon AI and I C are biſected in L & M; wherefore, L M is half AC; and conſequently, the Parallelogram L F G M (on half the Baſe AC, of the Triangle A B C, and half its Altitude) is equal to half the Triangle A B C. - - from Th. 19. 1. and 1. 6.

For the ſame reaſon, the Parallelogram L E H M is equal half the Triangle A C D; conſequently, the Parallelograms LG, LH, are equal half the Triangles A B C, A D C; i. e. the Parallelogram F F G H, is equal to half the Trapezium A B C D.

Thirdly. Having drawn the Diagonals, E G and F H.

I ſay, that AC ſquare, added to B D ſquare, is double the ſum of the Squares of E G and F H.

For, the Squares of the Diagonals, EG and FH, are equal to the Squares of all the Sides EF, FG, &c. 14.2.
But EF and GH are each equal to half BD; and FG, EH, are each equal half AC. } above.
Wherefore, the four Squares, of EF, FG, GH, and EH, are equal to half the Squares of AC and BD; Cor. to 4.2.
and, they are equal to the Squares of EG and FH.

Therefore, the Squares of AC and BD, are double the Squares of EG and FH.

This Propoſition is, by Stone and ſome others, given in the ſecond Book; conſequently, there is a neceſſity for a Lemma, previous to it; which Lemma, is only a particular Caſe of the ſecond Propoſition of the ſixth Book; which cannot be made general without that Theorem.

The third Part, of which he makes a ſeparate Propoſition, may be demonſtrated in the ſecond Book; but I did not think it of conſequence enough to make another Propoſition of it.

THEOREM XXXIV. 8.13. Euclid.

THE Diagonal of a regular Pentagon is, to its Side, as a Line divided in extreme and mean Proportion.

In the Pentagon ABCDE, let the two Diagonals, AD and BE, be drawn, cuting each other in F.
Then, AD is to AB, or AE, as AE is to AD—AE.

DEM. The Triangles ABE, ADE, are congruous. - 8.1.
For, AB=ED, eq. AE; and, the Angle, BAE=AED; 9.4.
wherefore, the Angle ABF=AEF, equal EAF; - 9.1.
conſ. the Triangles ABE, AFE, are ſimilar. - C. 2. 4.6.
Therefore, as BE : AE :: AE : FE. - - 4.6.

But, BE is parallel to CD, alſo, AD to BC; - C. to 9.4.
conſ. BCDF is a Parallelogram; wh. BF=CD (eq AE) 15.1.
Therefore, as BE : BF :: BF : FE;
i. e. as BE : AE, or AB, :: AE : FE.
Conſequently, AB, &c. is equal to the greater Segment of BE, divided in extreme and mean Proportion.

Cor. Two Diagonals, of a Pentagon, cuting each other, are divided in extreme and mean Proportion, in the point of their mutual Interſection.

As, AD and BF, in the Point F.
For, AD=BE, and FD=BC, equal BF.

THEOREM XXXV.

A Diameter or Perpendicular, of a regular Pentagon, is divided in extreme and mean Proportion, by the Diagonal which cuts it at right Angles.

In the Pentagon ABCDE, let the Diameter CF be drawn, and the Diagonal BD, cuting CF in G, perpendicularly.

I ſay, that CF is divided in extreme and mean Proportion, in G. Draw EH parallel to CF; alſo, the Diagonal BE, cuting the Perpendicular in I.

Dem. Becauſe BE is parallel to CD, the Ang. CDB=DBE; 4. 1.
and, CBD=CDB (9. 1.) wh. CBD=DBE. - Ax. 3. 1.
But, the Angle CGB=BGI (Ax. 9); wh. GI=GC. - 11. 1.
Now, ſince EH is parallel to FG (Con) and AE to BD, FGHE is a Parallelogram; wherefore, EH=FG. 15. 1.
And, becauſe EH is par. to IG, EH:IG::BE:BI. Th. 2.
But, BE is to BI, as BI is to IE (for, BI=BC).
And FG=EH; alſo, CG=GI; proved above;
wherefore, FG:GC::BE:BI; i. e. as BI:IE;
and therefore, CF is divided in extreme and mean Ratio, in the Point G; where it is cut by the Diagonal BD.

Cor. The Diagonal BE cuts FG, the greater Segment of CF, in extreme and mean Proportion; at I.

For, BD is par. to AE; conſ. the Tris. GBI, IEF are ſimilar;
wherefore, BI is to IE as GI to IF. - - Th. 4.
But, BI:IE::BE; BI (eq. AB) therefore, FG:GI::GI:IF.

Hence, if a Right line be divided in extreme and mean Proportion, and from either extreme of the greater Segment the meaſure of the leſs is ſet off, it will alſo be divided in the ſame Ratio; and the leſſer Segment of the firſt, will be the greater Segment of the other. e. g.

Let AB be divided in extreme and mean Proportion, in C.

Then, if CD be made equal to CB, AC will be divided in the ſame Ratio; and CD is the greater Segment.

For, ſince AB : AC :: AC : CB, it will be (by 7. 5.) AC : CB :: AB − AC : AC − CB, i. e. AC : CB :: CD : AD.

Alſo. If to the whole Line, divided in extreme and mean Proportion, there be added the greater Segment; then, the whole compounded Line is divided in the ſame Ratio, and the greater Segment, of the firſt, will be the leſſer.

Let AC be divided in extreme and mean Proportion, in D. Then, if CB be made equal CD, CB will be the leſſer Segment of the whole, AB, divided in the ſame Ratio.

For, ſince AC : CD :: CD : AD; it will be (by 6. 5.) AC : CD :: AC + CD : CD + AD; i.e. AC : CD :: AB : AC.

Hence, if either Segment be given, the other may be found.

Let AC, the greater Segment, be given.
Divide AC in the Ratio required, in D. Produce AC, make CB equal CD, and CB is the Segment ſought.

If CB, the leſſer Segment had been given.
To find the greater; divide CB as before, in E; make CD equal CB, and AD equal CE.
Then, AC is the greater Segment required.

ELEMENTS
OF
GEOMETRY.

BOOK VII. The XI. of Euclid.

THE ſeventh Book, of theſe Elements, treats firſt of the Elements or firſt Principles of plane Solids (i.e. Solids bounded by Planes only) viz. of the Poſitions of Right lines, and of Planes to Planes; of the Sections of Planes by Planes, and of Right lines by Planes.

Secondly, of ſolid Angles, generated by the interſections of Planes; their Conſtruction and Principles inveſtigated.

Thirdly, of Solids, bounded by ſix Planes, only; their Affections, Proportions, and Properties, are ſearched out, on which is founded the Theory of Menſuration of Solids.

THE Doctrine of Solids, contained in this Book and the following, is alſo of great uſe throughout the Mathematics; ſome of which have their very exiſtence in it. The Science of rectilinear Perſpective is wholly founded on the Sections of Planes by Planes, and, of Right lines cut by a Plane, and by parallel Planes; inſomuch that, without their aſſiſtance, Perſpective would be a very imperfect Science; which, to the immortal Fame of Dr. Brook Taylor, is now eſtabliſhed on the moſt permanent Principles.

Every Theorem, of the eleven firſt, is more or leſs uſeful in Perſpective; the 5th and 7th are of great uſe; which, with the 8th and 9th contain the Eſſence of it, in Theory; as the 10th and 11th, of both Theory and Practice.

To enumerate every branch of the Mathematics, in which this ſeventh Book is particularly uſeful, is needleſs, was it in my power; the Reader may depend on it, that its Doctrine is, at the ſame time, entertaining, and of extenſive utility.

DIFINITIONS.

Def. I. A SOLID is a BODY, having length, breadth, and thickneſs; and is bounded by Planes, or curved Surfaces, or both. As X.

Def. II. A PLANE (or RIGHT LINE) is parallel to another Plane, which, being produced infinitely, would never meet; but are, every where, equidiſtant.

As the Plane X, to the Plane Z;
or, the Right line AB, to both X and Z.

Def. III. A RIGHT LINE is perpendicular to a Plane, when it makes equal Angles with every Line drawn in the Plane, to or from that Point in which the Line cuts, or would cut, the Plane.

If AB makes Right angles with the two Lines, CD and EF, paſſing through B, in which Point, AB cuts the Plane CEDF; and if the Lines, CD and EF, are both in that Plane; AB is perpendicular to the Plane.

Def. IV. A PLANE, cuting another Plane, is perpendicular to the other, when it does not incline to the other on either Side.

If the Plane ABD cuts the Plane AED, in AD; and, if any Right line, BC, or CE, drawn in one, be perpendicular to the other Plane, thoſe Planes are perpendicular to each other.

N.B. It is uſual to define perpendicular Planes, by Right lines being drawn perpendicular to their common Section; which is taking for granted that their common Section is a Right line, which being granted, there would be no occaſion afterwards to demonſtrate it, in Propoſition 3rd.

Let the young Geometrician take particular notice, that no regard is had to the poſition of the Plane, or Line, in reſpect of the Horizon, to which another Line or Plane is ſaid to be perpendicular, but only to their poſition in reſpect of each other, for, if they make Right angles with each other, they are perpendicular, each to the other.

Def. V. If a PLANE (or RIGHT LINE) be neither parallel nor perpendicular to another Plane, it is ſaid to *incline* to the other Plane.

The Inclination of one Plane to the other is the acute Angle ABC, made by the Section of another Plane, DEC, cuting both the Planes, BE and BF, perpendicularly; i. e. at right-angles.

Alſo, the Inclination of a Right line, AB, to a Plane, BD, is the Angle ABC made by another Line (BC) drawn in the Plane, from the Point B, in which the inclined Line, AB, cuts the Plane; and paſſing through another Point, C, in which a Perpendicular, AC, from any Point (A) in the inclined Line, cuts the Plane.

N. B. Two Planes have an equal Inclination to two other Planes, when the Angles of their Inclination are equal.

Def. VI. A SOLID ANGLE is that which is made by more than two Plane Angles, being ſo applied to each other, at the ſame Point, that two of them are not in the ſame Plane.

If the three plane Angles A, B, and C, be turned over, they will form a ſolid Angle, at the Point where their Vertices meet each other.

N. B. If three Plane Angles, form a ſolid Angle, any two muſt be greater than the third; or (if four) three are greater than the fourth, &c.

Solid Angles are equal, which are conſtructed on an equal number of plane Angles; being equal, reſpectively, and equally inclined to the contiguous Planes, and ſituated in the ſame order.

Def. VII. A PYRAMID is a SOLID, bounded by any number of Planes more than three; of which, one (the Baſe) may have any number of Sides; all the other Planes are Triangles, meeting in a common Vertex. As AB and CD. A and C are their Vertices; B and D their Baſes.

N. B. Not leſs than four Planes can form a Solid; and that, muſt neceſſarily be a Pyramid, and the Planes are all Triangles.

Therefore, a Pyramid is the firſt of Plane Solids, it being the ſimpleſt; as having the leaſt number of Planes, Lines, and Angles.

Def. VIII. A PRISM is a SOLID, compriſed within any number of Planes not leſs than five, of which, two are parallel, equal, and ſimilar Figures (of any number of Sides) all the other Planes are Parallelograms.

As AB; or CD. A and B, alſo, C and D, are the ſimilar and equal Planes; the others, a b, b c, &c. are all Parallelograms.*

Def. IX. A PARALLELOPIPED is a PRISM, bounded by Six Planes; of which, two and two oppoſite Planes are parallel.

If the Plane ABCD be parallel to DEFG, ABDG to DCEF, and ADFG to BCED; the Solid, ACF, is a PARALLELOPIPED.

Def. X. A CUBE is a PARALLELOPIPED, whoſe Planes are all Squares. As A, B, C.

Therefore, a Cube is alſo a Priſm; - - by Def. 8.

Def. XI. SIMILAR PLANE SOLIDS are contained within an equal number of plane Figures, ſimilarly ſituated and ſimilar one to another, reſpectively, in each Solid.

The Angles are, alſo, neceſſarily equal.

If the Plane a, be ſimilar to A, b to B, and c to C; and if all the other Planes, of which each Solid is conſtructed, be alſo ſimilar and ſituate alike; then the Solid a b c is ſimilar to A B C.

All CUBES are ſimilar Solids.

All SOLIDS, contained within an equal number of equilateral and equiangular Planes, are ſimilar.

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* A Priſm may be conceived to be generated, by the direct motion of any right-lined Plane Figure; always parallel to its firſt poſition; but not in a continuation of the Plane of the Figure.

AXIOMS.

Ax. I. The first Proposition of Euclid.

Every part of a Right line is in the same Plane. i.e. One part of a Right line cannot be in any Plane, and another part of the Line out of that Plane; either, or both being produced.

For, if it was possible, a Right line does not agree with a Plane in every Point; agreeable to Def. 6. of Book 1st.

If the part AB, of a Right line, ABC, be in the Plane AEC, the whole Line is in the same Plane; for, the two Points, A and B, are in the Plane; and if any Point, D, be not in the Plane, ABD is not a Right line; seeing that, AB is a Segment of the Line AC.

II. Two Right lines, being parallel, are in the same Plane, i.e. a Plane, may pass through both Lines.

AB and CD, being parallel, are in the Plane ABCD.

Imagine a Plane, ABEF, revolving on AB as an Axis.

Then, since CD is parallel to AB, it is parallel to a Plane passing through AB; wherefore, the Perpendiculars CE and DF are equal (Def. 2.) and, consequently, when the Point C coincides with E, D will coincide with F, and the whole Line CD be in the Plane, ABCD.

III. The seventh Proposition of Euclid.

Two Right lines, being parallel, and cut by another Right line, are all in the same Plane.

For, if AB be parallel to CD, they are in the Plane ABCD (2nd); consequently, the Points, A and D, where AD cuts them both, are in that Plane.

Wherefore, since there are two Points, A and D, of a Right line AD, in a Plane, the whole Line is in that Plane (Ax. 1.) and, consequently, AD is in the same Plane with the two parallel Lines, AB and CD.

Hence, if two or more Lines, not parallel, are in the same Plane, and are both cut by another Right line, it is also in the same Plane with them. Consequently, two Right lines, AB and AD, or AD and CD cuting each other, are in the same Plane.

IV. The ſecond Propoſition of Euclid.

Three Right lines cuting each other (not in the ſame Point) are all in the ſame Plane; wherefore, the three Sides of every right-lined Triangle are in one Plane.

For, the two Right lines, AB and BC, are in the ſame Plane (3.). Conſequently, being both cut by another Right line (AC) ſeeing the two Points, A and C, are in the ſame Plane, the whole Line, AC, is alſo in that Plane. Ax. 1.

V. Solids, contained within an equal number of Planes, which are equal and ſimilar to each other, two and two, (i. e. one in each Solid) reſpectively; and being ſimilarly ſituated, in reſpect of each other, and alſo of the whole; ſuch Solids are equal in every reſpect. As ACE and ace.

If the Plane abde, be equal and ſimilar to ABDE; ac, to AC, and fd to FD; alſo afe, to AFE, and bcd, to BCD; then, the Solid abcdef is equal, in every reſpect, to ABCDEF.

THE eleventh Book of Elements is, in general, leſs underſtood than any of the foregoing; not owing to any defect or imperfection in the Demonſtrations, but very much ſo to the imperfection of the Diagrams. It has never happened, that a Mathematician, who had engaged in a Publication of the Elements of Geometry, was competent in Perſpective delineation; for want of which, the Schemes, which are intended to repreſent Planes cuting Planes, &c. are ſo badly deviſed, and worſe repreſented, that it is with the greateſt difficulty a juſt Idea of their meaning can be made out; nor is it poſſible to conceive the meaning of ſome, unleſs the Pupil has very acute Talents, or is aſſiſted in his ſtudy of them, by Apparatus, or an able Tutor.

I flatter myſelf that I have removed that difficulty, which has, hitherto, been the greateſt impediment, to a clear inveſtigation of the Doctrine of Solids. Where it is neceſſary, moveable Schemes are adapted for the purpoſe; and, where they may be diſpenſed with, the Diagrams, and Solids repreſented are perſpectively delineated; ſo that, the Student muſt not always expect (as in Plane Geometry) that, Angles which are ſaid to be right, or Lines parallel, &c. are really ſo, in the Diagram; but, that they are, or *would* be ſo, if ſuch a Figure was conſtructed, as the Diagram repreſents. For, unleſs the Reader can conceive this, it will be very difficult for him to ſuppoſe one Line to be greater than another, when it is, abſolutely and evidently, leſs; which, from the Premiſes given and the foregoing Elements, it is manifeſt muſt be greater.

IV. The ſecond Propoſition of Euclid.

Three Right lines cuting each other (not in the ſame Point) are all in the ſame Plane; wherefore, the three Sides of every right-lined Triangle are in one Plane.

For, the two Right lines, AB and BC, are in the ſame Plane (3.). Conſequently, being both cut by another Right line (AC) ſeeing the two Points, A and C, are in the ſame Plane, the whole Line, AC, is alſo in that Plane. Ax. 1.

V. Solids, contained within an equal number of Planes, which are equal and ſimilar to each other, two and two, (i. e. one in each Solid) reſpectively; and being ſimilarly ſituated, in reſpect of each other, and alſo of the whole; ſuch Solids are equal in every reſpect. As ACE and ace.

If the Plane abde, be equal and ſimilar to ABDE; ac, to AC, and fd to FD; alſo afe, to AFE, and bcd, to BCD; then, the Solid abcdef is equal, in every reſpect, to ABCDEF.

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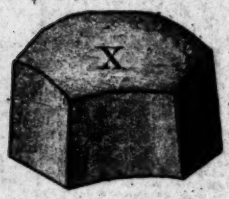
DEF. I. Book VII. *Definitions.*



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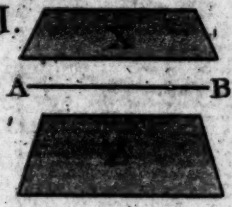

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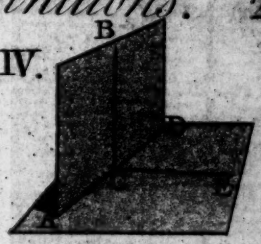

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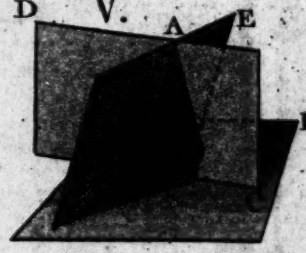

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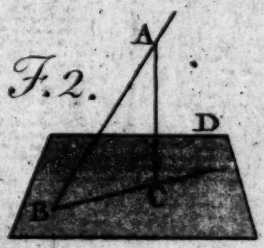

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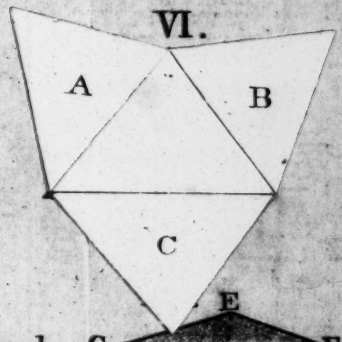

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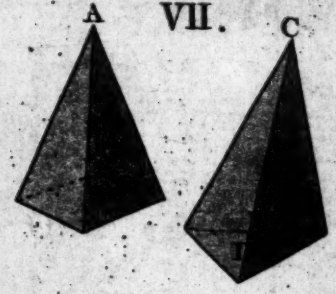

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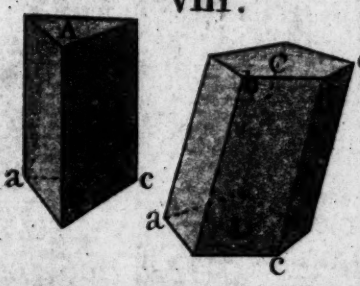

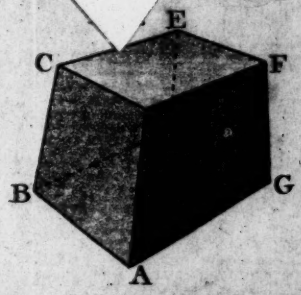

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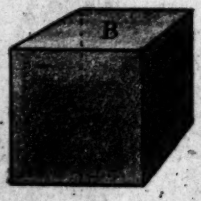

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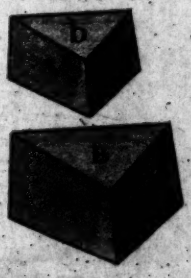

Axioms.

I

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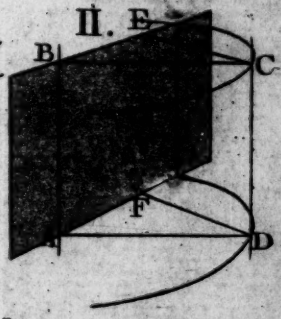

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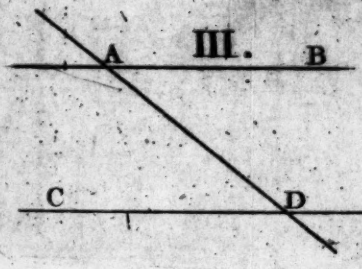

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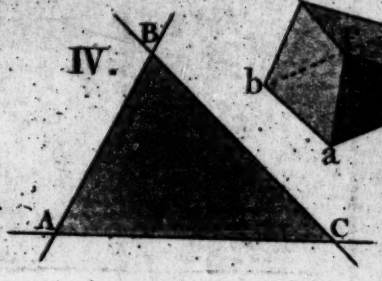

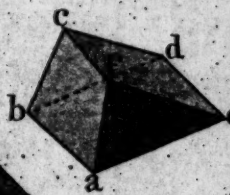

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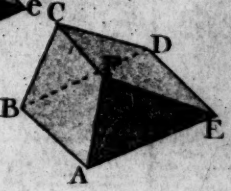

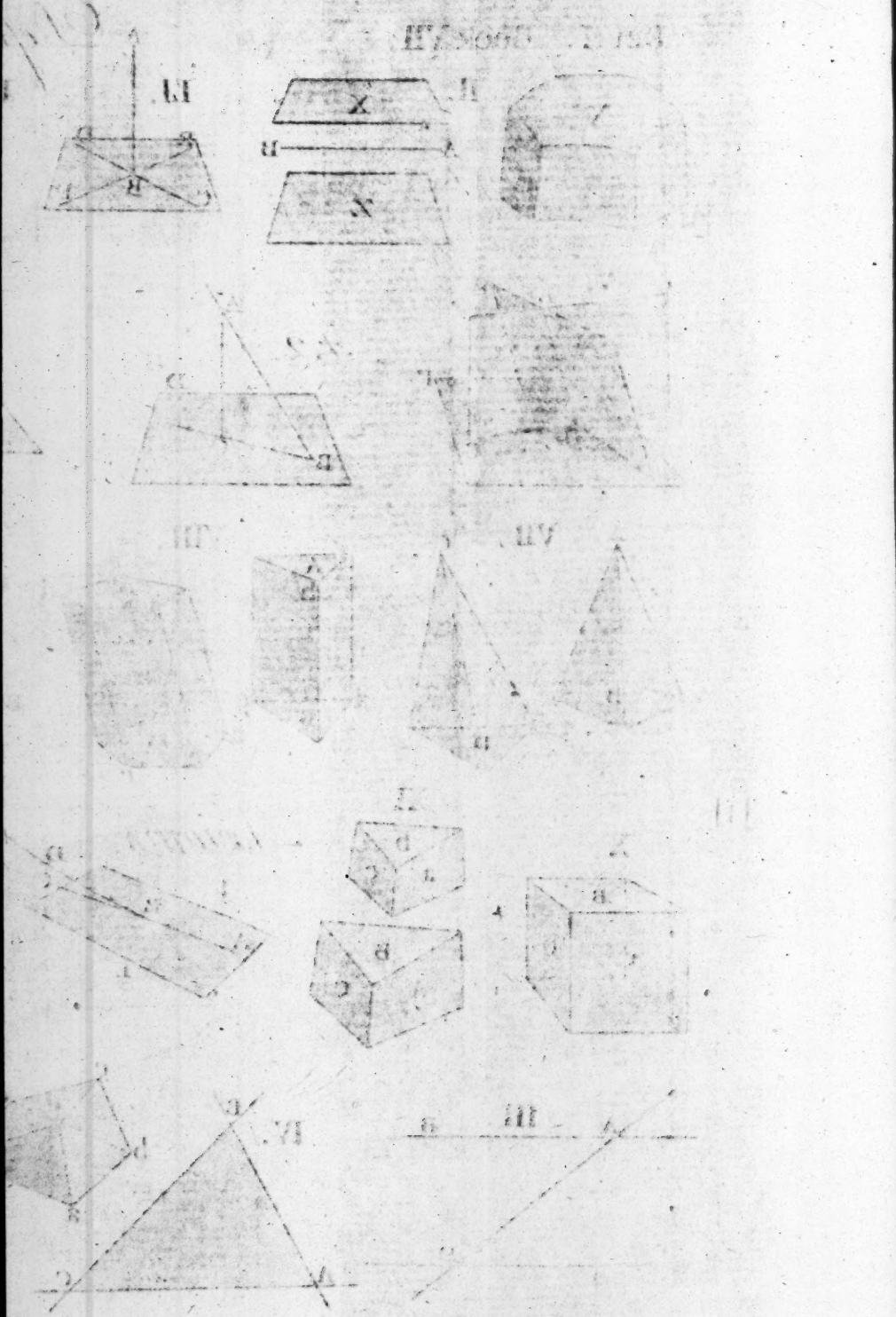

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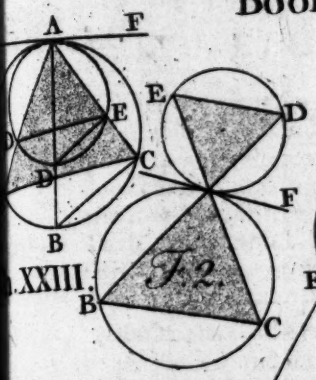

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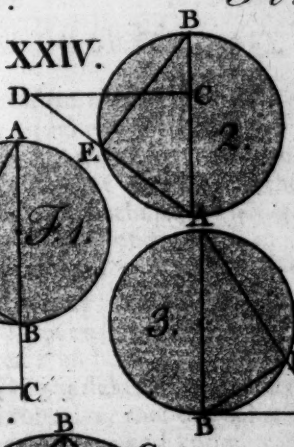

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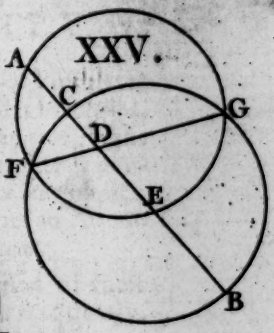

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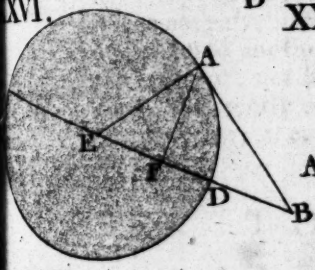

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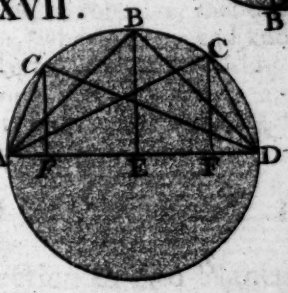

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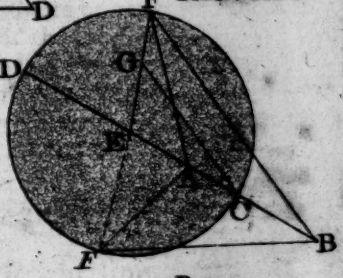

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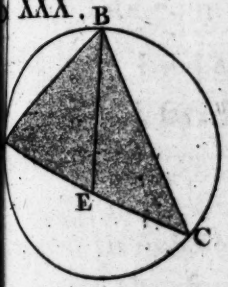

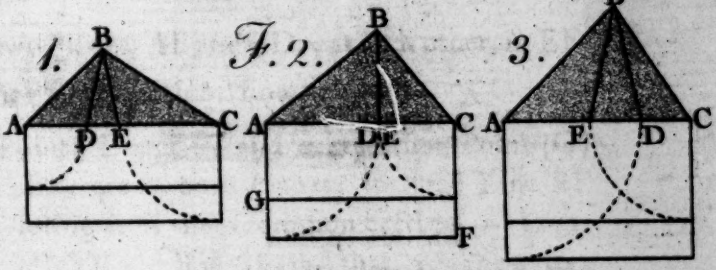

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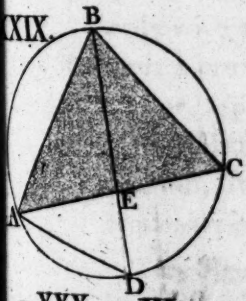

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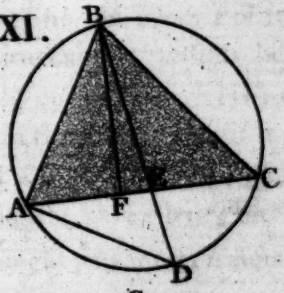

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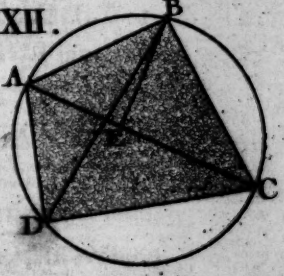

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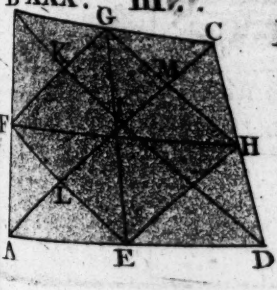

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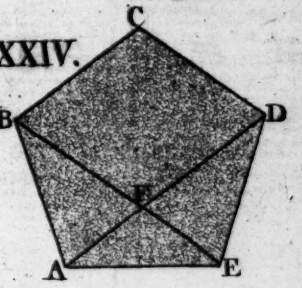

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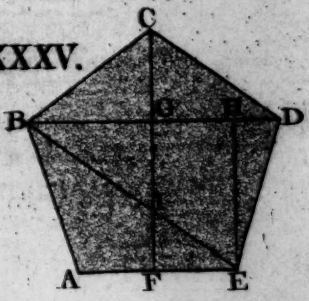
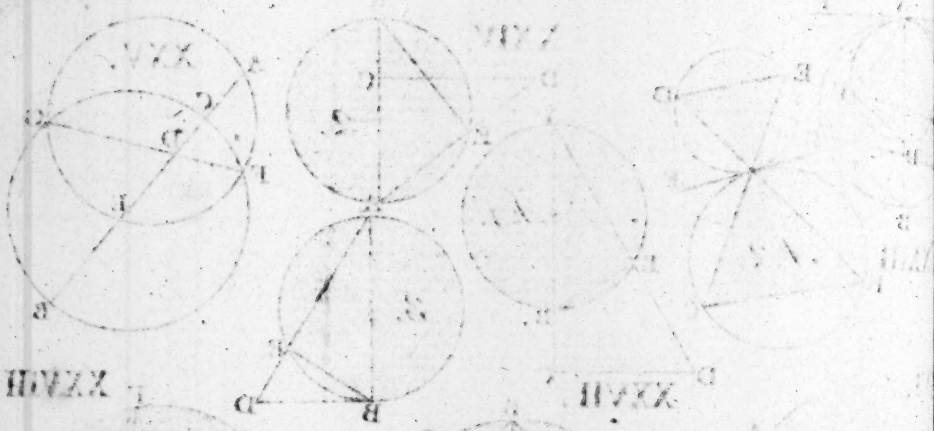
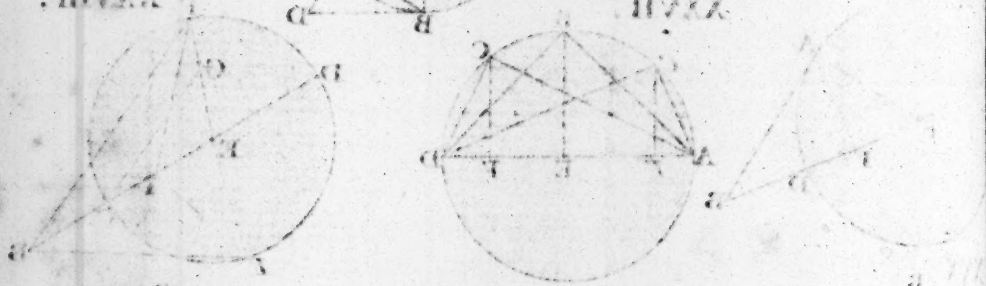
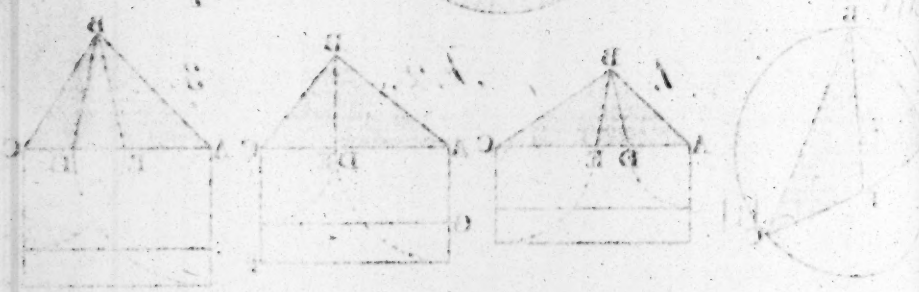
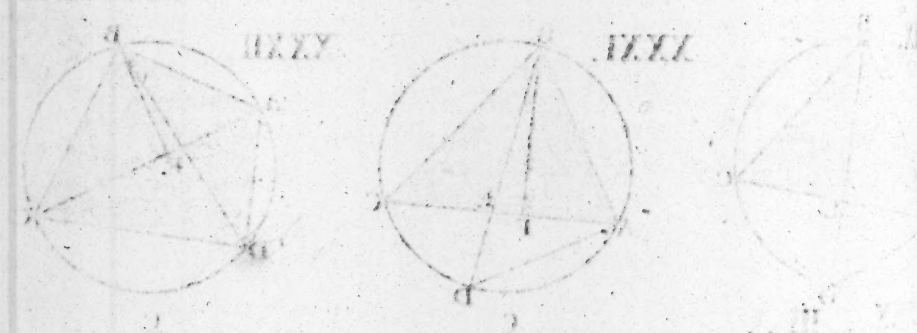
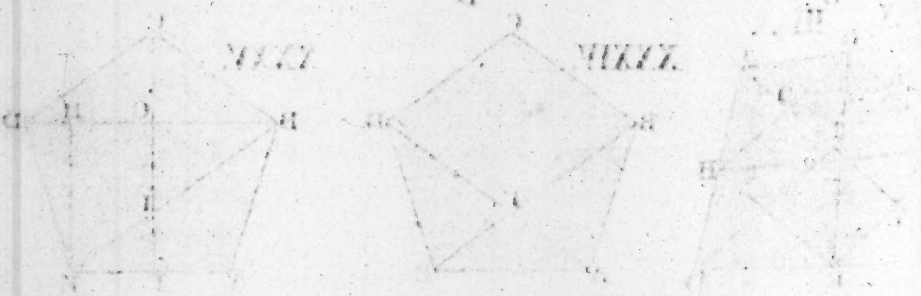












Therefore it is, in ſome meaſure, neceſſary (by way of Poſtulate) to requeſt, that one Line be equal to another; or at right angles, i. e. perpendicular to another Line, or Plane, &c; that two Right lines contain an Angle, equal to that which two other Lines contain; which, in the Diagram, is, perhaps, either greater or leſs. Theſe things muſt be granted and underſtood to be ſo, or no Demonſtration, of what is premiſed, can poſſibly follow.

It is neceſſary that the young Student be appriſed of theſe preliminaries; otherwiſe (without a Tutor) he would, frequently, be perplexed and bewildered; and would, very probably, conclude, that there muſt be ſome miſtake, or error in the Diagram; for it is not eaſy, at firſt, to conceive, that one Line ſhould be perpendicular to another, when we ſee and know, that the Angles they make are not Right ones; and, being acute, that another Angle, apparently contained within that, and manifeſtly leſs, is a Right angle; i. e. it is, or muſt be, underſtood to be ſo.

THEOREM. I. 3. Euclid.

THE common Section of two Planes is a Right line.

Let the two Planes, AB and CD, cut each other, in EF.

I ſay, that EF is a Right line.

DEM. The Points E and F, and every other Point (G) in the Line EF, are in both Planes; for, the Line EF is in both, ſeeing it is their common Section. - Ax. 1. Therefore, it is a Right line; ſeeing that, a Plane agrees only with a Right line, in every Point; - Def. 6. 1. and, a curved Line cannot poſſibly be in two Planes.

For, ſuppoſe the curve Line, EHF, drawn through the Points E and F, in the Plane AB; if it was their common Section, it muſt alſo be in the Plane, CD; and conſequently it would not be a Plane, but a convex Surface.

This Theorem might paſs for an Axiom, being ſelf-evident.

THEOREM II. 4. Euclid.

IF a Right line ſtands at Right angles, in the Point of interſection of two Right lines; it will be perpendicular to the Plane, in which thoſe Lines are.

This might alſo paſs for an Axiom; but there is a peculiar elegance in the Demonſtration, which cannot be diſpenſed with.

Imagine the Right line AB, at right angles to the two Right lines, CD and EF, in the Point B of their common Section.

Then, AB will be perpendicular to the Plane CHE, in which the two Lines, CD and EF, are ſituated.

Make BC equal to BD (at pleaſure) and imagine BE, BF, alſo equal to BC or BD.

AB ſtanding perpendicular to them, imagine the lines AC, AD, AE, and AF, drawn to any Point, A, in AB.

Join CE and DF; and, through B, draw any Right line, GH, at pleaſure, cuting CE and FD, at G and H; and draw AG, AH.

DEM. Then, becauſe CB, BD, BE, and BF are all equal, and AB is common; alſo, the Angles ABC, ABD, ABE, ABF, are all ſuppoſed Right, therefore equal; the Right lines AC, AD, AE, and AF, are all equal; - 8. 1, wh. the Angle ACE=AEC; and ADF=AFD. - 9. 1. And, becauſe the Angle CBE=DBF (2. 1.) and CB, BE are equal, reſpectively, to BD, BF; CE=FD; 8. 1. conſequently, the Angle BCE=BDF, - ſame. Alſo, the Triangle CAE, being equilateral to FAD*, they are conſ. equiangular; & the Angle ACE=ADF. 7. 1.

Now, the Angle BCE=BDF; and CBG=DBH; 2. 1. and CB=BD; wh. the Triangle CBG=DBH; } 11. 1. and, conſequently, CG=DH, and BG=BH. }

Wherefore, in the Triangles CAG, DAH; AC, CG, are reſpectively equal to AD, DH, and contain equal Angles; conſequently, AG is equal to AH; - - 8. 1. wh. the Triangles AGB, BAH, are equilateral, to each other; (for BG, was proved, equal to BH, AG equal AH, and AB is common) conſequently, the Angle ABG=ABH,

Therefore, AB is perpendicular to GH; - Def. 10. 1. and, consequently, to every Right line, drawn through B, in the Plane C H E; and therefore, AB is perpendicular to the Plane, C H E. Q. E. D. - - - Def. 3.

COROLLARY. Proposition V. of Euclid.

If three Right lines meet in one Point, and a fourth stands at right angles to each, in that Point, the three Lines are in the same Plane.

If AB makes Right angles with CB, and BE, and also with BG, they are necessarily in the same Plane. - Def. 3.

If ABG be less than a Right angle, BG will be on this side the Plane, CHE; and, if it exceeds a Right angle, BG will fall on the other Side. But, ABG is a Right angle; therefore, BG is in the same Plane with BC and BE.

* Triangles are equilateral to one another, when the Sides of one are equal, respectively, to the Sides of the other.

THEOREM III. 6. Euclid.

IF two Right lines be perpendicular to the same Plane, they are parallel to one another.

This might well pass for an Axiom; for, it is evident, the two Lines may be in the same Plane, which is thus proved.

Let AB and CD be at right angles with the Plane BGD.

I say, that AB is parallel to CD.

Draw BD, through the Points, B and D, in which the Lines, AB and CD, cut the Plane. And, let BE and DF, be perpendicular to BD.

DEM. Because AB and CD are both perpendicular to the Plane BGD, they are at right angles with every Line in that Plane, drawn through the Points B & D, respectively; therefore, to BE and DF (2.) and, consequently, they

do not incline to the Plane BGD, on any Side; Def. 3. wherefore, a Plane may paſs through both Lines. Def. 4. But, the Angles ABD and CDB are Right. - Def. 3. Therefore, AB is parallel to CD. - - Th. 4 1.

If the Plane BACD be turned up, perpendicular to BGD, the thing is manifeſt; ſeeing, the Lines AB and CD will make Right angles with BE and DF, reſpectively.

According to Euclid (AD being drawn) the three Lines, AD, BD, and CD, are proved to be at right angles with DG, ſuppoſed perpendicular to BD; but, the proceſs, though ſtrictly true, appears very lame and contradictory in the Figure.

COROLLARY. The 8th Propoſition of Euclid.

If two Right lines be parallel, and if one of them be at right angles with ſome Plane, the other Line is alſo at right angles with the ſame Plane.

This, being the Converſe of the Theorem, is manifeſt.

THEOREM IV. 9. Euclid.

Right lines being parallel to the ſame Right line, which is not in the ſame Plane with them, are parallel amongſt themſelves; i. e. each to the other.

Let AB and CD be each parallel to EF;

I ſay, that AB is parallel to CD.

In EF, take any Point, G; at which Point, let GH and GI, be at right angles with EF, and cuting the Lines AB and CD reſpectively.

By the Premiſes, EF is not in the ſame Plane with AB and CD; therefore, raiſe up EF, out of the Plane, but parallel to it, till CD coincides with *GD*, or any other Line, cd, parallel to EF.

DEM. Becauſe GH and GI are at right angles with EF, EF is perpendicular to a Plane paſſing thro' thoſe Lines. 2. But AB and CD are both parallel to EF; wherefore, they are alſo at right angles with the Plane HGI, paſſing through GH and GI. - Cor. to Th. 3. Therefore, they are parallel between themſelves; by Th. 3.

When the three Lines are all in the ſame Plane, as in Theo. 5. Book 1ſt. it is manifeſtly an Axiom; and, in this Caſe, it amounts to little more; ſeeing that, a Plane may paſs through any two Right lines which are parallel to one another. - - Ax. 2.

THEOREM V. 10. Euclid.

IF two Right lines, cuting or meeting each other, be reſpectively parallel to two other Right lines, alſo meeting or cuting one another, not in the ſame Plane with the firſt, they ſhall contain equal Angles.

Let the two Right lines, AB and BC, be reſpectively parallel to DE and EF; viz. AB to DE, and BC to EF; and, ſuppoſe they are not in the ſame Plane.

I ſay, the Angle ABC is equal to DEF.

Take BA equal to ED, BC equal to EF, and draw the Right lines AD, BE, and CF.

DEM. Becauſe AB is equal and parallel to DE, - Hyp. AD is equal and parallel to BE; - - Cor. to 15. 1. and, for the ſame reaſon, BE is equal and parallel to CF; wherefore, CF is equal and parallel to AD. Ax. 3. 1 & Th. 4.

Join AC and DF, by Right lines.

Now, becauſe AD is equal and parallel to CF, AC is equal (and alſo parallel) to DF; as above.

Then, in the Triangles ABC, DEF, the three Sides, AB, BC, and AC, of the one, are reſpectively equal to DE,

EF, and DF of the other; wherefore, the Angles of one are alſo, reſpectively, equal to the Angles of the other. 7. 1. Therefore, the Angle ABC is equal to DEF; being oppoſite equal Sides, AC and DF. Q. E. D.

THEOREM VI. 14 Euclid.

Planes, to which the ſame Right line is perpendicular, are parallel to one another.

Let the Right line AB be perpendicular to both the Planes, CGD and FIE.

I ſay, thoſe Planes are parallel.

Imagine a Plane, CDEF, to paſs through the Line AB, cuting both Planes, in CD, and EF; which Lines, neceſſarily paſs through the Points A and B, in which the Line AB cuts thoſe Planes, reſpectively; by Ax. 1. and Theorem 1ſt.

At the Points A and B, let AG and BI, in the Planes CGDH, and FIEK, be perpendicular to CD and EF, reſpectively, and produce them to H and K.

Dem. Then, becauſe GH and IK are perpendicular to CD and EF, and thoſe Lines are in the Plane CDEF, GH and IK are perpendicular to that Plane; - Th. 2. (For AB being perpendicular to both CGD and FIE, it is ſo to every Line drawn thro' A and B; th. to GH and IK) wherefore, they are parallel to one another; - Th. 3. and, conſequently, the two Planes cannot meet; being produced, either way, towards G and I, H and K. Ax. 1.

Let other Lines paſs through A and B, in the Planes CGD and FIE, at right or equal angles with, and inclined the ſame way towards, GH and IK; as CD and FE.

Then, becauſe the Angle GAD is equal to IBE, and AG is parallel to IB; and thoſe Lines are perpendicular

to AB; CD is parallel to EF; - - - Ax. 10. 1.
conſequently, the Planes CGDH and FIEK cannot meet being produced towards D and E, or F and C.
Therefore CGDH and FIEK are parallel. - - Def. 2.

THEOREM VII. 15. Euclid.

If two Right lines, cuting or meeting each other, be parallel to two other Right lines, alſo meeting each other, not in the ſame Plane with the firſt two; Planes paſſing through each two Lines, reſpectively, are parallel.

Let, the Right lines AB and BC, meeting at B, be parallel, reſpectively, to DE and EF, meeting at E, and not in the ſame Plane with AB and BC.

I ſay, the Planes ACB, DFE, paſſing through thoſe Lines, are parallel.

From the Point B (in which AB meets BC) let BG be perpendicular to the Plane DFE; and, through the Point G, in which BG cuts that Plane, let GH and GI be drawn parallel, reſpectively, to ED and EF.

Dem. Then, becauſe BG is perpendicular to the Plane DFE, GH, and GI, make Right angles with BG. - Th. 2.
But, AB is parallel to GH, and BC to GI - - 5.
(for, GH and GI are reſpectively parallel to ED and EF);
And, BGH, BGI are Right angles; - - - Th. 2.
conſequently, ABG and CBG are Right angles. - 4 1.
Wherefore, BG is perpendicular to the Plane ACB; 2. 7.
and, conſequently, the Plane ACB is parallel to DFE, in which the Lines AB, BC, and DE, EF are ſituated. } 6.

THEOREM VIII. 16. Euclid.

IF two, or more, Planes, which are parallel, be cut by another Plane; their common Sections are parallel.

This Theorem might paſs for an Axiom, or be deduced from the 6th, as a Corollary; but it is eaſily demonſtrated.

Let the Planes ABC and DEF be parallel; and, imagine a Plane, ABF cuting them, in AB and DE.

I ſay, the Sections, AB and DE, are parallel.

DEM. If AB (being in the ſame Plane, ABF, with DE) be not parallel to DE, they will meet, if produced, towards one extreme or the other. - - - Def. 7.1.
But, AB is in the Plane ABC; and DE, in DEF. Th.1.
And, the Plane ABC is parallel to DEF. - - Hyp.
Therefore, ſince both Lines are in the Plane ABF, and are, alſo, reſpectively, in the Planes ABC, DEF, AB is parallel to DE; ſeeing they cannot go out of thoſe Planes, and conſequently can never meet. - - Ax. 1.

THEOREM IX. 18. Euclid.

IF a Right line be perpendicular to a Plane; every Plane, which paſſes through that Line, is alſo perpendicular to the Plane.

Let AB be perpendicular to the Plane CEDF, and let any Plane CAD, or EAF, paſs through the Line AB.

I ſay, the Planes, CAD and EAF, are perp. to CEDF.

DEM. Becauſe AB is perpendicular to CEDF, - Hyp.
AB does not incline to it on any Side. - - Def. 3.

But, AB is in the Plane CAD, and alſo in the Plane EAF; conſequently, thoſe Planes do not incline, on either Side, to the Plane CEDF; ſeeing that, every Line, BC, BE, BD, and BF, make Right angles with AB. - - Th. 2.
Th. the Planes, CAD, EAF are perp. to CEDF. - Def. 4.

COROLLARY. The 19th Propoſition of Euclid.

If two Planes, cuting each other, be perpendicular to the ſame Plane; their common Section is perpendicular to that Plane. The Converſe.

For, ſince the Planes CAD, EAF, cuting each other in the Line AB, are perpendicular to the Plane CEDF, AB is perpendicular to that Plane; for, it is in both the other.

COR. 2. The 38th Propoſition of Euclid.

If from any Point in a Plane, which is perpendicular to another Plane, a Right line be drawn, perpendicular to that other Plane; it will cut the other in the common Section of the two Planes.

For, ſince one Plane is perpendicular to the other (Hyp.) every Line drawn from any Point in one, perpendicular to the other, is wholly in the firſt Plane, as AB; and conſequently, it muſt cut the other Plane in their common Section, CD, or EF.

THEOREM X. 17. Euclid.

Right lines, being cut by parallel Planes, are cut proportionally.

Let the Right lines, AB and CD, be cut by the Planes X, Y, and Z, which are parallel between themſelves, in A, E, B, and C, G, D.

I ſay, AE is to EB, as CG to GD.

Join the Points A and C, B and D; and, AD being drawn diagonal-wiſe, cuting the Plane EG in F, draw EF and FG, joining the Points where they cut the Plane, Y.

DEM. Then, becauſe AB, AD, and BD are three Right lines, cuting one another, they are in the ſame Plane ABD; and ADC is alſo a Plane, by the ſame. - Ax. 4.

Now, ſince the Planes Y and Z are parallel, and they are both cut by the Plane ABD, their common Sections, BD and EF are parallel; and, for the ſame reaſon, AC is parallel to FG. - - - - Th. 8.

But, in the Triangle ABD, becauſe EF is parallel to BD, the Sides AB, AD are cut, as AE:EB::AF:FD. - 2.6.

And, becauſe FG is parallel to AC, as AF:FD::CG:GD.

Therefore, as AE is to EB, ſo is CG to GD. - Ax. 13. 5.

COR. Hence it is manifeſt, that any Number of Right lines, proceeding from one Point, are cut proportionally (from their common Point of Section) by parallel Planes.

For, ſince, DF:DA::DG:DC; by 2. 6. and 6. 5. conſequently, any other Right line, DH, is cut in the ſame Ratio, at H and I; ſo that, DI:DH::DG:DC.

THEOREM XI.

IF two Right lines are perpendicular to a Plane, (whether they are on the ſame Side, or on contrary Sides) and are both cut by another Right line, which alſo cuts the Plane; that Line will cut the Plane, ſomewhere in a Right line paſſing through the Points where the Perpendiculars cut the Plane; and, it will be cut in the ſame Ratio by the Plane, and the Perpendiculars, as the Perpendiculars by that Line; and the Right line, joining or paſſing through the Points where the Perpendiculars cut the Plane, will alſo be cut proportionally, by that Line and the Perpendiculars.

First; let the two Lines AB and CD (on the ſame Side) be perpendicular to the Plane aBf; cuting it in the Points B and D.

From any Point A, in AB, draw a Right line AC, cuting CD in C, and produce it, till it cuts the Plane; ſuppoſe at E.

Then, a Right line, BD, being produced, will paſs through the ſame Point, E; and they will, both, be cut, in the Points A, C, E, and B, D, E, in the Ratio of the two Perpendiculars; viz. as AB to CD.

Dem. For, becauſe AB and CD are both perpendicular to the ſame Plane, they are parallel (3.); wherefore, AC and BD are in the ſame Plane with them. - Ax. 3.
And, BD is the common Section of the two Planes.
Conſ. every Line (as AC) not parallel to BD, in the Plane AEB, will cut the other Plane, in the Interſection of both.
Th. AC, produced, will cut the Plane aBf, in BD produced.

2nd. Becauſe CD is parallel to AB, the Sides EA, EB, of the Triangle AEB, are cut proportionally; - - 2. 6.
wh. as EC: CA::ED: DB; conſ. EC: EA::ED: EB. 6.5.
i.e. EC: EC+CA::ED: ED+DB; viz. as CD: AB. 4.6.

Again. Let FG (on the other Side of the Plane) be perpendicular to the ſame Plane (aBf) with AB; they are parallel (3.) which is evident, producing either; as FG, to H.
Then, if AF be drawn, cuting AB and FG, it will cut the Plane aBf in the Line BG, joining the Points B and G.
And, they alſo, mutually, cut each other, in the proportion of AB to FG.

Dem. Becauſe AB is parallel to FG, a Plane may paſs through them both (Ax. 2.); conſequently, AF and BG are both in the ſame Plane with AB and FG. Ax. 3 and 4.

Wherefore, in the Triangles, EAB, EFG, the Angles at E being vertical, are equal (2. 1.) the Angles at B and G are right (2. 7.) therefore equal; - - - Ax. 10. 1. the Angles at A and F, are conſ. equal; and alſo, by 4. 1. wherefore, thoſe Triangles are ſimilar; - - - 4. 6. and therefore, as AB : FG :: AE : EF, and BE to EG.

Cor. Hence it is manifeſt, that if a Right line, ab, be drawn, at pleaſure, through the Point B, in the Plane a Bf; and (through G) if fG be drawn parallel to aB; aB and fG being made equal to AB and FG, reſpectively, or in the Ratio of AB to FG; then, a Right line, drawn through a and f, will cut BG in the ſame Point E, as before.

For, becauſe aB is parallel to fG, the Triangles EaB, EfG, are ſimilar; or, becauſe aB is parallel to fG, af and BG are cut proportionally, in E. - - - - Caſe 3. 2. 6.
And, becauſe the Triangles are ſimilar, aE : Ef :: BE : EG, and, alſo as aB : fG. - - - - 4. 6.
But, aB : fG :: AB : FG; conſ. aE : Ef, and BE : EG, :: AB : FG; and therefore, BG is cut in the ſame Point (E) as before.

THEOREM XII. 21. Euclid.

Every plane ſolid Angle is contained under plane Angles, which, taken together, are leſs than four Right angles.

Let ABCDE be a Section of the Planes AFB, BFC, &c. forming a ſolid Angle, at F; i. e. let ABCDE be the Baſe of a Pyramid, whoſe Vertex is F; ſuppoſed to be out of the Plane of the Baſe.

I ſay, all the Angles, AFB, &c. will, in that caſe, be leſs than four Right angles.

Dem. All the Angles AFB, BFC, &c. being in a Plane, are equal to four Right angles. - - - Cor. 2. 2. 1.

But, if F be raiſed out of that Plane, ſuppoſe equal to FG, perpendicular to AF, BF, &c. and AG, or BG, drawn, AG is longer than AF. - - - 12. 1.

For the ſame reaſon, all the Right lines, from each Angle, A, B, &c. of the Baſe, to the Vertex, at F, elevated perpendicularly, at F (eq. FG) are longer than BF, CF, &c.

But, if two Sides of a Triangle be, reſpectively, greater than two Sides of another Triangle, and the remaining Sides are equal, they ſhall contain a leſs Angle. - 14. 1.

Therefore, the Angles of the Triangles AFB, BFC, &c. being conſidered as raiſed out of the Plane ABCDE, are leſs, reſpectively, than thoſe Angles in the Plane; which, being equal to four Right angles, conſequently, the Angles raiſed out of the Plane, are *leſs* than four Right angles.

If the Triangles AGF, BGF, be turned up, on AF and BF, till FG, in one, coincides with FG in the other; and from the Point G, ſo elevated (FG being perpendicular to the Plane ABD) if Right lines are drawn to all the other Angles, C, D, and E; each two will contain a leſs Angle than thoſe in the Plane, to which they are inclined (as AGB than AFB) and are therefore, together, leſs than four Right angles.

THEOREM XIII. 20 and 22. Euclid.

If any two, of three given Plane Angles, be greater than the third, and the Lines containing the Angles all equal; and if thoſe three Angles are, together, leſs than four Right angles, they will form a ſolid Angle; and three Right lines, joining the extremes of the equal Sides, will form a Triangle.

Let ABC, ABD, and CBE be three Plane Angles; of which, any two are greater than the remaining Angle; and, the Sides, AB, BC, BD, and BE are all equal; alſo, the three Angles, together, are leſs than four Right angles.

Then, a ſolid Angle may be formed of thoſe three Angles, ABC, &c. Alſo, of the three Lines, AC, AD, and EC, a Triangle may be formed.

Turn over the Triangles ABD and CBE, on AB and BC, till the two Sides BE and BD coincide.

DEM. Becauſe they are leſs than four Right angles (the deficiency being the Angle EBD) it is manifeſt they will form a ſolid Angle, at B, their common Vertex; the two leſſer Angles, ABD, CBE, being together greater than ABC.

For, it is evident, if they were leſs, as AB*F*, CB*G*, the Sides B*F* and B*G* would not meet each other; and conſ. with ABC, they cannot form a ſolid Angle, at B.

Therefore, a ſolid Angle formed of three Plane Angles, any two are, together, greater than the third. Q.E.D.

Or, a ſolid Angle formed of any number of Plane Angles, any one muſt be leſs than all the other, added together.

Notwithſtanding, B*F* and B*G* being equal to AB & BC reſpectively, altho' the Angles (AB*F*, CB*G*) are, together, leſs than ABC, A*F* & C*G*, together, may be greater than AC; and conſequently, a Triangle may be formed of the three Lines AC, A*F*, and C*G*; and therefore, of AC, AD, and CE, which are greater than A*F* and C*G*. - 13. 1.

N. B. The laſt part of this Theorem, which (according to Euclid) ſeems to imply, that, unleſs two of the plane Angles are greater than the third, a Triangle cannot be formed of the three Lines, joining the extremes of equal Sides; whereas, the contrary is manifeſt. It appears however, to me, of ſo little conſequence, as not worthy of being made a diſtinct Theorem, of itſelf.

N. B. 2. To make a Solid Angle of three given Plane Angles (the 23rd Prop. of Euclid) is to place them ſo, together, that the Vertices, of all the three, meet in one Point; and the Sides, which contain the plane Angles, coincide, two and two in one Line, as in the Figure. (See Defin. 6.)

With this Problem, Euclid's Expoſitors take up four full Pages; but, for what purpoſe I cannot deviſe.

THEOREM XIV. 24. Euclid.

If a Solid be bounded by ſix Planes, of which two and two are parallel; the Planes are all Parallelograms, and thoſe oppoſite are equal and ſimilar.

Let ABEG be a Parallelopiped. (See Def. 9.)

I ſay, the Planes ABCD, DEFG, BCED and ADFG, alſo, ABDG and DCEF, are Parallelograms; and each two are, reſpectively, equal and ſimilar.

Dem. The Planes, AC, GE, being parallel, and both cut by the Planes BG, and CF, alſo parallel; their Sections, AB and DG, CD and EF, are parallel amongſt themſelves. - - - - - by Th. 8.

But, they are alſo cut by the parallel Planes, BE and AF; their Sections, AD, BC, DE, and FG, are conſequently parallel.

Therefore, the Figures formed by thoſe Sections are Parallelograms, for their oppoſite Sides are parallel. Def. 33

2nd. Becauſe AB is parallel to GD, and AD to GF, and are in parallel Planes, the Angle BAD is equal to DGF; wh. the oppoſite Angles, BCD and DEF, are equal. 15. 1. But AB=CD, and alſo to DG; and DG=EF - ſame; conſ. they are all equal amongſt themſelves. - Ax. 3. 1. Alſo, AD, BC, DE, and GF are equal amongſt themſelves; therefore, the Parallelogram ABCD=DEFG, and ſimilar; alſo, BE=AF, and BG to CF, and reſpectively ſimilar.

THEOREM XV. 25. Euclid.

IF a Parallelopiped be cut into two Parallelopipeds, they will have that Proportion to each other, as their Baſes, i. e. as the Parts of thoſe Planes, cut by the other; or, as the Segments of the Sides.

This Theorem is manifeſtly an Axiom.

For, ſince the Sections made by parallel Planes are equal, every where, which is manifeſt, ſeeing the oppoſite Planes are equal and ſimilar; conſequently, if each Part be again cut into Parts, by parallel Sections, which are equal amongſt themſelves (as in Fig. 1.) the Demonſtration follows from the 9th Axiom of the 5th Book.

Or, being incommenſurable, they may be divided into an equal Number of Parts, (Fig. 2.) and demonſtrated as Theorem 1ſt. 6.

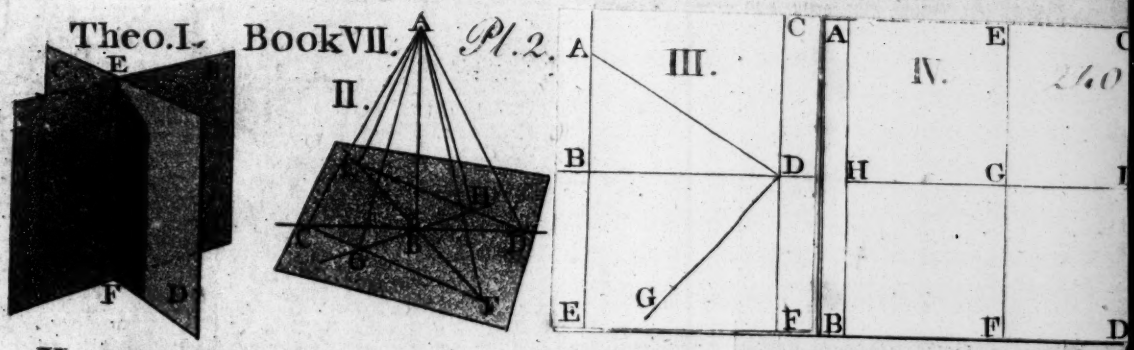
If the Baſe (AKMC) or Sides (AC, or DF) are biſected, the Parallelopiped is biſected; and conſequently, in whatever Ratio one is divided, (by a Plane BEHL) the other is the ſame.

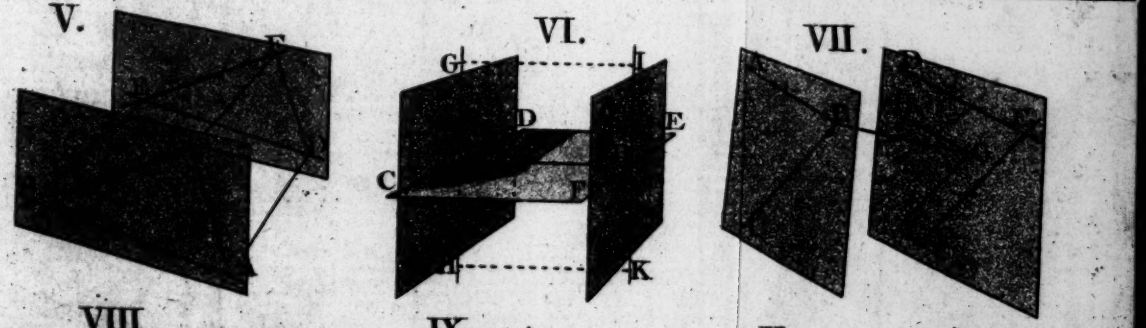
DEM. Let the Baſe AM, or any other Plane, AF, be cut by a Right line, parallel to its Sides, AK or AD; and imagine a Plane (BH) to paſs through BE or BL, parallel to AG; then, AE : EC, or AL : LC :: AB : BC. Th. 1. 6. And, ſeeing the oppoſite Planes, DGIF and GM, are equal and ſimilar, to AKMC, and AF, reſpectively; and are all cut by the Plane BEHL, parallel to AG and CI; conſequently, the Solid AGHB : BHIC :: AE : EC, or, AL : LC, i. e. as AB : BC. Q.E.D.

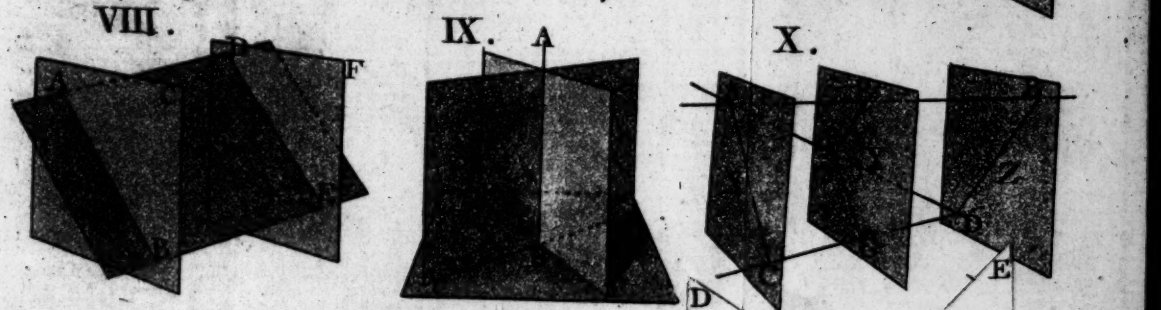
N. B. The two following Propoſitions, 26th and 27th of Euclid, are Problems; which, I do not conceive to be at all neceſſary, or elementary.

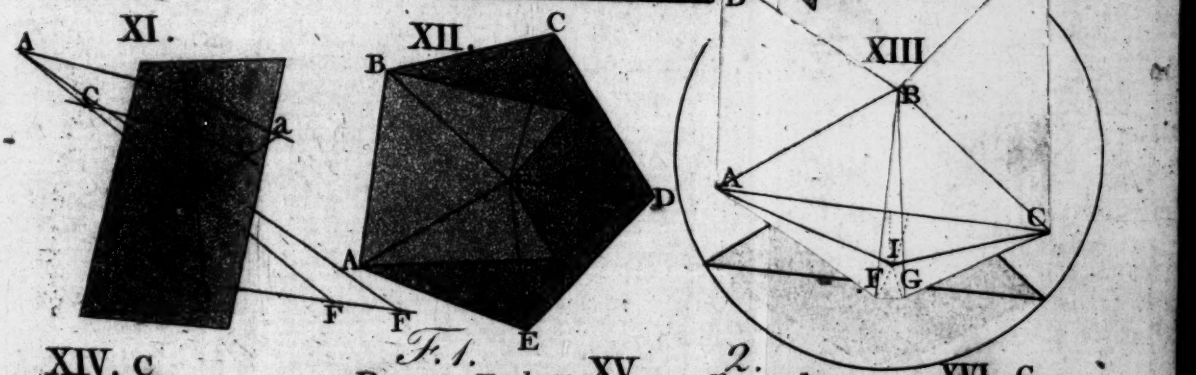
To make Solids is rather a mechanical Proceſs than geometrical; 'tis ſufficient, for the purpoſe of Demonſtration, to ſuppoſe that one ſolid Angle, or Solid is equal to another; for, an attempt to make them really ſo, on a Plane, is inconſiſtent; and, unleſs to delineate them perſpectively ſo, abſurd.

Theo. I. Book VII. Pl. 2.









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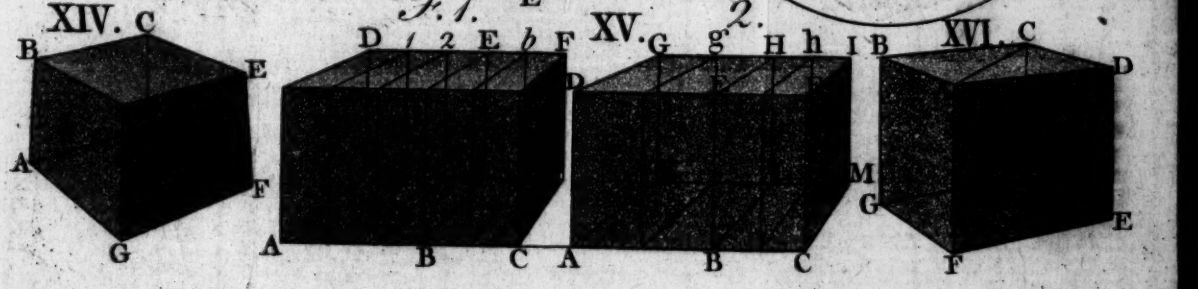

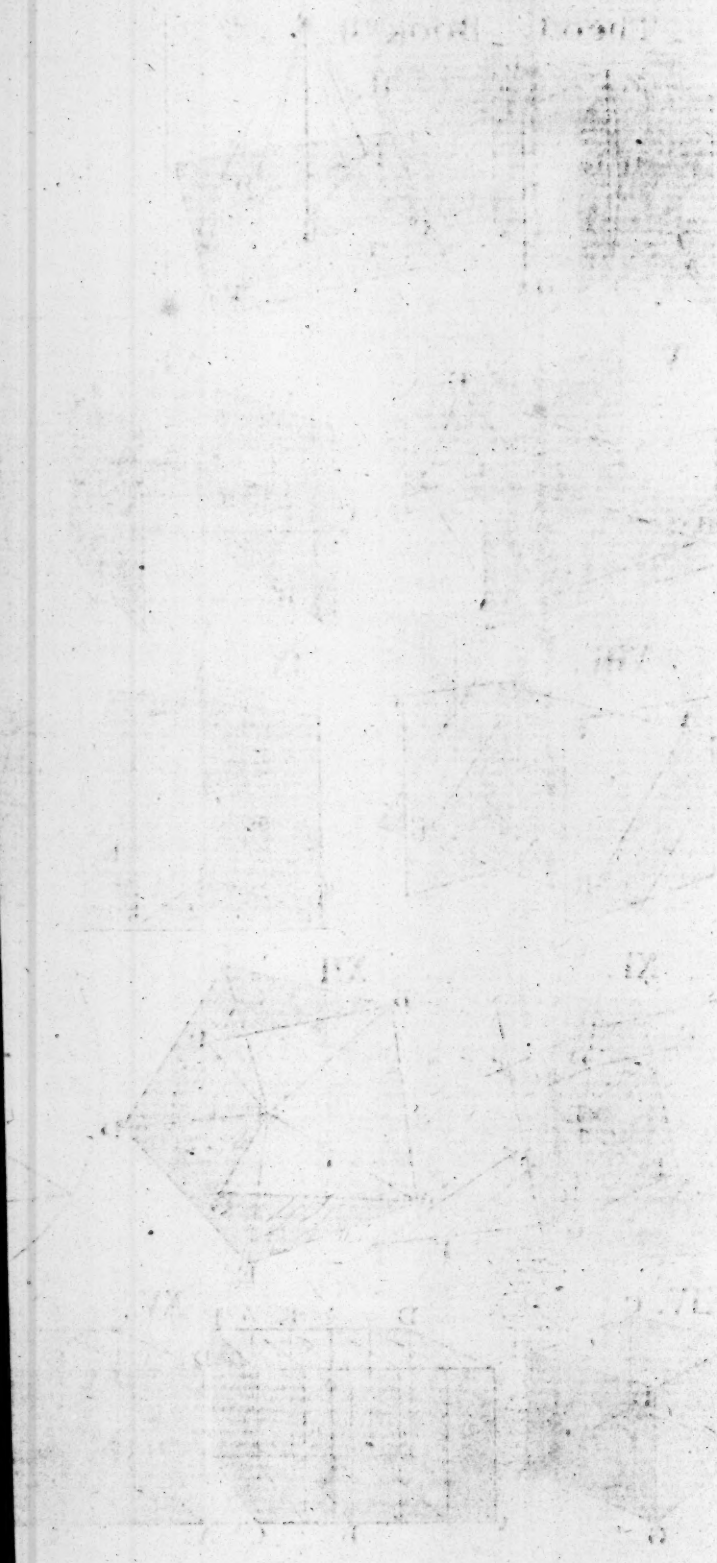

THEOREM XVI. 28. Euclid.

If a Parallelopiped be cut by a Plane, paſſing through the Diagonals of two oppoſite Planes, it will be cut in two equal and ſimilar Parts.

Let ABDF be a Parallelopiped, and AC, FH, Diagonals of the oppoſite Planes, ABCD and EFGH.

Then, becauſe AF and CH are both parallel to BG, they are parallel between themſelves (Th. 4.); wherefore, AF and CH are in the ſame Plane. - - Ax. 3.

I ſay, the Parallelopiped, ABDF, is biſected by the Plane ACHF.

Dem. For, the Triangle ABC = ADC; FGH = FEH. 15. 1.
And the Parallelogram ABCD is equal to EFGH; Th. 14.
wh. the Triangles are all equal amongſt themſelves.
But, the Parallelogram BF = CE, and AE = BH; - 14.
wherefore, the Priſm FBH is equal to AEC. - Ax. 5.
(For, they are contained within the ſame number of Planes which are equal, and ſimilar to each other, reſpectively)
Therefore, the Plane, ACHF, paſſing through oppoſite Diagonals, (AC and FH) biſects the Parallelopiped, ABDF.

Otherwise.

By the parallel motion of the Parallelogram ABCD along the Right Lines AF, DE, &c. the Parallelopiped is ſuppoſed to be generated. (See Note, Def. 8.).

Wherefore, ſince the Triangles, ABC, ACD, are equal, and the Priſm, FBC, CEF, are deſcribed in equal Time, that is, by the ſame motion of both Triangles together, conſequently, they are equal.

The Definition of a Priſm, as it is given by moſt Authors, is ſomewhat obſcure and unſatisfactory, being left to imagine, that any one of the Planes may be its Baſe; we may as reaſonably con-

clude, that any Plane of a Pyramid may be its Baſe, but it is never ſuppoſed ſo, unleſs they are all Triangles; nor in a Priſm, except they are all Parallelograms. For, to ſuppoſe the two equal, ſimilar, and parallel Planes, in a Priſm, Triangles or Poligons, and any other Plane to be its Baſe, is inconſiſtent, and abſurd.

Hence, the 40th Propoſition of Euclid, which, in reality, is but a Corollary to this Theorem, is an abſurd Propoſition.

Becauſe, I do not conceive, that, unleſs a Priſm be alſo a Parallelopiped, its Baſe can be a Parallelogram.

THEOREM XVII. 29, 30, & 31. Euclid.

Parallelopipeds having the ſame Baſe,* or equal Baſes, and the ſame Altitude, are equal to one another.

Firſt; let the two Parallelopipeds, AHFC and AIGC, have the ſame Baſe, ABCD; and, let their oppoſite Faces have a common Side, GH.

I ſay, the Parallelopiped AHFC is equal to AIGC.

DEM. The Parallelopiped AHFC, being cut by a Plane, CGHD, through the oppoſite Diagonals, DH and CG, is cut into two equal Parts CAG=DFH. - Th. 16.

And, for the ſame reaſon, the Plane AHGB biſects the Parallelopiped AIGC; in AIG equal AGD.

But, AGD+DFH=AGD+AIG. - - Ax. 6.1.

Therefore, the Parallelopiped AHFC=AIGC. Q.E.D.

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* By the Baſe, of a Solid, literally, is underſtood the Plane, or Face, on which it is ſuppoſed to reſt; but 'tis not always confined to it in Geometry. For, when a Face of various Parallelopipeds are in one Plane, or in parallel Planes; whether they coincide with each other, entirely, or are apart, and diſtinct from each other (no regard being had to the poſition of the Plane they are in) the upper Face is ſometimes called the Baſe.

N. B. Parallelopipeds, having the ſame Altitude, are ſuppoſed to be contained between the ſame parallel Planes; i. e. two of their oppoſite Faces, in each Solid, are in the ſame Plane; or, that there is the ſame perpendicular diſtance between them.

Secondly. Let the Parallelopipeds, AHFC and AMKC, have the ſame Baſe, ABCD; their oppoſite Faces, EFGH and IKLM, are in the ſame Plane, and have not a common Side. AMCK is equal to AHFC.

Let the ſpace, GHIK, between the upper Faces, be equal to the Baſe ABCD; conſ. equal to EFGH, equal IKLM. - - - - Th. 14.

DEM. The Parallelopiped AHFC is equal to AIGC; proved. And, for the ſame reaſon, AIGC is equal to AMKC. Th. the Parallelopiped AMKC is equal to AHFC. Ax. 3. 1.

Suppoſe GHIK, either greater or leſs than EFGH. The Triangles, AMH, DIE, are congruous; - 7. 1. (for AM is equal to DI, MH = IE, and AH = DE; by 15. 1.) and the Sides, AB, DC, EF, GH, IK, & LM, are equal; 14. the Par. AG = DF; ABLM = IKCD; & EFKI = MLGH; conſ. the Solid AGLM is equal to DFKI. - Ax. 5. Now, if the Solid, aIKGb (common to both) be taken away, AMKba is equal to DaGC. - - - - Ax. 7. 1. And, if the common Solid AabCD be added, to both; the Parallelopiped, AMKC, is equal to AHFC. Ax. 6. 1.

The ſecond Part, viz. when they have equal Baſes only, and have not one common to both, is readily proved.

Let the Baſe, ABCD, of the Parallelopiped AHFC, be equal to NOPQ, of the Par. NMKP; and ſimilar. Then, the Parallelopiped, AHFC, is equal to NMKP.

Let them be ſo ſituated, that the Planes AE, NI, are in the ſame Plane, DEMN; draw AM, BI, BL, and CK.

DEM. Now, becauſe MI is parallel to AD, and LK to BC, and alſo equal, AMID and BLKC are Parallelogms. C. 15. 1. And, for the ſame reaſon, AMLB and DIKC are Parallelograms; and they are, alſo, parallel and ſimilar, reſpectively;

wherefore, AMKC is a Parallelopiped. - - Def. 9.
(For, AM is parallel to DI; and ML to IK; wherefore, the Angle AML is equal to DIK, &c. - by Th. 5)

But, the Parallelopiped AHFC has the ſame Baſe with AMKC; and they are contained within parallel Planes; wherefore, they are equal Parallelopipeds. - - proved.

Alſo, the Parallelopiped NMKP, has the ſame Baſe with AMKC (i. e. the Plane or Face IKLM is common to both) conſequently, they are equal Parallelopipeds. - ſame.

But, the Parallelopiped AMKC is, alſo, equal to AHFC.

Therefore, the Par. NMKP, is equal to AHFC - Ax. 3. 1.

After the Proofs already given, it may, by ſome, be thought unneceſſary to add more; but, there are other Caſes, in which there is abundantly more room to diſpute the Affirmation, than in the preceding; and, as this Theorem is very eſſential, it cannot be too much enforced.

CASE II. When no more than two Faces of the Parallelopipeds (i. e. two in each) are in the ſame parallel Planes.

Let AGHD and FHIL be Parallelopipeds; whoſe Baſes, ABCD and IKLM, are equal, and in the ſame Plane; and the Face, EFGH, common to both; but have no other Plane common.

I ſay, that the Parallelopiped AGHD, is equal to FHIL.

Produce AD and BC, IM and KL, (being in the ſame Plane) they will cut each other, in NOPQ, which is a Parallelogram; equal and ſimilar to ABCD, and IKLM.

DEM. Becauſe EFGH is common to both Parallelopipeds, and becauſe the oppoſite Planes are equal and ſimilar, AC is ſimilar to LI; and, being ſimilarly poſited, IK, LM, are parallel to AD and BC; and IM, KL, to AB and DC; therefore, NOPQ, being between the ſame Parallels, is conſequently equal and ſimilar to AC, & LI, 4. & 18. 1. and conſequently to EFGH; and alſo ſimilarly ſituated.

Join EO, FP, GQ and HN.

Then, PFHN is a Parallelopiped; which, being equal to each of the other, AGHD and FHIL (part 1ſt) they are conſequently equal between themſelves. Ax. 3. 1.

Alſo; if the Parallelopipeds, ASFD and FHIL have their Baſes equal; and have no Face common; being between the ſame parallel Planes, AOI and RGE, they are equal Parallelopipeds; which is evident, from the foregoing; each being equal to PFHN.

CASE III. When the Parallelopipeds have no Face common nor ſimilar Figures.

Let the Parallelopipeds AOMD and EMKG, be between the ſame parallel Planes, ARH and NQI; conſequently, they have equal Altitude; and, let the Baſe, ABCD, be equal to the Baſe, EFGH, of the other, which are not ſimilar Figures.

I ſay, the Parallelopiped, EMKG, is equal to AOMD.

DEM. Let the Sides NM and ML be ſo placed, at the common Angle M, that NML is a Right line.

Compleat the Parallelopiped, DMKT; having a common Side, DM, with AOMD.

The Parallelopiped DMKT is equal to EMKG; Caſe 2nd. (For they have the ſame Altitude, and IMLK is common)

Produce KI and TU, PM & CD, meeting at X and V. Draw LY and SZ, parallel to MX and DV, cuting IK and TU, at Y and Z. Join VX and YZ.

Then the Parallelopiped VMYZ=UMKT; - 1ſt. conſ. the Parallelopiped, VMYZ=HMKG. Ax. 3. 1.

Produce OP and YL, BC and ZS, meeting at Q and R; and, joining QR, there is formed a Parallelopiped DPQS; equiangular to both the Parallelopipeds, AOMD and VMYZ; conſ. they are equiangular amongſt themſelves.

Now, as the Parallelogram IMLK=NOPM (Hyp.) XMLY (eq. IMLK, 18.1.) is equal to NOPM, Ax.3.1. conſequently, (the Angle PMN being equal XML, 2.1.) as NM:ML::XM:MP. - - - - - - - by 8. 6.

But, the Parallelopiped AOMD : DPLS :: NM : ML
And, the Parallelopiped VMYZ : DPLS :: XM : MP; } 15.

Now, the Parallelopipeds AOMD and VMYZ having an equal Ratio to DPLS; are therefore equal; Ax.5.5.

But, the Parallelopiped HMKG is equal to VMYZ, above.

Therefore, HMKG, is equal to AOMD. Q.E.D.

COROLLARY. The 33rd Propoſition of Euclid.

Parallelopipeds having equal Altitudes, have that Ratio to each other as their Baſes.

For, ſince by this Theorem, all Parallelopipeds, having equal Baſes and equal Altitudes are equal; and, by the 15th, if a Parallelopiped be cut in two, by a Plane, parallel to *its* Planes, the two Parts have that proportion to each other as their Baſes; conſequently, all Parallelopipeds, having equal Altitudes, are to each other as their Baſes.

COR. 2. Parallelopipeds, having equal Baſes, have that Ratio to each other as their Altitudes.

For, having equal Baſes and equal Altitudes, they are equal; conſequently, if the Altitude of one, be half, or a third part, &c. of another; or, in whatever Ratio the Altitude of one Parallelopiped is to the other, whoſe Baſes are equal, they are in that Proportion, to each other, as their Altitudes.

THEOREM XVIII. 33. Euclid.

THE Proportion of ſimilar Parallelopipeds, to one another, is the triplicate Ratio of their correſponding Sides.

Let ADK and a c d be ſimilar Parallelopipeds; in which, let AB, ab, BC, bc, and BD, bd be correſponding Sides, in each, reſpectively.

I ſay, the Proportion, which ADK has to a c d, is the triplicate Ratio of AB to ab, BC to bc, or BD to bd.

In the Solid ADK, produce the Sides, AB, CB, & DB.

Make BE equal to ab, BF equal bc, and BG equal bd; draw E H and F H, reſpectively, parallel to BF and BE; and compleat the Parallelopipeds GH, DH, and AFD; equiangular to A D K.

DEM. Becauſe BH is a Parallelogram (by Conſtruction) and the Angle EBF=ABC (2.1.) EBG=ABD, and FBG=CBD; alſo, BE=ab, and BF to bc; BH is equal and alſo ſimilar to ac; for, ac is ſimilar to AC, by Hyp.

And, for the ſame reaſon, G E is equal and ſimilar to a d (which is ſimilar to AD); alſo, FG is equal and ſimilar to c d (ſimilar to CD) conſequently, the Parallelopiped GH, is equal and ſimilar to a c d (ſimilar to AKD).

Now, AB:ab::BC:bc, and as BD is to bd, Def. 11.7. and, the Parallelopiped GH is equal and ſimilar to a c d; BE=ab, &c. wh. as AB:BE::CB:BF, and DB:BG.

But, the Parallelopiped, A F D and D F H, have equal Altitude, with A D K; wherefore, they have that Proportion to each other as their Baſes;

i.e. the Solid ADK : AFD :: Par. AC : AF; i.e. as CB to BF } Cor. 1.
and, - - AFD : DFH :: Par. AF : FE; i.e. as AB to BE } of 17.
alſo, - - DFH : GEH :: Par. DE : GE; i.e. as DB to BG } & 1.6.

Wherefore, the Parallelopiped ADK : AFD : DFH : GEH :: for the Ratio of CB to BE, of AB to BE, and of DB to BG, is the ſame, by Hypotheſis.

Therefore, the Ratio of ADK to a c d (equal GH) is triplicate of ADK to AFD; i. e. of CB to BF (eq. bc) of AB to BE (eq. ab) or, of DB to BG (eq. bd). Def. 12.5.

THEOREM XIX. 34. Euclid.

The Baſes and Altitudes, of equal Parallelopipeds, are reciprocally proportional.

Let ABD and FEG be equal Parallelopipeds.

If the Baſe of one be equal to the Baſe of the other, their Altitudes are alſo equal (Th. 17.); becauſe the Parallelopipeds are equal, by Hypotheſis. Let them be unequal.

Being acute-angled and unequal, let them be ſo placed together, that the Angle C, of one, touches the Side EF, of the other.

Produce the Plane, BC to H, parallel to FG; and, through C, let EF be drawn, perpendicular to their Baſes, AD, FG (which are ſuppoſed to be in the ſame Plane). EF and CF are the Altitudes of the Parallelopipeds.

I ſay, as AD is to FG, ſo is EF to CF, reciprocally.

DEM. ABD : FCG :: AD : FG, their Baſes. Cor. 1. 17.
And, FEG : FCG :: EF : CF, their Altitudes. Cor. 2. 17.
But, ABD is equal to FEG; by Hypotheſis.
Therefore, ABD : FCG, i. e. AD : FG :: EF : CF
i. e. as the Baſe AD, of one, is to FG of the other;
ſo is EF, the Altitude of the laſt, to CF, of the firſt.

THEOREM XX.

Parallelopipeds, whoſe ſolid Angles are equal, have that Proportion, to one another, which is compounded of the Ratio of their Sides.

Let ADK and BH be equiangular Parallelopipeds; and let them be ſo placed together, at the equal Angles, B, that the Side AB, of the one, is in the ſame Right line with BE, of the other, and CB with BF; conſequently, the two Sides BD and BG are in one Right line.

It is not material which Plane, EF, GE, or GF, is made the Baſe.

On EF, and BD, compleat the Parallelopiped DFI.

Take any Right line, V, at pleaſure; and let X be made to V, as BE to AB; and as BF is to CB, let Y be to X; alſo, as BG is to BD, let Z be to Y.

Then, as V is to Z, ſo is the Parallelopiped, ADK to BH.

Dem. The Solids, ADK and DFI, have equal Altitudes; wherefore, they are to each other as AC to EF; C. 1. 17. that is, as the Ratio which is compound of AB to BE, and CB to BF; i. e. as V to Y. - - - 11. 6.
And, the Solid DFI is to BH, as the Par. DE is to EG; 15; that is, as DB is to BG, i. e. as Y to Z. - - 1. 6.
But, ADK : DFI :: V : Y, and DFI : BH :: Y : Z.
Therefore, the Parallelopiped ADK : BH :: V : Z; 9. 5. i. e. in the compounded Ratio of AB to BE, CB to BF, and of DB to BG. Q.E.D.

THEOREM XXI.

In equal ſolid Angles, contained by three Plane Angles; any two Planes, which contain equal Angles, have equal Inclination to one another.

Let B and F be two ſolid Angles, each contained by three Plane Angles; viz. ABC equal to EFG, ABD equal to EFH, and CBD equal to GFH.

I ſay, the inclination of ABD, or CBD, to ABC, is equal to the inclination of EFH, or GFH, to EFG; alſo, the inclination of ABD to CBD is the ſame, as EFH to GFH.

In AB take any Point, A, at pleaſure, and take EF equal AB; and, in the Planes ABC ABD, EFG EFH, draw AC and AD, EG and EH, at right Angles with AB and EF, reſpectively. Draw CD and GH.

DEM. The Angle ABC = EFG (Hyp.) and BAC = EFG, Con. and AB = EF; wh. the Triangles ACB, EGF are congruous; AC is equal to EG, and BC is equal to FG. - 11. 1.
Alſo, the Angle ABD = EFH (Hyp.) & BAD = FEH, Con. wherefore (ſeeing AB = EF) BD = FH, and AD to EH, ſame.
And, becauſe BC, BD, are reſpectively equal to FG, FH, and the Angle CBD is equal to GFH; CD = GH. 8. 1.
Laſtly, becauſe the Triangle ACD is equilateral to EGH, they are alſo equiangular; and, the Angle CAD = GEH, 7. 1
But, thoſe Angles are the Inclination of the Planes, ABD to ABC, and EFH to EFG. - - - Def. 5.
Therefore they have equal Inclination. Q. E. D.

After the ſame manner, it may be proved that any other two have equal Inclination; therefore, &c.

COR. Solid Angles, contained by three Plane Angles, of which, two and two (one in each) are equal, reſpectively, are equal to one another.

For, the Planes have equal Inclination to each other, reſpectively (by Theo.); therefore the Angles are equal.*

THEOREM XXII.

IN equiangular Parallelopipeds (which are not right angled) Perpendiculars, to equiangular Planes, are proportional to the Sides.

This Theorem, though very different, apparently, is the ſame, in Subſtance, as the 35th of Euclid.

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* Profeſſor Simſon ſays (Prop. B.) "and alike ſituated" making that a Condition of their equality, which it is not; for, it is impoſſible, that three Plane angles, equal reſpectively to three other Plane angles (containing a ſolid Angle) can be ſo placed, in forming a ſolid Angle, which will not be equal to the other.

For, they muſt neceſſarily have the ſame Inclination each to the other, reſpectively, however ſituated.

Let BFG, bfg be equiangular Parallelopipeds, and let the Angle A be equal to a; conſequently, B is equal to b.

Let AC and ac be perpendicular to the Baſes, containing the equal Angles, EBG, ebg; cuting them at C and c.

I ſay, that, AC is to ac, as AB to ab, and as BG to bg.

From C and c, let CD and cd, be perpendicular to BE and be, reſpectively; and join AD and ad.

DEM. Becauſe AC is perpendicular to the Plane EG, Con. the Plane CAD, paſſing through AC, is perp. to EG; 9. and, for the ſame reaſon, cad is perpendicular to eg. But, CD is perpendicular to BE; and cd to be. Con. wh. BE is perpendicular to the Plane CAD, and be to cad; conſ. AD is perpendicular to BE, and ad to be. Th. 2.

Then, in the Triangles BAD, bad, the Angles ADB, adb are right, therefore equal; - - - Ax. 10. 1. and the Angle ABD=abd, (Hyp.) conſ. BAD=bad; 10. 1. and therefore, as AB : AD :: ab : ad. - - Th. 4. 6.

But, the Angle ACD=acd (becauſe, AC is perpendicular to the Plane EG, and ac to eg, ACD, acd are Right angles) and, the Angle ADC=adc; - by 21. for, ADC is the Inclination of the Plane AE to EG; and, adc is the Inclination of the Plane ae to eg, Def. 5. wherefore, the Triangles DAC, dac are ſimilar; and therefore, as AC : AD :: ac : ad. - - Th. 4. 6. But, AD : AB :: ad : ab (above) Th. AC : AB :: ac : ab, 9. 5. and, by alternation, AC : ac :: AB :: ab. Alſo, as AB : BG :: ab : bg; and conſequently, AC : ac :: BG : bg.

THEOREM XXIII. 36. Euclid.

IF three Right lines are Porportionals; a Parallelopiped conſtructed on all three, is equal to an equi-

angular Parallelopiped, whoſe Sides are each equal to the Mean.

If the Parallelopipeds be right angled, that which is conſtructed under the Mean is a Cube.

Let the three Right lines X, Y, and Z, be Proportionals; as X is to Y, ſo let Y be to Z.

And, let ACD be a Parallelopiped, whoſe Side AB is equal to X, BC equal to Y, and BD equal to Z.*

Alſo, let EGH be a Parallelopiped equiangular to ACD.

Then, if the Sides EF, FG, and FH, are each equal to Y;* I ſay, the Parallelopiped ACD is equal to EGH.

DEM. Becauſe, X:Y::Y:Z, the Rectangle, under X and Z, is equal to the Square of Y, i.e. $X \times Z = Y\square$. Cor. to 9. 6.

But, Parallelograms, having equal Angles, are in the ſame Ratio as Rectangles, whoſe Sides are equal to the Sides of the Parallelograms, reſpectively; - - C. 2. 11. 6.

wherefore, the Parallelogram AD is equal to EH.

But, BC = FG; and conſ. their Altitudes are equal. 22.

Therefore, the Parallelopiped ACD is equal to EGH.

For they have equal Baſes, and equal Altitudes. Q.E.D.

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* In thoſe Figures (which are but repreſentations of Solids, on a Plane) if the Lines, BD, and FH, were made equal to Z, and Y, reſpectively, they could not poſſibly have that appearance, but they would appear much greater; becauſe they are ſuppoſed to recede from the Planes of the Faces, AC and EG.

Therefore, they are ſuppoſed to be equal, only; which is the ſame thing, and holds equally true, in the Demonſtration. If one be, the other is a neceſſary Conſequence.

Note. As the Ratio, between two ſimilar Plane Figures, is diſcovered by a third Proportional, ſo, between Solids, a fourth determines it. And, to deſcribe plane Figures, in any given Ratio to others, a Mean Proportional is found; ſo, in reſpect of Solids, two Means are requiſite.

For, ſince the Ratio between ſimilar Solids is triplicate of their correſponding Sides; conſequently, ſince AB : DH : BF : AD ::; and, if the Ratio of ſimilar Solids be as AB to AD, AB, and DH are in the Ratio of their correſponding Sides. Hence, ſimilar Solids may be conſtructed in any Ratio to one another.

THEOREM XXIV. 37. Euclid.

If four Right lines are Proportionals; ſimilar Parallelopipeds, and ſimilarly conſtructed on each Line, are alſo proportionals.

Let A, B, C, and D be Proportionals; as A is to B, ſo let C be to D.

And let the Parallelopipeds V, X, Y, and Z, be ſimilarly conſtructed, on A, B, C, and D.

I ſay, that, as V is to X, ſo is Y to Z.

Dem. Let E and F be ſo taken, as A : B : E : F ∴;
and, let G and H be taken, as C : D : G : H ∴.
Then, the Parallelopiped V : X :: A : F;
and the Parallelopiped Y : Z :: C : H. } - Th. 18.
But, as A : B :: C : D; wherefore, B : E :: D : G; and,
as E : F :: G : H (Ax. 13 5.); conſ. as A : F :: C : H. - 9.5.
Therefore, as the Parallelopiped V : X :: Y : Z. Q.E.D.

It is not neceſſary that the Parallelopipeds Y and Z ſhould be ſimilar to V and X; for, if V be ſimilar to X, and Y to Z, the Demonſtration is evidently the ſame.

Or, if V be not ſimilar to X, nor Y to Z; but, let V be ſimilar to Y, and X to Z; it ſtill holds true.

For, ſince A : B :: C : D, ſo A : C :: B : D (4.5.); conſequently, ſimilar Parallelopipeds, conſtructed on A and C, and others, not ſimilar to them, on B and D, will ſtill be Proportionals.

For, ſince the Parallelopipeds, on A and C, B and D, are, as V : X :: Y : Z, ſimilar; ſo V : Y :: X : Z, not ſimilar.
The ſame holds true, in ſimilar Solids of any kind.

ELEMENTS

OF

GEOMETRY.

BOOK VIII. The XII. of Euclid.

THE laſt Book, or Section, treats only of Plane Solids, and determines their proportions, and Relation to each other; in this, the Doctrine of Solids, bounded by regular curved Surfaces, as the Cylinder, Cone, and Sphere, are conſidered; the Solution of which is truly admirable.

IT muſt appear, to all who conſider it, before they have gone through ſome Book on the ſubject, a matter not merely of difficulty, but of abſolute impoſſibility, to aſcertain the true Area of the Surface, much leſs the ſolid Contents of a Sphere; to reduce ſuch a Solid to right angled and cubical Meaſure, with any degree of certainty, ſeems to be beyond the power of Art, or the reach of finite Knowledge.

And yet, nothing more is requiſite thereto, than the true meaſure of its Diameter, which may be acquired; at leaſt of any Sphere conſtructed by Man. In reſpect of the Earth, the Diameter can only be obtained by its Circumference. It is poſſible to come ſomewhat near the truth; but it is beyond all human abilities to aſcertain it, with abſolute certainty.

How much leſs, then, are we able to come at the true Diſtances, by which, the magnitudes of diſtant Planets are reduced to the ſame ſcale of Proportion; viz. by the known meaſure of the Earth. For, if that cannot be had, with certainty, how much leſs exact muſt the other be, from Parallaxes, on ſuch imperfect Data?

Although the many ingenious contrivances, by Inſtruments, curiouſly conſtructed for that purpoſe, are worthy of the greateſt praiſe, and redound highly to the Fame of their ſeveral Inventors; and notwithſtanding all other means, which, by Tranſits, &c. may ſtill approach nearer to the truth, they only evince, and prove the imperfection of all human Inventions, for the purpoſe; we muſt (or we may as well) then, remain ſatisfied, that we have acquired ſo much, as to anſwer various great and notable purpoſes, in Trigonometry, Navigation, &c. but, that we ſhall never be able, by any means, to come at the Truth is moſt certain.

The Affinity between the Pyramid and the Priſm, and the Proportion one bears to the other, are, in the firſt place, determined; by means of which, the Proportion between the Cone and Cylinder, is alſo inveſtigated.

Laſtly, of the Sphere; ſhewing the Proportion it alſo has to a Cylinder and Cone, as given by Archimedes, for, on that Head, Euclid is totally ſilent; the 12th Book only determines the Ratio one Sphere has to another, in the laſt Theorem. To obtain which, he has two Problems, the 17th and 18th, the latter, the moſt prolix of the whole; containing, in four or five full Pages, the moſt tedious deſcription of the Conſtruction of a Solid, which is not of any other uſe, whatever; and previous to it, Profeſſor Simſon alſo gives a preparatory Lemma.

The manner, in which the Ratio of one Sphere to another is here inveſtigated, and deduced from the 9th Theorem, is ſolid and convincing; and alſo ſhews, how the quantity of every Sphere is aſcertained, which the other does not; and therefore, what it does is of little uſe

DEFINITIONS.

FOR the Definition of a PYRAMID, ſee Def. 7th of the 7th; which, altho' it is not uſed there, is the firſt of Plane Solids (having the leaſt number of Faces) and as a PRISM (Def. 8th) includes all Parallelopipeds whatever, it could not be diſpenſed with, though of but little uſe in that Book.

SOLIDS bounded by curved Surfaces are as follow.

DEF. I. A CYLINDER is a SOLID, having two parallel Plane Surfaces, which are equal Circles; and a convex Surface, bounded, in length, by the circular Planes, and returning into itſelf in width, without bounds. As AB, or *AB*.

DEF. II. The BASES, of a CYLINDER, are the two circular Planes; A and B.

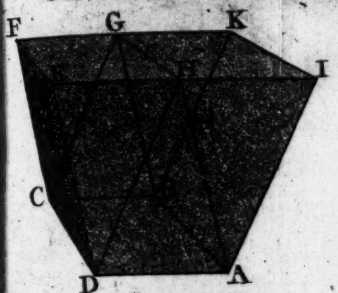
DEF. III. The AXE, or Axis of a CYLINDER, is a Right line paſſing through the Centers of its Baſes. As AB, or *AB*.

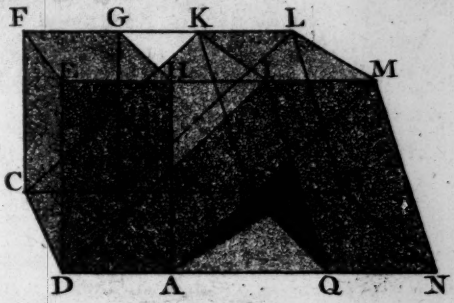
N. B. A CYLINDER may be conceived to be generated by the direct Motion of a Circle (as A, or *A*) along its Axe, always parallel to itſelf.

Let the Circle A, or *A*, be ſuppoſed to be applied, at its Center, to the Right line AB, or *AB*; and being moved, parallel to its firſt poſition, at A or *A*, to B or *B*; the Line AB or *AB*, being always in the Center; it will, in that motion, have deſcribed the Cylinder AB or *AB*.

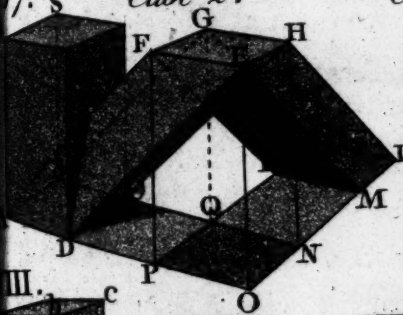
If any Right line as ab, or *ab*, be drawn in the Circle, cuting the Circumference, at a, or b, *a* or *b*; and, whilſt the Circle deſcribed the Cylinder (the Right line ab, or *ab*, being always parallel to its firſt poſition) the Point a or b, *a* or *b*, will deſcribe a Right line, ac, or bd, which is a SIDE of the Cylinder.

The two Extremes, a and b, *a* and *b*, will deſcribe parallel Lines, ac, bd, or *ac*, *bd*, conſ. equal; and cd, or *cd*, is parallel to ac, or *ac*; wherefore, abcd, or *abcd*, is a Parallelogram.

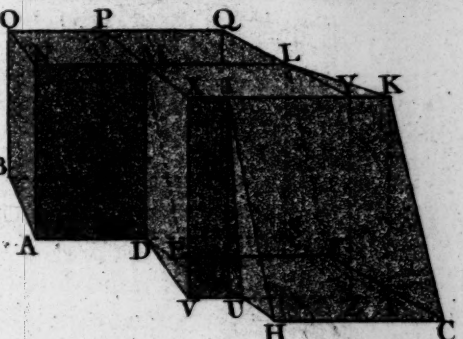




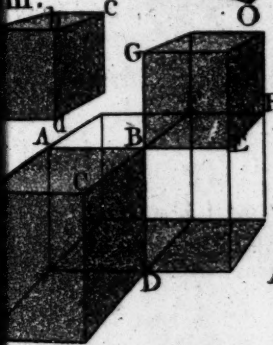
VII. Case 2.



C. 3.



III.



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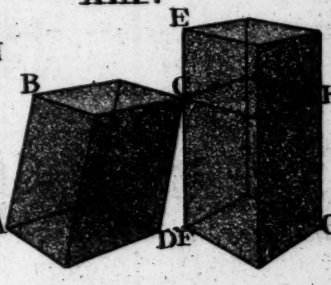

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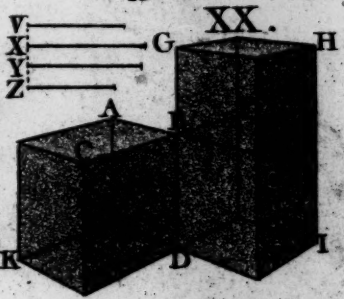

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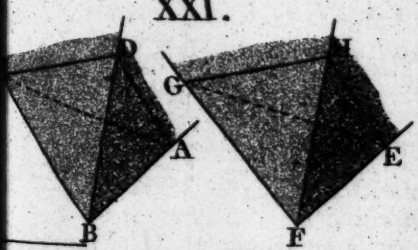

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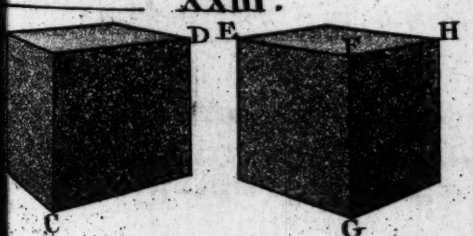

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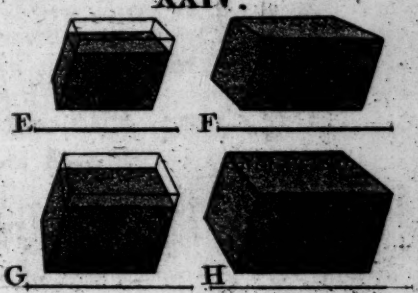
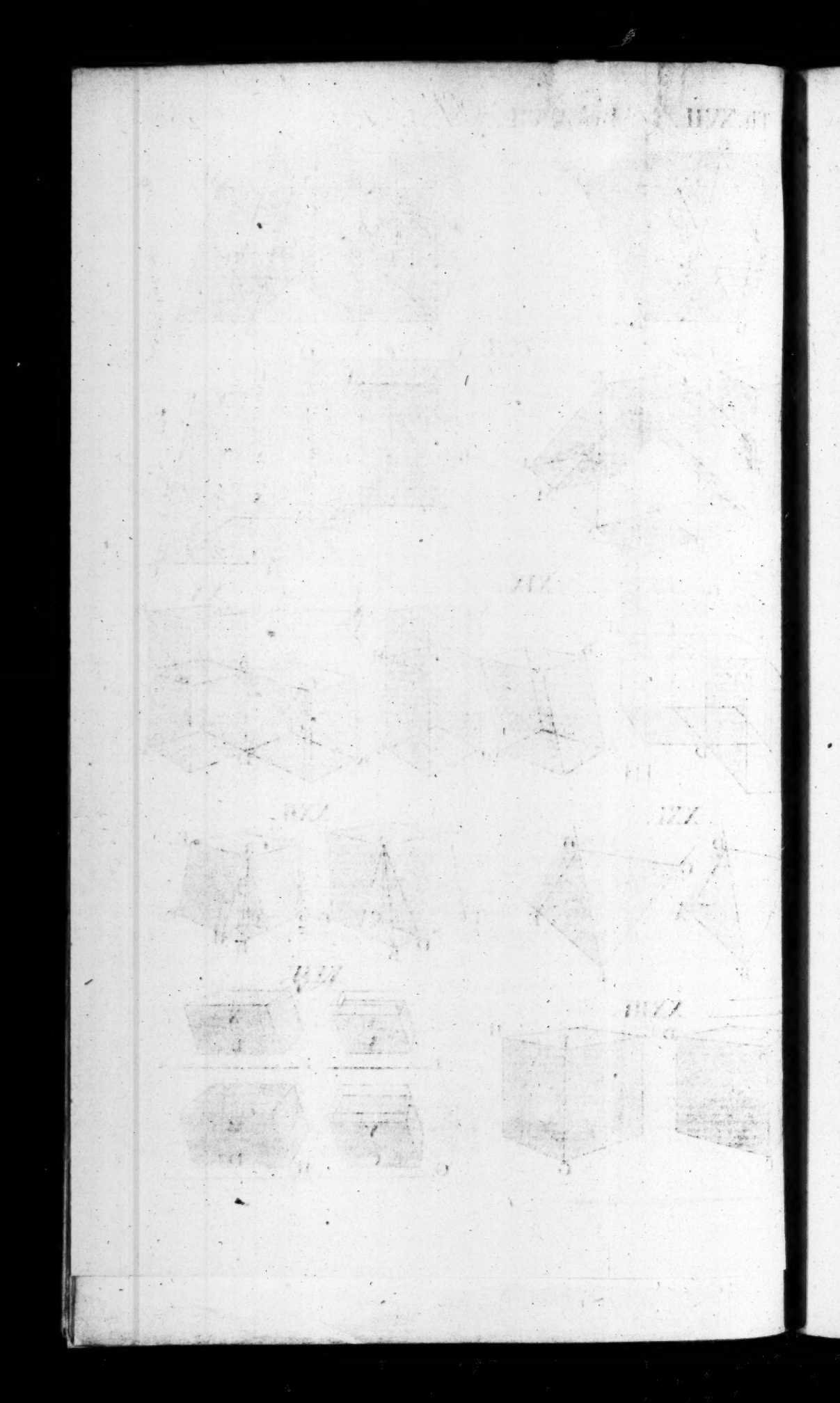




DEF. IV. A RIGHT CYLINDER is one whose Axis, AB, is perpendicular to the Planes of its Bases, A and B. (See the 1st Figure)

A Right Cylinder may be conceived to be generated by the revolution of a right angled Parallelogram, or Rectangle (CABC) on one Side (BD) which remains fixed, and is therefore called its Axe, or Axis; on which the Rectangle revolves, and on which the Solid, revolving, would appear at rest.

The two Sides, AB and CD, being equal, describe the equal Bases, whilst AC describes its convex Surface.

DEF. V. An OBLIQUE CYLINDER, is when its Axe is inclined to the Bases; (see Fig. 2).

DEF. VI. A CONE is a SOLID having but one Plane Surface, which is a Circle, and a convex Surface, returning into itself; continually varying in convexity, from the Circle which bounds it, till it terminates in a Point. As ACD, or *ACD*.

DEF. VII. The BASE of a CONE is the circular Plane CD, or *CD*.

DEF. VIII. The VERTEX (or Apex) of a CONE is the Point A or *A*.

DEF. IX. The AXIS, of a CONE, is the Right line AB, or *AB*; from the Vertex A, or *A*, to the Center B, or *B*, of its Base.

N. B. A Cone may be conceived to be generated thus:

If CD or *CD* be a Circle, and any Point, A, or *A*, be taken out of its Plane; then, if a Right line be applied from the Point A or *A*, to any Point C, or *C*; and being fixed, at A, or *A*, let it be revolved around the circumference of the Circle; in which revolution, it will describe the Cone ACD, or *ACD*.

DEF. X. A RIGHT CONE is when the Axe (AB) is perpendicular to the Plane of its Base.

A Right Cone may be conceived to be generated by the revolution of a right angled Triangle, ACB, on either Side contain-

ing the Right angle, as AB, which remains fixed; and is therefore called its Axe, or Axis, on which it revolves.

The other Side, BC, deſcribes the Baſe, in its revolution; whilſt the Hypothenuſe, AC, deſcribes the Surface of the Cone, and preſcribes its Bounds.

Every Section, by a Plane, through the Axis of a Right Cone, is an Iſoſceles Triangle, as CAD; CD is its Baſe; and a Right Line, AC, or AE, from the Vertex to any Point in the Circumference of its Baſe, is a Side of the Cone.

DEF. XI. A SCALENE or OBLIQUE CONE is when the Axe, *AB*, inclines to the Baſe, *CD*.

Every Section of an oblique Cone, through its Axis, is a Scalene Triangle, except in one poſition, it is Iſoſceles (when the Axis is at right angles with the Line of ſection with the Baſe); it is, therefore, called a Scalene Cone.

When the Diameter of its Baſe, *CD* is equal to the ſhorteſt Side *AD*; the Section, *CAD*, through that Side, perpendicular to the Baſe, is an Iſoſceles Triangle, and *AC*, the longeſt Side, is its Baſe; nevertheleſs *CAD* is a Scalene Cone.

DEF. XII. A SPHERE is a SOLID, bounded by one uniform convex Surface; which Surface has no Bounds.

As the Circle is bounded by one Line, its Circumference, ſo the Sphere is bounded but by one Surface, without begining or end.

DEF. XIII. The CENTER of a SPHERE is the middle Point of the Solid; which is equally diſtant, every way, from its Surface. As C.

DEF. XIV. The DIAMETER of a SPHERE is any Right line paſſing through its Center, and terminated by its Surface. As AB.

N. B. A Sphere may be conceived to be generated, by the revolution of a Semicircle, as ADB, on its Diameter AB, which remains fixed, whilſt the Semicircumference revolves; and, in its revolution, deſcribes, or preſcribes the Bounds of, the Sphere.

The Diameter, AB, being at reſt, on which, if the Sphere revovled it would appear at reſt, is therefore called an AXE, or Axis of the Sphere.

DEF. XV. A SEGMENT, of a SPHERE, is any portion, cut off by a Plane. As ACB.

Its Baſe, AB, is a Circle, made by the Section.

DEF. XVI. A HEMISPHERE is a Segment of a Sphere. When the Plane of the Section paſſes through its Center, each Segment is a Hemiſphere.

As ADB, and AEB, made by the Plane AB.

DEF. XVII. A SECTOR of a Sphere contains a Segment, and a Right Cone on its Baſe, whoſe Side is a Radius.

DEF. XVIII. SIMILAR CYLINDERS, and CONES, are ſuch as have their Axes, and the Diameters of their Baſes Proportionals; being Right.

If they are ſcalene or oblique; Sections through the Axes of Cylinders, perpendicular to their Baſes, or equally inclined to their Baſes, are ſimilar Parallelograms; as ABCD, a b c d; and, in Cones, they are ſimilar Triangles; as a bc, ABC.

All Spheres are ſimilar to one another.

THEOREM I.

IF a Pyramid be cut in two parts, by a Plane parallel to its Baſe; the leſſer Pyramid, made by that Section, will be ſimilar to the whole Pyramid.

Firſt; let A be the Vertex, and, BCD the Baſe of a triangular Pyramid; BADC.

Let FGH be a Section made by a Plane, parallel to BCD.

I ſay, the Pyramid, FAHG, is ſimilar to the Pyramid BADC.

DEM. Becauſe the Planes BCD, FGH are parallel, and they are both cut by the Plane BAC, the Sections, FG and

BC, are parallel; for the ſame reaſon, the Sections GH and CD, FH and BD are parallel. - - 8. 7.
Wherefore, AF:FB::AG:GC, and as AH:HD; 2. 6.
conſ. AF:AB::AG:AC, &c. (6.5.) i.e. as FG:BC, &c. 4. 6.
and conſequently, the Triangle AFG is ſimilar to ABC, AGH to ACD, and AFH to ABD;
alſo, the Baſe, FGH, is ſimilar to the Baſe, BCD.
Therefore, the Pyramid FAHG is ſimilar to BADC.

After the ſame manner, the Pyramid fAhg may be proved ſimilar to either of the other.

Again, Let CBDE be the Baſe of a quadrangular Pyramid, and, let GFHI be a Section, parallel to the Baſe.

It may be demonſtrated, in the ſame manner, that each Triangle AFG, AGI, &c. is ſimilar to ABC, ACE, &c.
alſo, FGH is ſimilar to BCD, and GHI to CDE;
conſequently, GFHI is ſimilar to CBDE. - 13. 6.

Wherefore, ſince a Pyramid, whoſe Baſe is a Poligon of any number of Sides, may be divided, by diagonal Sections, into triangular Pyramids, the ſame Demonſtration will hold true in every Pyramid whatever.

THEOREM II.

Pyramids, having equal Baſes and equal Altitudes, are equal. (See the laſt Figure.)

Firſt; let the Pyramids ABCD and CADE have triangular Baſes, BCD and CDE, and a common Vertex, A; conſequently, they have equal Altitude.

I ſay, they are equal Pyramids.

Let there be made ſeveral Sections, GFHI, gfhi, parallel to the Baſes, BCD, CDE, which are in one Plane.

DEM. Then, becauſe they are both cut by a Plane, parallel to their Baſes, and the Baſe BCD=CDE, FGH=GHI.

For, FGH is ſimilar to BCD, and GHI to CDE; by 1. and each has that Ratio to the other, reſpectively, which is duplicate of AC to AG; i. e. of CD to GH. - 12. 6.

After the ſame manner, the Section fgh, may be proved equal to ghi; and conſ. every Section, made by Planes par. to CBDE, will be equal, each to the other, reſpectively. Therefore, the Priſm BACD is equal to CADE.

For, the Sections being equal, every where; conſ. if each Pyramid be conceived to be compoſed of an infinite number of Triangular Priſms, whoſe height (or thickneſs) are all equal, and the leaſt that can be conceived, laid on one another, as at BCD, CDE, being each eq. to the contiguous one, the whole is eq. to the whole.

Again. Suppoſe the Baſes of the Pyramids to be different in Figure, as ABC and DEFG; their Vertices, H and I, being equally diſtant from the Plane of their Baſes (AKF) conſequently, every Section, abc, or *abc*, and defg or *defg* (made by the Planes X and Z, parallel to AKF) being in the ſame Ratio, each to the other, reſpectively, as the Baſes, are equal. Therefore, the Pyramid AHBC is equal to DEIFG.

THEOREM III. 7. Euclid.

Every Priſm, having a triangular Baſe, may be divided, into three Pyramids, having triangular Baſes, which are equal to one another.

Firſt; let ACF be a Priſm, whoſe Baſe ABF, and conſequently CED, are right angled Iſoſceles Triangles.

Let the Planes, ABCD and ADEF, be Squares, and BAF, CDE, Right angles; conſ. BCEF is a Rectangle, under the Side of a Square and its Diagonal (BC and CE).

Draw the Diagonals AE, BE, and BD, and imagine Planes to paſs through AE and BE, BD and BE. Ax. 4. 7.

I ſay, the Pyramids, ABEF, AEDB, and BCED, made by the Sections of thoſe Planes, are equal and ſimilar.

DEM. The Triangle ADE=AEF, and ADB=BCD. 15. 1. But, ABCD and ADEF are Squares, and they have a common Side, AD, wherefore they are equal. Ax. B. 2. Alſo, BAF and CDE are Right angles; wherefore, the Triangles AFB, CED, ADB, BCD, ADE, and AEF are all equal, and ſimilar to one another. - - 8. 1.

But, AEB and BED (the Planes of the Sections) are each common to two Pyramids; and they are equal and ſimilar to each other, and alſo to CEB, BEF. - - 7. 1. (For, AE, BD, CE, and BF are Diagonals, AB, DE, CB, EF, Sides, of equal Squares; and BE is common to them all. Wherefore, the Pyramids ABEF, AEDB, and BCED (being bounded by an equal number of Planes, equal and ſimilar to each other, reſpectively) are equal. - Ax. 5. 7.

For two Angles in each Pyramid, at A, C, D, and F, are contained by two R. angles each, and half a R. angle; as AFE, BFE, & AFB. The remaining two, in each, meeting at B and E, are each contained by two half R. angles, (AEF, BEF, &c.) and an Angle AEB, equal DBE (each common to two Pyramids) each equal CEB, or EBF; being contained by the Diagonal of a Square, (CE) and the Hypothenuſe (BE) of a right angled Triangle, under the Side and Diagonal, BC, and CE.

2nd. Let ABCE be a triangular Priſm; i. e. whoſe Baſes, ADE and BCF, are Triangles; ſuppoſe them Scalene, and the Sides AB, &c. of the other Faces, not perpendicular to them; the Priſm is conſ. oblique in every reſpect.

Draw the Diagonals, BE, BD, & EC; and ſuppoſe a Plane to paſs thro' BE & BD, and, another, thro' BE & EC. Ax. 4. 7

I ſay, the Priſm will be divided into three Pyramids, ABDE, BCEF, and BECD, equal to one another.

DEM. The Pyramids, ABDE and BCEF, have equal and ſimilar Baſes, ADE & BCF, and equal Altitudes. Def. 8. 7. Conſequently, every Section, made by parallel Planes, at

equal distances from their Bases, are equal, and also similar; and therefore, the Pyramids are equal. - Th. 2.

Again. Suppose the Pyr. BCEF (eq. ABDE) taken away; the Face BCE is common to the Pyramid BECD (see Fig. 3) Then, in the Pyramids ABDE, BECD; because ABCD is a Parallelogram (Def. 7.) it is divided by the Diagonal, BD, into congruous Triangles, ABD and BCD. - 15. 1. Let them be considered as the Bases of the two Pyramids; E being their com. Vertex, they have, conf. equal Altitude; wh. every Section, parallel to ABCD, will be equal, and also similar; and consequently, the Pyramids are equal. Th. 2.

If ABDE be supposed removed (Fig. 2.) then, the Pyramids BCEF, & BECD are equal; for their Bases CEF, & CED are equal (15. 1.) and B is their common Vertex. Therefore, they are all equal to one another. Q.E.D.

THEOREM IV.

Every Prism is triple of a Pyramid, having equal Bases, and equal Altitudes.

First. Let the Prism ACHF be a Cube.

Then, a Plane drawn through the opposite Diagonals CE and BF, will cut the Cube into two equal triangular Prisms; whose Faces FD and AC, &c. are Squares, and the Angles BAF, CDE, CHE, and BGF, are Right.

Draw, AE, BE, and BD; dividing the hither Prism, ABCEF, into three equal and similar Pyramids ABEF, AEDB and BCED (as in Case 1. of the 3rd). Draw EG.

BEFG is a Pyramid, equal and similar to ABEF; and the Prism ACHF is triple of the Pyramid ABEFG.

Dem. For, the Base BFG is equal and similar to ABE, 15. 1. and the Face BEG is congruous with AEB; - Ax. 7. 1.

(for EG biſects the Square FH, and AE biſects FD). But, BEF is common; and BEG, AEB are congruous; 7. 1 (for, BG is equal to AB, EG = AE, and BE is common) wherefore the Solid, BEFG, is contained within four triangular Planes, equal and ſimilar to thoſe of AEBF; and conſequently, they are equal Pyramids. - Ax. 5. 7.

But, the Pyramid, AEBF, is equal one third of the Priſm, ABCEF; and BEFG one third of BHF, equal CAE. 3.

Therefore, the quadrangular Pyramid, ABEFG, is equal one third of the Cube ACE; or, the Cube triple of the Pyramid, on the ſame Baſe, ABGF, and having the ſame Altitude, EF.

2nd. After the ſame manner (as in Caſe 2nd. of the laſt) every triangular Priſm may be proved triple of a Pyramid, having the ſame, or an equal Baſe and Altitude.

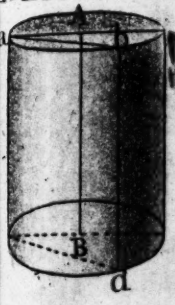
Wherefore, ſince every Priſm, having a quadrangular Baſe, may be divided into two triangular Priſms; and each triangular Priſm, into three equal Pyramids having triangular Baſes; conſ. a Pyramid on the whole quadrangular Baſe of the Priſm, and having equal Altitude, is equal one third of the Priſm, or, the Priſm triple of the Pyramid.

3rd. Let the Baſe, AIHED, of the Priſm, AKFD, be a Pentagon; and let the Diagonals, CG, CK, DH, DI, be drawn.

Then, becauſe CD is parallel to AB and EF, GH to EF, and IK to AB (Def. 33.) DC is parallel to IK & GH; 4. 7. wh. Planes may paſs through DC & GH, DC & IK, Ax. 2. and conſ. the Priſm, AKFD, will be divided into three triangular Priſms, by the Planes, DCKI and DCGH; viz. AKD, KIDHG, and DGE.

Draw the Diagonals BD, DF, DG, and DK; KBDFG is a Pyramid having the ſame Baſe, BCFGK, and Altitude as the Priſm.

F. 2.



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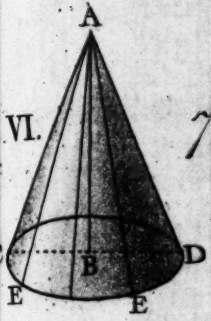
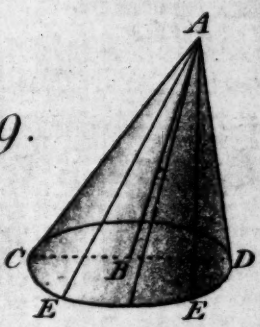


7. 8. & 9.



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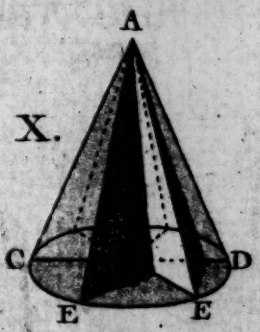

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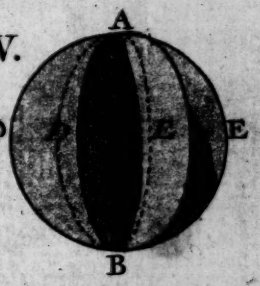

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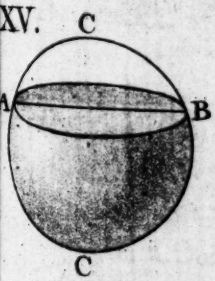

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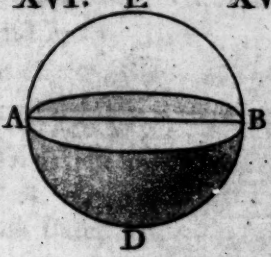

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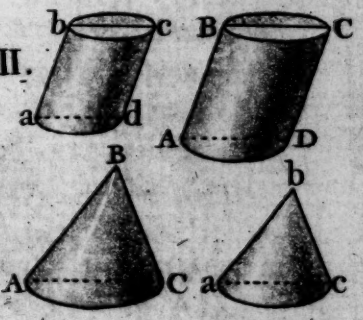

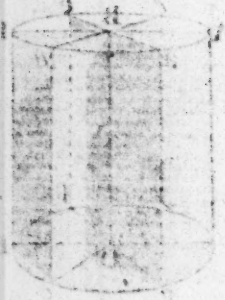

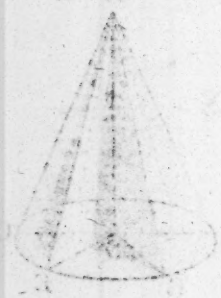

But, BDCK is a Pyramid, having the ſame Baſe, BCK, and Altitude, as the Priſm AKD;
AKD is therefore triple of the Pyramid, BDCK. - 3.
For the ſame reaſon, KIDHG is triple of KDGC; and, the Priſm DGE is triple of the Pyramid CDFG.
But, the Pyramid BDFGK is equal to the three Pyramids BDCK, KDGC, and CDFG; and the Priſm AKFD is equal to the three Priſms AKD, KIDHG, and DGE.
Th. the Priſm AKFD is triple the Pyr. BDFGK. Q.E.D.

THEOREM V. 10. Euclid.

Every Cone is equal to the third part of a Cylinder, having equal Baſes and equal Altitudes.

Let the Circle ACE be conſidered as the Baſe of a Cylinder or Cone; and, let a regular Poligon, ABCDEF, of any number of Sides be inſcribed; it is manifeſt, that the Poligon is leſs than the Circle.

Let the Arks, AGB, &c. be biſected, as at G; and draw AG and GB, &c. This Poligon will be greater than the firſt, but it is alſo leſs than the Circle.

Wherefore, it is evident, that, the more the Sides of the Poligon are multiplied, the nearer it approaches to the Circle, till the difference is leſs than any aſſignable Quantity whatever. The difference is the Segments AG, GB, &c.

And, by the ſame reaſoning, every Poligon, circumſcribed, will be greater than the Circle; conſequently, they will at laſt end in the Circle; i. e. the Perimeter of the Poligon, in the Circumference of the Circle, and their Areas will be equal; or the difference will be leſs than any other Quantity, aſſigned.

Let AGKD be a Cylinder whoſe Baſe is AD; and a g d a Cone, whoſe Baſe, a d, is equal to the Baſe A D, of the Cylinder; and let them alſo have equal Altitudes.

I ſay, the Cylinder AGKD, is triple of the Cone, a g d.

DEM. Let there be regular Poligons, of any number of Sides, inſcribed; as ABCDEF, and abcdef, whoſe Sides AB, BC, &c. ab, bc, &c. are equal; and conſequently they are equal Poligons.

Let a Priſm be conſtructed on the Poligon, ABCDEF, whoſe Sides AG, BH, CI, &c. are in the Surface of the Cylinder; and from every Angle of the Poligon, abd, let there be drawn Right lines ag, bg, &c.

Then, abgdc is a Pyramid, whoſe Baſe is equal to the Baſe of the Priſm (by Conſtruction); and, having equal Altitudes, the Priſm, AMKC, inſcribed in the Cylinder, is triple of the Pyramid, inſcribed in the Cone. - Th. 4.

Again. Let the Arks AB, BC, &c. be biſected, at N and O, &c. and let a Priſm be inſcribed, having twice the number of Sides.

Alſo, let the Pyramid, inſcribed in the Cone, have the ſame number of Sides, as the Priſm; their Baſes are equal; and, conſequently, the Priſm is triple of the Pyramid. 4.

And ſo it will continually be; till, by multiplying the Sides of the Priſm, its Surface will be the ſame as the Surface of the Cylinder, and the Surface of the Pyramid as the Cone; the Baſes of the Priſm and Pyramid will be the ſame as the Baſes of the Cylinder and Cone.

But, they are equal, and they have equal Altitudes.

Therefore, the Cylinder is triple of the Cone; or, the Cone equal to one third of the Cylinder. Q. E. D.

THEOREM VI. 5 and 6, Euclid.

Priſms, or Pyramids, having equal Altitudes, have that Ratio to each other, as their Baſes.

First. Let the Priſms have quadrangular Baſes.

If they are Parallelopipeds, it is already proved. Cor. 1. 17. 7. For, every Parallelopiped is a Priſm (Def. 8. 7.) and may be divided, by a Plane paſſing through its oppoſite Diagonals, into

two equal and ſimilar triangular Priſms; conſequently their Baſes and Altitudes are equal. - - - - Th. 16. 7.

And it is manifeſt, that if the Baſe of one be double, or triple of the other, ſeeing they have equal Altitudes, they are as their Baſes; i. e. the Priſm, whoſe Baſe is double or triple of the others Baſe, is double or triple of the other Priſm; and conſequently, whatever Ratio one Baſe has to the other, the Priſms have neceſſarily the ſame. For, whether the Baſes be ſimilar Figures or not, being equal, it is the ſame; as it is proved, in Caſe the 3rd of the 17th, being Parallelopipeds.

Alſo, if their Baſes be equal, they are as their Altitudes; C. 2. 17. 7. if the Priſms are equal, their Baſes and Altitudes reciprocate. 19. 7. Conſequently, every triangular Priſm is to another, of equal Altitude, in the ſame Ratio as their Baſes; ſeeing, they are each half a Parallelopiped or Priſm, whoſe Baſe is double the Baſe of the triangular Priſm, and having equal Altitudes.

Let the Priſms AFD and EGKM have equal Altitudes; the one on a triangular Baſe, the other pentagonal.

Let the Diagonals GL, GO, HM, and HN be drawn, between oppoſite Angles in the equal Pentagons; and imagine Planes OGHN, and GLMH, drawn through thoſe Diagonals, dividing the Priſm, EGKM, into three triangular Priſms, EGN, NGM, and MGI.

Dem. Then, becauſe the triangular Priſms, AFD and EGN, have equal Altitudes, they have that Ratio to each other, as their Baſes ADE to EHN; - proved above;

i. e. the Priſm AFD : EGN :: ADE : EHN
And, the Priſm AFD : NGM :: ADE : NHM
alſo, the Priſm AFD : MGI :: ADE : MHI
} their Baſes

Wh. as the Baſe ADE : EHN + NHM + MHI
ſo is the Priſm AFD : EGN + NGM + MGI
} - 2. 5.

i. e. as the Baſe ADE : EHIMN :: the Priſm AFD : EGKM.

2nd. Becauſe the Priſm AFD is triple the Pyramid ACED; and the Priſm EGKM is triple of the Pyramid NEGIM, 4. Alſo, as Baſe ADE : EHIMN, :: the Priſm AFD : EGKM
conſ. as Baſe ADE : EHIMN, :: the Pyr. ACED : NEGIM

Cor. 1. Cylinders, having equal Altitudes, have the ſame Ratio to each other as their Baſes.

For, ſimilar Poligons, inſcribed in Circles, are, to each other, as the Squares of their Diameters. - - - - - 14. 6.

And it has been proved, that all Priſms, having equal Altitudes, are in the Ratio of their Baſes to each other; conſequently, Cylinders are to each other in the ſame Ratio.

Becauſe, every Cylinder, whether it be right or oblique, is equal to a Priſm, whoſe Baſe is the ſame, or equal to the Baſe of the Cylinder; i. e. having its Sides multiplied continually till it ends in the Surface of the Cylinder; and its Baſe, in the Circumference of the Cylinder.

COR. 2. Cones, having equal Altitudes, are to each other, in the Ratio of their Baſes.

Becauſe, a Cone is equal to one third of a Cylinder, having equal Baſes and Altitudes; and Cylinders are in that Ratio to each other; therefore Cones have the ſame Ratio as Cylinders.

COR. 3. Priſms and Pyramids, having equal Baſes, are to each other as their Altitudes.

Becauſe Parallelopipeds are Priſms; and it has been proved, that Parallelopipeds are to each other in that Ratio; and conſequently, all Priſms, having equal Baſes, are as their Altitudes.

COR. 4. Cylinders and Cones, having equal Baſes, are to each other as their Altitudes.

Becauſe there is the ſame Ratio between Cylinders and Cones, as between Priſms and Pyramids, having equal Baſes. Therefore, &c.

THEOREM VII. 8. Euclid.

Similar Priſms and Pyramids are, to one another, in the triplicate Ratio of their correſponding Sides.*

In reſpect of Priſms (being alſo Parallelopipeds) this property is already proved (18.7.) and conſequently, in Priſms having triangular Baſes; ſeeing, every ſuch Priſm is half a Parallelopiped, whoſe Baſe is double of the Priſm and having equal Altitudes.

Let the Pentagonal Priſms AKD, akd be ſimilar.

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* I cannot conceive the reaſon, why Euclid confines this Propoſition, and the following, to Pyramids only; and to ſuch, only, as have triangular Baſes; ſeeing that, it holds equally true in Priſms; and alſo in Pyramids, having poligonal Baſes, of any kind.

Take any Right line, V; and as ab, or bc, is to AB, or BC, &c. let X be to V; and as X is to V, ſo let Y be to X; alſo, as Y is to X, or X to V, let Z be to Y.

I ſay, that, as V is to Z, ſo is the Priſm AKD to akd.

Let them be divided by diagonal Planes, BK, KC, bk, kc, paſſing through the oppoſite Diagonals; dividing the Priſms into three triangular Priſms, each.

Dem. The Poligons ABCDE, abcde, are ſimilar, Hyp. wh. the Triangles ABE, abe, &c. are ſimilar; - 13.6. and conſ. the Priſms AKB, akb, &c. are ſimilar. Def. 11.7.

For, as AB:ab, or as BC:bc, :: BG:bg, and GK:gk; conſequently, the diagonal Planes, BK and bk, KC and kc, are ſimilar.

Wherefore, ſince each triangular Priſm is equal to half a Parallelopiped, whoſe Baſe is double of the triangular Priſm, and Altitudes equal (above) they have that Ratio to each other, which is triplicate of their correſponding Sides; i. e. as V to Z. (Ax. 8. 5.) - - - 18.7.

But the Priſm AKB:akb :: BKC:bkc, and as CKD:ckd; wh. as AKB:akb :: AKB+BKC+CKD:akd+bkc+ckd.* i. e. as the Priſm AKB:akb :: the Priſm AKD:akd.

But, AKB:akb :: V:Z. Th. the Priſm AKD:akd :: V:Z.†

2nd. Let the Pyramids ABCD, abcd be ſimilar. (Fig. 2.)

The Pyramids ABCD, abcd, being ſimilar, are each equal to the third part of a Priſm on the ſame Baſe and Altitude.

But, Quantities are in the ſame Ratio, to each other, as their Equimultiples or equal Parts. - - Ax. 8. 5. and the triples of the Pyramids ABCD, abcd, are as V to Z. therefore, the Pyramid ABCD:abcd :: V:Z. Q.E.D.

Cor. Similar Cylinders are, to one another, in the triplicate Ratio of their Diameters or Sides.

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* By Theo. 2nd of the 5th. † By Axiom 13th of the 5th.

For, they are in the ſame Ratio, to each other, as ſimilar Priſms inſcribed; or, as Priſms, whoſe Baſes are equal to the Baſes of the Cylinders, and having equal Altitudes.

COR. 2. Similar Cones are, to each other, in the triplicate Ratio of the Diameters of their Baſes, or of their Axes.

Becauſe a Cone is equal to the third part of a Cylinder, on equal Baſes and Altitudes.

THEOREM VIII. 9. Euclid.

Equal Priſms, or Pyramids, have their Baſes and Altitudes reciprocally proportional.

If the Priſms are Parallelopipeds it is already proved. 19. 7.

Let the triangular Priſm AKC be equal to the quadrangular Priſm FLG; whoſe Baſes, ACD, FGHI, are in the ſame Plane; to which, BF is perpendicular. BF and EF are their Altitudes.

I ſay, the Baſe, ADC, is to the Baſe, FGHI, as EF to BF.

Let the Plane of the upper Face, LE, be produced, cuting the triangular Priſm, at EMN.

DEM. The Priſm AMC: AKC :: EF: BF, their Altitudes.
And - - - AMC: FLG :: ADC: FGHI, their Baſes.
But, the Priſm FLG is equal to AKC; by Hypotheſis, conſequently, AMC: FLG :: EF: BF, i.e. as ADC: FGHI.
Therefore, as ADC: FGHI :: EF: BF. Q. E. D.

2nd. Draw the Diagonals AB, DB; EG, EH, and EI.

The Pyramid ABCD is equal to one third of AKC. 4.
And the Pyramid FGEHI is equal one third part of FLG.
But, Quantities are in the ſame Ratio as their Equimultiples wherefore, the Pyramid ABCD is equal to FGEHI.

But, they have the ſame Baſes and Altitudes, as the Priſms, AKC and FLG. Therefore, their Baſes and Altitudes are reciprocally proportional. Q. E. D.

Cor. Equal Cylinders, or Cones, have their Baſes and Altitudes reciprocally Proportional.

Becauſe, Cylinders are equal to Priſms whoſe Baſes and Altitudes are equal; and Cones to Pyramids.

In theſe eight Theorems, and their Corollaries, with the Corollary to the next, is contained the whole ſubſtance of Euclid's 12th Book. Thoſe which follow, are ſelect Theorems from Archimedes, which ſhew how the ſurface of the Sphere may be aſcertained; alſo, of any Segment, or Portion of its Surface, intercepted between parallel Circles, with great accuracy.

The next Theorem (which is not from Archimedes) ſhews, in a brief and elegant manner, the Ratio between the Sphere and a circumſcribing Cylinder; and conſequently, if the Quantity of the Cylinder be known, or attainable, the Quantity of the Sphere may alſo be obtained. According to Archimedes, it is determined by the Cone, in the 17th of this Book; which, though perfectly true, is not ſo brief; and though perhaps more geometrical, yet not more convictive and ſatisfactory.

In the following is contained all that is really uſeful or valuable, in reſpect of the Cylinder, Cone, and Sphere; nor is it poſſible, I think, to obtain the Area of the Surface of a Sphere in a more conciſe manner.

In ſhort, without the following Theory, I cannot but think the Doctrine of Solids imperfect; ſeeing that, the Quantity of a Sphere, the moſt perfect Solid, cannot, by Euclid, be obtained, in reſpect either of its Superficies or ſolid Contents; for want of which, Artificers are frequently at a loſs, in meaſuring a Dome or other ſpherical Object.

For, they are in the ſame Ratio, to each other, as ſimilar Priſms inſcribed; or, as Priſms, whoſe Baſes are equal to the Baſes of the Cylinders, and having equal Altitudes.

COR. 2. Similar Cones are, to each other, in the triplicate Ratio of the Diameters of their Baſes, or of their Axes.

Becauſe a Cone is equal to the third part of a Cylinder, on equal Baſes and Altitudes.

THEOREM VIII. 9. Euclid.

Equal Priſms, or Pyramids, have their Baſes and Altitudes reciprocally proportional.

If the Priſms are Parallelopipeds it is already proved. 19. 7.

Let the triangular Priſm AKC be equal to the quadrangular Priſm FLG; whoſe Baſes, ACD, FGHI, are in the ſame Plane; to which, BF is perpendicular. BF and EF are their Altitudes.

I ſay, the Baſe, ADC, is to the Baſe, FGHI, as EF to BF.

Let the Plane of the upper Face, LE, be produced, cuting the triangular Priſm, at EMN.

DEM. The Priſm AMC: AKC :: EF: BF, their Altitudes.
And - - - AMC: FLG :: ADC: FGHI, their Baſes.
But, the Priſm FLG is equal to AKC; by Hypotheſis, conſequently, AMC: FLG :: EF: BF, i.e. as ADC: FGHI.
Therefore, as ADC: FGHI :: EF: BF. Q. E. D.

2 nd. Draw the Diagonals AB, DB; EG, EH, and EI.

The Pyramid ABCD is equal to one third of AKC. 4.
And the Pyramid FGEHI is equal one third part of FLG.
But, Quantities are in the ſame Ratio as their Equimultiples wherefore, the Pyramid ABCD is equal to FGEHI.

But, they have the ſame Baſes and Altitudes, as the Priſms, AKC and FLG. Therefore, their Baſes and Altitudes are reciprocally proportional. Q. E. D.

Cor. Equal Cylinders, or Cones, have their Baſes and Altitudes reciprocally Proportional.

Becauſe, Cylinders are equal to Priſms whoſe Baſes and Altitudes are equal; and Cones to Pyramids.

In theſe eight Theorems, and their Corollaries, with the Corollary to the next, is contained the whole ſubſtance of Euclid's 12th Book. Thoſe which follow, are ſelect Theorems from Archimedes, which ſhew how the ſurface of the Sphere may be aſcertained; alſo, of any Segment, or Portion of its Surface, intercepted between parallel Circles, with great accuracy.

The next Theorem (which is not from Archimedes) ſhews, in a brief and elegant manner, the Ratio between the Sphere and a circumſcribing Cylinder; and conſequently, if the Quantity of the Cylinder be known, or attainable, the Quantity of the Sphere may alſo be obtained. According to Archimedes, it is determined by the Cone, in the 17th of this Book; which, though perfectly true, is not ſo brief; and though perhaps more geometrical, yet not more convictive and ſatisfactory.

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In ſhort, without the following Theory, I cannot but think the Doctrine of Solids imperfect; ſeeing that, the Quantity of a Sphere, the moſt perfect Solid, cannot, by Euclid, be obtained, in reſpect either of its Superficies or ſolid Contents; for want of which, Artificers are frequently at a loſs, in meaſuring a Dome or other ſpherical Object.

THEOREM IX.

Every Sphere is equal to two-thirds of a circumſcribing Cylinder; i. e. of a Cylinder whoſe Diameter and Altitude are equal, each to the Diameter of the Sphere.

Let AB be the Diameter of a Sphere; biſect AB, at C.

On C, deſcribe the Semicircle AKB; and, on AB, let the Rectangle AEDB, circumſcribing it, be conſtructed.

Draw AD.

Then (AB remaining fixed) imagine them all revolved about AB, as an Axis; the Semicircle, AKB will deſcribe a Sphere, AKBL (Def. 14.); alſo, the Rectangle, AEDB, will deſcribe a Right Cylinder, DEMN, containing the Sphere (Def. 4.); and, the right-angled Triangle, ADB, will generate a Right Cone, DAN (Def. 10.); whoſe Baſe, DN, is the ſame as the Cylinder, and equal to the Diameter of the Sphere.

The Cone, is equal to a third part of the Cylinder. Th. 5.

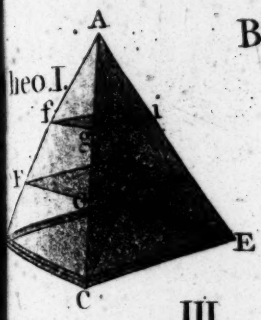
I ſay, that the Sphere, AKBL, is equal to two-thirds of the Cylinder circumſcribing it.

Draw CD; and FG (at diſcretion) parallel to CK, cuting CD at H, and the Circumference of the Sphere at I; and draw CI.

DEM. In the R. angled Tri. CFI, $CI\square = CF\square + FI\square$, 20. 1 and, the Areas of Circles are as the Squares of their Diameters (Cor. 1. 14. 6.); wherefore, a Circle, deſcribed on CI, is equal to two Circles, on CF and FI.

Now, in the revolution of the Square CKDB, there will be deſcribed the Cylinder KDNL; the Ark BIK will deſcribe a Hemiſphere; and, CD will deſcribe the

heo. I.



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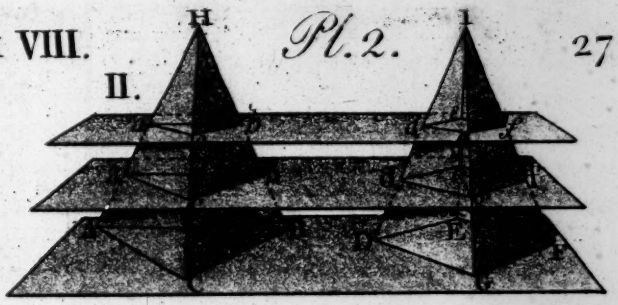

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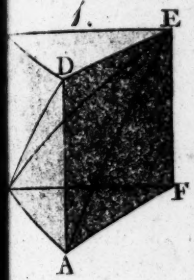

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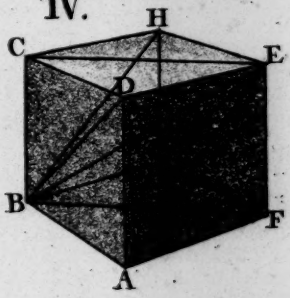

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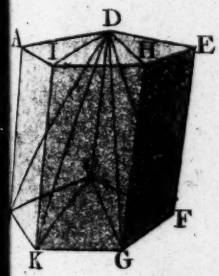

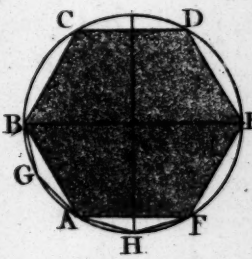

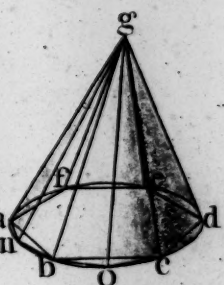

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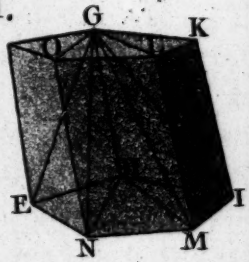

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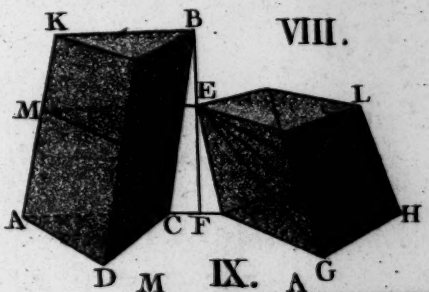

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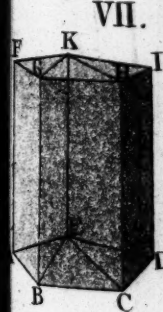

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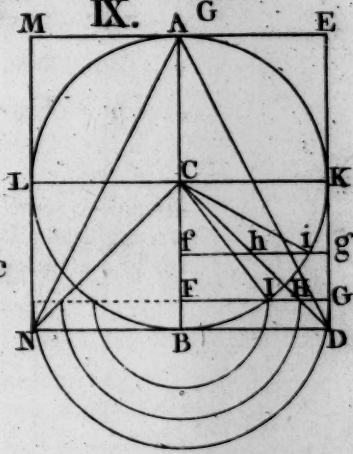

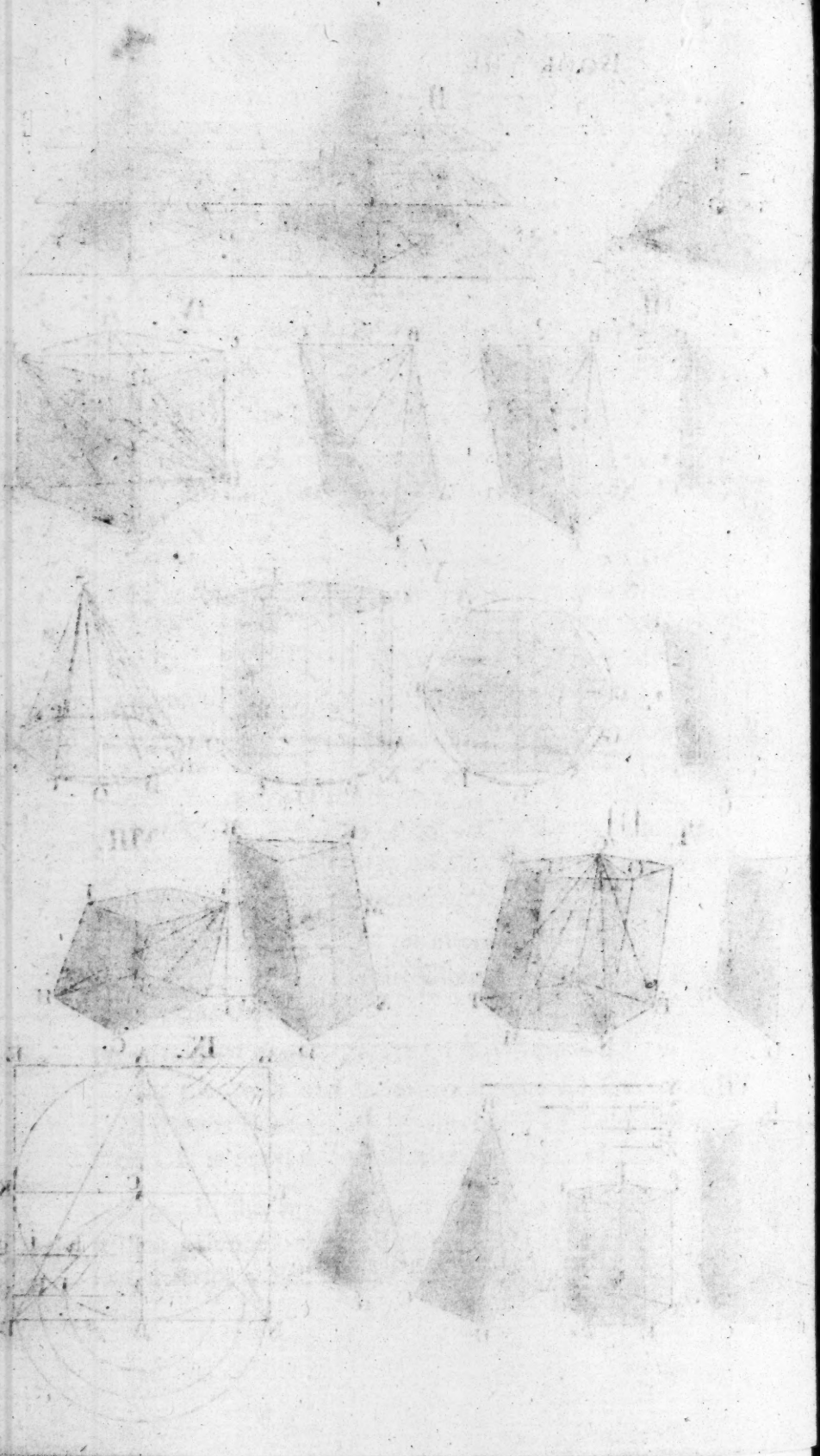

Surface of a Right Cone, DCN; alſo, the Points G, H, and I, will deſcribe Circles, with the ſeveral Radii FG, FH, and FI; FG is equal to CI (equal CK), and FH is equal to CF (9. 1.).

For, the Angle CFH is Right; and the Angles FCH, FHC, are half Right; ſeeing that, the Diagonal, CD, biſects the Angles BCK, KDB.

Wherefore, the Circle deſcribed by FH (equal CF) is equal to the difference between the Circles deſcribed by FG (equal CI) and FI. Conſequently, the Circle, deſcribed by FH, is equal to the Annulus, or Ring, deſcribed by GI; and, conſequently, every Section (FG, or fg) of the Cone DCN, by a Plane parallel to its Baſe, is equal to a Section of the concave Solid, remaining when the Sphere is ſuppoſed to be taken out of the Cylinder.

Wherefore, the Cone, DCN, is equal to the Solid deſcribed by the mixed Triangle KiIBD.

But, the Cone, DCN = one third of the Cylinder KDNL conſ. the concave Solid (taking away the Hemiſphere, KBL, out of the Cylinder, KDNL) is equal to one third of that Cylinder; and conſequently, the Hemiſphere, KBL, is equal to two thirds of the Cylinder.

But, the Cone, DCN, is half the Cone DAN; and, the Cylinder, DKLN, is half the Cylinder DEMN; C. 4. 6. alſo, the Sphere, AKBL, is double the Hemiſphere, KBL. Therefore, the Sphere AKBL, whoſe Diameter, AB or KL, is equal to the Diameter, and Altitude, of the Cylinder DEMN, is equal to two thirds of the Cylinder. Q.E.D.

COROLLARY. The 18th Propoſition, of Euclid.

Spheres have that Proportion to one another, which is triplicate of their Diameters;

For, all Cylinders, being ſimilar, are in that Ratio to one another; and conſequently, Spheres, inſcribed in Right Cylinders whoſe Diameters and Altitudes are equal to the Diameters of the

Spheres, reſpectively, being two thirds of ſuch Cylinders, are in the ſame Ratio to each other as the Cylinders; i. e. in the triplicate Ratio of their Diameters.

THEOREM X.

A Circle, whoſe Radius is a mean Proportional between the Diameter of the Baſe, and the Side of a Right Cylinder, is equal to the cylindrical Surface.

Let AB be the Diameter of the Baſe of the Cylinder AD, and let BD be its Side. Take FG a mean Proportional, between AB and BD, and, on FG, deſcribe a Circle.

Let C be the Center of the Baſe of the Cylinder; produce BD, and make DE equal to BD.

Then, CB, FG, and BE are Proportionals; for, CB : AB :: BD : BE; wherefore, $CB \times BE = AB \times BD$; 9.6. i. e. to FG ſquare; for $AB \times BD = FG\square$. - Cor. to 9.6.

Let ſimilar Poligons (X and Z) be circumſcribed about the Baſe of the Cylinder, and the Circle whoſe Radius is FG; and let a Priſm, be ſuppoſed to circumſcribe the Cylinder.

Dem. A Triangle, whoſe Baſe is equal to the Perimeter of the Poligon, and its Altitude equal CB, is equal to the Poligon;* and a Rectangle under the Perimeter and BD, a Side of the Priſm, is equal to the Superficies of the Priſm. 1.2 Alſo, a Triangle under the Perimeter and BE is equal to a Rectangle under the Perimeter and BD, half BE.

But, a Triangle, under the Perimeter and CB, is to a Triangle under the ſame Perimeter & BE, as CB is to BE, 1.6 conſ. the Poligon is to the Surface of the Priſm, as CB : BE.

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* This follows from the 18th of the firſt.
For, all the Sides of the Poligon are equal; and all Triangles having equal Baſes and Altitudes are equal.

But, CB to BE is duplicate of CB to FG; - Def. 12.5. wherefore, the Poligon X is to the Superficies of the Prism, in the Duplicate ratio of CB to FG.

But, the Poligon X is to Z in the Duplicate ratio of their Sides, or Perpendiculars, CB to FG, i. e. as CB to BE. wh. the Poligon X has the same Ratio to Z, as X has to the Superficies of the Prism, AD; conf. they are equal. Ax. 5.5

After the same manner, it may be proved that the Superficies of any other Prism, whose Faces being multiplied infinitely, till they end in the Surface of the Cylinder, AD, is equal to the Poligon circumscribing the Circle, Z, whose Sides, being multiplied, end at last in the Circle.

Therefore, the Circle, Z, whose Radius is a Mean, between AB & BD, is equal to the Surface of the Cylinder, AD.

Cor. 1. The Superficies of a Right Cylinder is to its Base, as the Side of the Cylinder to the fourth part of the Diameter of the Base.

For the Superficies of the Cylinder is equal to the Circle, whose Radius is FG; and that Circle, is to the Base of the Cylinder in the Duplicate ratio of FG to CB; i. e. of BE to CB; i. e. of BD to half CB.

Cor. 2. The Superficies of a Cylinder whose Side is equal to the Diameter of its Base, is equal to four times the Area of the Base; conf. if the Side be a fourth part of the Diameter, the cylindrical Superficies and the Base are equal.

Cor. 3. The Superficies of a Right Cylinder is equal to a Rectangle, under its Circumference and Side.

Cor. 4. Cylindrical Surfaces, of equal height, are to each other as the Diameters of their Bases; and, having equal Bases, they are as their Altitudes.

Cor. 5. The Surfaces of similar Cylinders have that Ratio between them; which is duplicate of their Diameters.

THEOREM XI.

A Circle, whoſe Radius is a mean Proportional between the Side of a Right Cone and the Radius of the Baſe, is equal to the Superficies of the Cone.

Let ABD be a Right Cone, the Diameter of whoſe Baſe is AD, and C its Center.

Take FG, a mean Proportional between the Side, BD, and the Radius CD (equal BE and CE) and on FG deſcribe a Circle.

Let regular Poligons (X and Z) be circumſcribed about the Baſe of the Cone and the Circle, whoſe Radius is FG, of the ſame kind; and ſuppoſe a Pyramid, on the Poligon X, circumſcribing the Cone.

DEM. Becauſe CE:FG::FG:BE; by Hypotheſis; CE to BE is duplicate of CE to FG. - Def. 12. 5. But, as CE is to BE, ſo is the Triangle, whoſe Altitude is CE, and its Baſe the Perimeter of the Poligon X, to the Triangle under BE and the ſame Perimeter. - 1.6. Alſo, as CE is to BE ſo is the Triangle whoſe Altitude is CE and the Perimeter of the Poligon X, to a Triangle, under FG and the Perimeter of Z. - Th. 6. and 12. 6.

But, a Triangle, under CE and the Perimeter of X, is equal to the Poligon X; and, the Triangle under FG and the Perimeter of Z is equal to that Poligon; alſo, the Triangle under BE and the Perimeter of X, is equal to the Superficies of the Pyramid.*

Wherefore, the Poligon X has the ſame Ratio to the Poligon Z, and to the Superficies of the Pyramid; conſ. the Poligon Z = to the Superficies of the Pyramid. Ax. 5. 5.

This alſo follows from the 18th of the firſt Book of Elements.

After the ſame manner it may be proved, that the Superficies of any other Pyramid (whoſe Baſe is a regular Poligon, ſimilar, to one circumſcribing the Circle, whoſe Radius is FG, till by multiplying the Sides, equally, the Perimeters, of both, fall into the Circumferences of the Circles, and the Superficies of the Pyramid into the Cone) is equal to the Poligon Z; conſequently, the Superficies of the Cone is equal to the Circle whoſe Radius is FG, a mean Proportional between the Side BD & the Radius CD.

Cor. 1. The Superficies of a Right Cone is to its Baſe, as the Side of the Cone to the Radius of the Baſe.

For, they are, reſpectively, equal to Triangles, whoſe Baſes are equal to the Circumference of the Baſe of the Cone, and Altitudes equal to BD and CD reſpectively; conſequently, they are to each other, as BE to CE. - - - 1. 6.

Cor. 2. The Superficies of a Right Cone is equal to a Triangle, whoſe Baſe is equal to the Circumference of the Baſe of the Cone, and its Altitude equal to a Side.

Hence it is manifeſt, that the Surfaces of Right Cones have all the ſame Properties as Triangles.

If their Sides be equal, they are, to each other, as the Diameters of their Baſes; and having equal Baſes, they are as their Sides; or as their Axes.

If they are ſimilar, the Ratio between their Surfaces is duplicate of their Sides, or Diameters of their Baſes.

If they are equal, their Sides and Diameters are reciprocally Proportional.

Cor. 3. The Superficies of a Right Cylinder, is to a Right Cone, having equal Baſes and Altitudes, as the Side of the Cylinder, to half the Side of the Cone.

For, the Superficies of the Cone is to its Baſe, as the Side to the Radius; or, as half the Side to half the Radius. And the Superficies of the Cylinder, to its Baſe, is as the Side to half the Radius; therefore, the Surface of the Cone is to the Surface of the Cylinder, as the Side of the Cylinder to half the Side of the Cone.

COR. 4. An Equilateral Cone, i. e. whoſe Side is equal to the Diameter of its Baſe, has it Superficies double of its Baſe; and if the Vertex be right angled, the Ratio is as the Diagonal of a Square to its Side.

THEOREM XII.

IF a Right Cone be cut by a Plane, parallel to its Baſe; a Circle whoſe Radius is a mean Proportional, between that Part of the Side (betwixt the Baſe and the Section) and the Radius of that Section added to the Radius of the Baſe, is equal to the conical Superficies, between the Section and the Baſe.

Let BAF be a Right Cone; and let DJ be a Section, parallel to the Baſe, BF.

Take GH a mean Proportional, between AB, and BC; and GI a Mean between AD and DE; on which, deſcribe Circles, X and Z.

The Circle X, on GH, is equal to the whole Superficies; and, that on GI = to the Surface of the Cone, DAJ - 11. and, AB × BC = AD × DE, + DB × BC, + DB × DE.*

Take GK a Mean, between DB and DE + BC; on which, deſcribe a Circle; let it be called Y.

I ſay, that Circle (Y) is equal to the conical Superficies BDJF; between the parallel Circles.

DEM. Becauſe GH is a Mean, between AB, & BC - by Con. the Square of GH = AB × BC; i.e. to the Rect. ABcd. } 9.6
For the ſame reaſon, GI □ = AD + DE; i.e. to ADea. }

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* This is deducible from the ſecond Book of Elements; having formed a Conſtruction.

Let ABcd be a Rectangle, under AB and BC; Ac being drawn, and Df parallel to Bc, cuting it at e; and a b (through e) parallel to AB.

Then, the whole, Bd (equal AB × BC) = Da (equal AD × DE) + Bf (equal BD × BC) + ed (equal BD × DE, i. e. Be, 17. 1.)

And, becauſe GK is a Mean, betwixt BD and DE + BC;
GK □ = BD × BC, + BD × DE; (equal Bf + Be)
i. e. to a Rectangle under BD and DE + BC.
Conſequently, GH □ = GI □ + GK □.

(Becauſe, the Rectangle equal to GH Square, is equal to two Rectangles, which are reſpectively equal to the two Squares, of GI and GK).

But, Circles are to each other, as the Squares of their Diameters, and conſequently as their Radii - C. 1. 14. 6.
And the Circle X is equal to the whole conical Surface;
alſo, the Circle Z is equal to the Surface DAJ.
Therefore, the Circle Y (being equal to the Difference between the two, X and Z) is equal to the Superficies BDFJ; between the the Section, DJ, and the Baſe.

THEOREM XIII.

If a regular Poligon, having an equal number of Sides, be inſcribed in a Circle, and if a Diameter (which is alſo a Diagonal of the Poligon) be drawn, and another Diagonal, from either extreme of the Diameter to the adjacent Angle; the Rectangle, under the Diameter, and that Chord, is equal to a Rectangle under a Side of the Poligon and all the parallel Chords, joining two and two contiguous Angles.

In the Circle AGF, let a regular Oct. ACEGBHFD be deſcribed; draw the Diameter AB, and the Diagonal BC; alſo, draw CD, EF, and GH.

I ſay, the Rectangle, under AB and BC, is equal to a Rectangle under AC (or AD) and CD, added to EF, added to GH. Join DE and FG.

Dem. Becauſe CE = DF, and EG to FH, the Chords CD, EF, and GH, are all parallel - Cor. to 10. 3.

And, for the ſame reaſon, AC, DE, FG, & HB are parallel; conſ. the Triangles ACI, IDK, KEL, &c. are ſimilar.
Wh. AI:IC::IK:ID,::KL:LE,::LM:LF, &c. - 4. 6.
i. e. as AI:IC::AB:CD+EF+GH, - - 2. 5.
for, the Diameter, AB,=AI+IK+KL, &c.
and CD+EF+GH=CI+ID+EL, &c. - Ax. 1. 1.
But, as AI:IC::AC:CB (7.6) for ACB is a R. angle. 12. 3.
Conſequently, AC:CB::AB:CD+EF+GH. Ax. 13. 5.
Therefore, AC×CD+EF+GH=CB×AB. - Th. 9. 6.

COR. Hence it is manifeſt, that if in any Segment of a Circle, as GAH, a Poligon be inſcribed, on GH, whoſe Sides are equal, and equal in number; a Rectangle, under the Chord CB, and AN (part of a Diameter, perpendicular to GH) is equal to a Rectangle under a Side AC, and the Chords, CD+EF+GN, half the Baſe.

THEOREM XIV.

IF a regular Poligon be inſcribed in a Circle (as in the foregoing Theorem) and Chord Lines be drawn (as there deſcribed); a Circle, whoſe Radius is a mean Proportional between the Diameter (AB) and the Chord CB, is equal to all the conical Superficies deſcribed by the revolution of the Poligon, on the Diameter or Axis, AB.

The ſame Conſtruction remaining, as in the former Figure.

Let X be a mean Proportional between AC and CI; alſo Y, a Mean, between CE and CI added to EL.

DEM. Now, AC, CE, EG, and GB are equal, by Con.
And, AB×BC=AC×CD+EF+GH (Th. 13.) i. e. to AC×CI,+CE×CI+EL,+EG×EL+GN,+GB×GN.

Take Z, a mean Proportional between AB and BC.
The ſquare of Z is equal to the Squares of X and Y, taken twice; i. e. to AC×CD,+CE×EF, +EG×GH - 13.
On X and Y deſcribe Circles; the Circle X is equal to the

Surface of the Cone DAC (11.); and, the Circle Y is equal to the conical Surface ECDF; - by the 12th.

But, Circles are in the ſame Ratio as the Squares of their Diameters (C. 1. 14. 6.); wherefore, a Circle, deſcribed on Z, is equal to X and Y, taken twice.

But, the Cone DAC=GBH; and ECDF=EGHF; for, CD is equal to GH, and CE to EG, &c.

Therefore, a Circle whoſe Radius is a mean Proportional between AB and BC, is equal to the conical Superficies, deſcribed by the revolution of ACEGB, on AB.

COR. After the ſame manner it may be proved, that a Circle, whoſe Radius is a mean Proportional between CB and AN, is equal to all the conical Superficies deſcribed by ACEG on AN, in the Segment GAH.

THEOREM XV.

THE Surface of a Sphere is quadruple of the greateſt Circle, made by a Section through its Center; i. e. to a Circle whoſe Radius is equal to the Diameter of the Sphere.

Let ADBK be a Circle, the Section of a Sphere through its Center.

Let a regular Poligon be inſcribed, having an even number of Sides; draw the Diameter, AB, and the Diagonal BC, to the adjacent Angle.

Then, imagine the Circle ADBK, with the Poligon, revolved on AB; the Circle will generate a Sphere (N.B. Def. 14.) and the Sides of the Poligon AC, CD, &c. will generate Right conical Surfaces, inſcribed in the Sphere.

AC and EB will deſcribe equal Cones (for they are equal, and equally inclined to the Axe AB); CD and DE will alſo deſcribe equal Surfaces; for they are equal, and CH is equal to EI.

The whole conical Superficies, is leſs than the Superficies of the Sphere; for the Arks, AFC, CGD, &c. are greater than the Chords AC, CD, &c; alſo, a mean Proportional, between AB and BC, is leſs than the Diameter, AB.

Let the Arks, AC, CD, &c. be biſected, at F and G; and draw AF, FC, &c.

It is manifeſt, that the conical Surfaces, deſcribed by the revolution of AFCGD, will be greater than thoſe deſcribed by

AC, CD; but they are leſs than the Surface of a Hemiſphere, deſcribed by the Ark ACD; and, a mean Proportional between AB and BF, is greater than a Mean, between AB and BC, but it is alſo leſs than the Diameter AB.

By the ſame reaſoning, i. e. by multiplying the Sides of the inſcribed Poligon, the Superficies of Cones, inſcribed in the Sphere, will, at laſt, end in the Surface of the Sphere; when the Sides, of the Poligon end in the Circumference of the Circle. For, if AF be biſected, at O, the Chords AO, OF deviate but very little from the Arks they ſubtend; and a mean Proportional between AB and BO, will differ very little from the Diameter AB; and conſequently, it will at laſt be the ſame as the Diameter.

DEM. But, a Circle, whoſe Radius is equal to a Mean, between AB and BF, or between AB and BO, is equal to all the conical Superficies, deſcribed by the revolution of the Poligon, whoſe Sides are equal to AF or AO; Th. 14.

Conſequently, a Circle whoſe Radius is equal to the Diameter, AB, is equal to the Surface of a Sphere, deſcribed by the revolution of the Semicircle ADB.

But, Circles are to each other, or amongſt themſelves, as the Squares of their Diameters; - - Cor. 1. 14. 6. and the Square of any Right line is equal to four times the Squure of half that Line. - - - Cor. to 4. 2. Conſ. a Circle whoſe Radius is equal to the Diameter of another, is quadruple, i. e. equal to four times that Circle.

Th. the Superficies of a Sphere is equal to four great Circles, equal, in Diameter, to the Diameter of the Sphere.

COR. The Superficies of the Segment of a Sphere is equal to four Circles, whoſe Diameter is a Right line drawn from the middle Point in its ſurface (i. e. the Point where a Perpendicular, from the Center of its Baſe, cuts it) to the Circumference of its Baſe.

For, by the preceding Theorem, it is proved, that the conical Superficies, in every Segment, is equal to a Circle, whoſe Radius is equal to a mean Proportional, between the Chords CB, FB, or OB and AI, the Altitude of the Segment EAL.

Let DAE be a Segment, and AG its Axis. Draw AE.

Then, becauſe all the conical Surfaces, deſcribed by AF, FC, &c. is equal to a Circle, whoſe Radius is a mean Proportional between AG and BF; conſequently, by multiplying the Sides of the Poligon, BF will coincide with AB.

But, AE is a mean Proportional between AB and BG. (Cor. 2. 7. 6.). Conſequently, a Circle, whoſe Radius is AE, isequal to the ſpherical Surface, deſcribed by the revolution of DAG.

Cor. 2. The Superficies of a Right Cylinder, whoſe Diameter and Altitude are equal, each to the Diameter of a Sphere, is equal to the Superficies of the Sphere.

For, the cylindrical Surface, being equal to a Circle whoſe Radius is a mean Proportional between its Side and Diameter, which are in this Caſe equal (Theo. 10) is equal to four times its Baſe, i. e. to a great Circle of the Sphere.

But, the Surface of the Sphere is alſo equal to four ſuch Circles. Therefore, the cylindrical Surface is equal to the ſpherical. Ax. 3. 1

THEOREM XVI.

If a Sphere be inſcribed in a right Cylinder, and being both cut by Planes, parallel to the Baſes of the Cylinder; each Segment of the ſpherical Surface will be equal to the portion of the cylindrical, intercepted between the ſame parallel Planes.

Let the Sphere, AEBF, be circumſcribed by the Cylinder GIKH, and let them be both cut by the Planes CD, EF, parallel to the Baſes, GH, IK, of the Cylinder.

I ſay, the ſpherical Surface NAO, or ENAOF, is equal to the cylindrical Surface ICDK, or IEFK; alſo, the ſpherical Surface NEFO, is equal to the cyl. Surface, CEFD.

Let AB, be the Axe of the Cylinder, and Sphere; and let the Diameters CD, EF, of the Cylinder, be drawn, cuting the Axe perpendicularly, at L and M, and the Surface of the Sphere at N and O, E and F. Draw AN, NB, and AE, EB.

Dem. Then becauſe ANB is a Right angle (12. 3) AN is a mean Proportional between AL and AB; i. e. between CI and CD. - - - - - - C. 2. 7. 6.

Alſo, becauſe AEB is a Right angle, AE is a mean, between AM and AB, i. e. between EI and EF. - ſame.

But, the Circle, whoſe Radius (AN) is a mean Proportional, between CI and CD, is equal to the cylindrical Superficies, ICDK (10.); and, it is equal to the Superficies of the ſpherical Segment NAO. Therefore, the Superficies of the two Segments, of the Cylinder and Sphere, are equal.

After the ſame manner it may be proved, that the Superficies IEFK, of the Cylinder, is equal to the Surface of the Segment, ENAOF.

Conſ. the Surface of the Cylinder, CEFD, between the parallel Circles, CD and EF, is equal to the Segment of the ſpherical Surface, NEFO, between the ſame parallel Planes.

And, it is alſo manifeſt, that the remaining Superficies, EGQHF, of the Cylinder, is equal to EBF of the Sphere.

From this, and the foregoing moſt excellent Theorems, is deduced a certain Rule for aſcertaining the true Area of the Surface of any Segment of a Sphere or other Portion, contained between parallel Circles.

COR. The Segments of a ſpherical Surface, intercepted between parallel Circles, have the ſame Ratio, between themſelves, as the Segments of the Axe, or Diameter (AB) cut by theſe Planes.

For, each Segment, or portion of the ſpherical Surface, NAO, NEFO, and EBF, is reſpectively equal to the correſponding portion of the Cylinder, ID, CF, and EH; and thoſe Portions are, to each other, as IC to CE, to EG, i. e. as AL: LM: MB; conſequently, the Surfaces, NAO, NEFO, and EBF, are to each other, as AL, LM, and MB.

THEOREM XVII.

A Sphere is equal, in its ſolid Contents, to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius.

Imagine the Sphere ADF circumſcribed by any regular Body whatever; and, from every Angle of the circumſcribing ſolid, ſuppoſe Right lines, AC, EC, &c. drawn to the Center, C; there will be formed as many Pyramids, as the Solid has Faces, whoſe common Vertex is at the Center of the Sphere; which, together, is greater than the Sphere.

Imagine further, that every Angle, A, B, E, &c. of this Solid, be cut off by a Plane, touching the Sphere, and perpendicular to the Right line from the Angle cut off. It is evident, that there will be formed another regular ſolid, containing the Sphere, touching every Face; which Solid is leſs than the firſt, but it is greater than the Sphere; and, by drawing Lines from every Angle, a and b, *a*, *b*, *c*, &c. as before, there will be formed as many Pyramids as this Solid has Faces.

By proceeding thus, *ad infinitum*, it is manifeſt that the exterior Surfaces of this Solid, which conſtitute all the Baſes of the innumerable Pyramids, muſt, at laſt, end in the Surface of the Sphere; and conſequently, all the Pyramids, or whole Solid, is the ſame as the Sphere; or, the difference leſs than any aſſignable Quantity.

Then, let BEF be a Circle, and C its Center. If the Radius BC be equal to the Diameter of the Sphere, the Circle BEF is equal to the Surface of the Sphere. Th. 15.

Make AC (perpendicular) equal to its Radius, and imagine the Right angled Triangle ABC revolved around; the Hypothenuſe, AB, will deſcribe the Surface of a Cone, BADF.

I ſay, the Cone is equal to the Sphere.

DEM. For, becauſe the Baſe of the Cone is equal to the Surface of the Sphere, it may be divided into as many plane Figures, as the circumſcribing Solid has Faces; each equal, reſpectively, to the other.

Conſequently, if Right lines be drawn from every Angle of theſe Figures, to the Vertex (as at G) there will be formed as many Pyramids, as the Solid has Faces.

But, they are, reſpectively, equal to them, by Hypotheſis; and they have, alſo, equal Altitudes; i. e. the Radius of the Sphere. Conſ. each Pyramid is equal to its correſponding one in the Solid; and, conſ. all the Pyramids contained in the one are equal to the other. But, Cones and Pyramids, whoſe Baſes are equal to one another, and having equal Altitudes, are equal. Therefore, the Cone BADF is equal to the Sphere. Q. E. D.

COR. 1. Sectors of Spheres, are equal to Cones, whoſe Baſes are equal to the ſpherical Superficies of the Sectors, and the Altitude of the Cone to the Radius of the Sphere.

For, ſince the whole Sphere is equal to a Cone, whoſe Baſe is equal to the whole Surface, and Altitude to the Radius; conſ. a Cone having the ſame Altitude will be to the other, as the Baſe of one to the Baſe of the other; i. e. as the portion of the ſpherical Surface to the whole; and, conſ. the Sector is equal to ſuch a Cone.

COR. 2. A Hemiſphere is double of a Cone, having the ſame Baſe and Altitude.

For, becauſe the Surface of a Sphere is equal to four large Circles; the Surface of the Hemiſphere is double its Baſe.

But, a Cone whoſe Baſe is equal to the ſpherical Superficies of the Hemiſphere, and equal Altitude, is equal to the Hemiſphere; conſequently, if the Baſe of the Cone be equal to the Baſe of the Hemiſphere, it will be half the Hemiſphere.

COR. 3. A Segment of a Sphere is equal to the Difference between a Sector, whoſe ſpherical Superficies is the Surface of the Segment, and a Right Cone, whoſe Baſe is equal to the Baſe of the Segment, & its Side, to the Radius of the Sphere.

FINIS.

AN

APPENDIX;

ON THE THEORY

OF MENSURATION.

MENSURATION is the Art of determining the Meaſure of LINES, of SURFACES, and SOLIDS, by ſome fix'd and determined ſtandard; of which, a FOOT, or an INCH, may be conſidered as the primary. How ſuch a ſtandard as a Foot or an Inch is, or may be, aſcertained, I ſhall not enquire; but only conſider it as it is and has been made uſe of, for obtaining the *Meaſure* of Length, the *Areas* of Surfaces, the *Contents* of Solids, and the *Capacities* of Veſſels, their hollow Contents.

An INCH is conſidered as the length of three Barley-Corns; but that is vague and undeterminate: Being the leaſt, it may be conſidered as the firſt, the ſtandard of all other. A FOOT is the Length of twelve Inches; or, the Meaſure of a Foot being firſt determined, an Inch is a twelfth part of it. Three Feet, or thirty-ſix Inches, is called a YARD; by which three Meaſures, all other, of every Denomination, is determined.

In reſpect of Length, ſimply, 'tis but to apply any of thoſe three Meaſures, which is leſs than the Length to be meaſured, as often as one can contain the other: For inſtance; if the Length to be meaſured be leſs than a Foot, it is meaſured by an Inch; if leſs than a Yard, 'tis meaſured by a Foot, or an Inch, or both; being one Foot, or two Feet, and three, five, eight, or any number of Inches leſs than a Foot. If the Length be more than a Yard, any Length leſs than a Furlong (the 8th part of a Mile,

220 Yards) it is meaſured by a Yard, or Foot, or both; except ſome kinds of Cloth, which is meaſured by the Ell (5-quarters of a Yard); Land is meaſured by Chains and Links. A Mile is eight Furlongs, 1760 Yards. Great Diſtances are meaſured by Miles, or Leagues (3 Miles); but, in almoſt every Country, they differ in their Miles and Leagues, ſome conſiderably. A German Mile is $4\frac{3}{4}$ of ours; eleven Iriſh Miles is equal to fourteen of ours; ſo that, 'tis 2240 Yards. Two Yards is a Fathom, by which, Seamen meaſure the Deep.

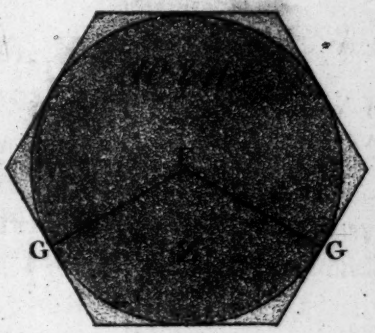
MENSURATION OF SUPERFICIES.

THE whole Art of Menſuration of Superficies, conſiſts in determining a Rectangle, which is equal to the given Figure.

In Theo. 19, Book 1ſt. it is demonſtrated, that every Triangle is equal to half a Parallelogram, having the ſame Baſe and Altitude; and, in the 18th. is ſhewn and demonſtrated, that, all Parallelograms or Triangles, having the ſame or equal Baſes, and the ſame Altitude, are equal. Conſequently, ſince Rectangles are Parallelograms (Def. 33. and 34) and it is, there, fully demonſtrated that they are all equal, having equal Baſes (i. e. having any two Sides equal) and the perpendicular diſtance between thoſe Sides and the oppoſite alſo equal, it is evident, that the meaſure of one is alſo the meaſure of the other. And, becauſe Triangles on equal Baſes with Parallelograms, and being of equal height are equal half ſuch Parallelograms, the Area of a Triangle is readily obtained.

Hence is deduced the general Rule (well known to all Surveyors, or Artificers, concerned in meaſuring Superficies) for finding the Area of any triangular plane Figure; which is, to multiply the Baſe, i. e. any one Side, by half its height, from that Side, or half its Baſe by the whole

Theo. X.



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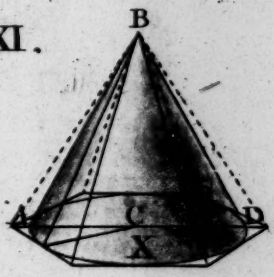

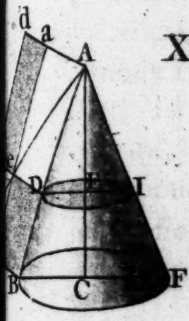

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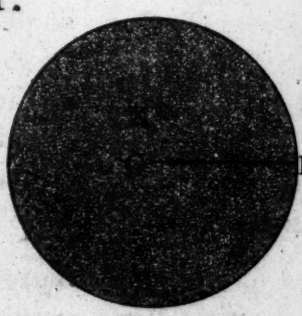

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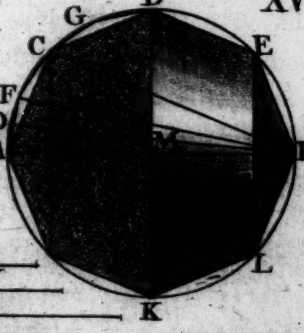

XV.

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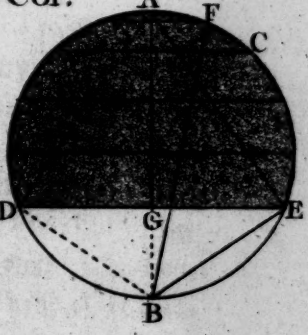

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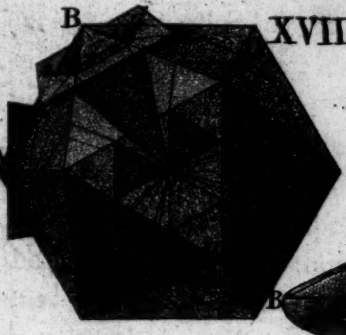
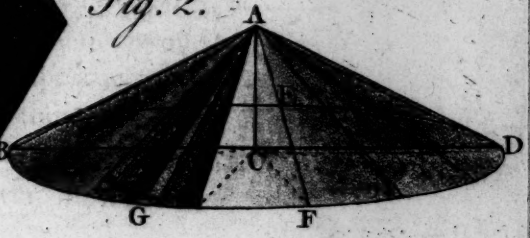
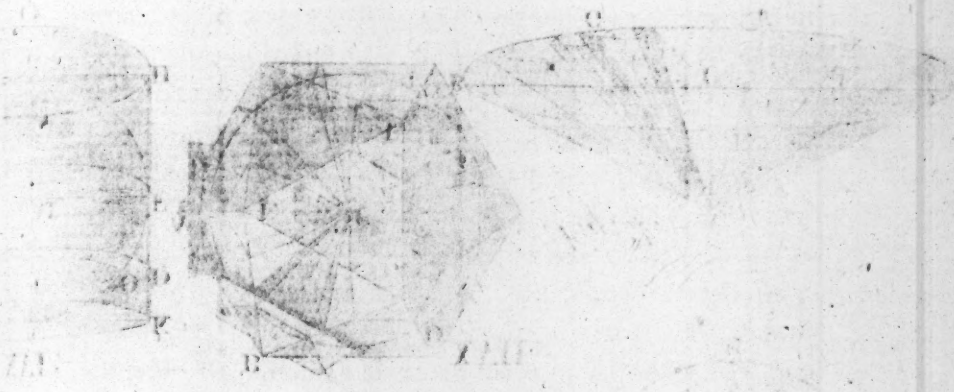
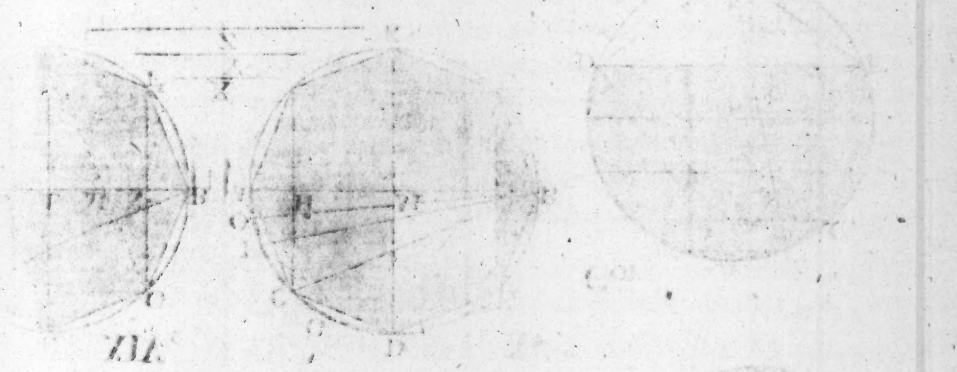


Fig. 2.







height; the reaſon for which is, in thoſe Theorems, clearly accounted for; and, in that conſiſts the whole of ſuperficial Menſuration.

On thoſe Principles, a Rectangle may be conſtructed equal to any Parallelogram, and extended to Triangles and Trapezia; to any right-lined Figure whatever.

If the conſtruction of thoſe Figures be well underſtood; practical Menſuration is eaſily attained; which conſiſts wholly in finding a Rectangle, i. e. in knowing how to obtain the dimenſions of a Rectangle, equal to any given Figure; for, the multiplication of any two Numbers, being applied to meaſure, denotes a Rectangle under ſuch dimenſions, and produces its Area; which I ſhall, in the firſt place, endeavour to make clear and intelligible.

Most People, at the firſt thought of theſe matters, imagine, that if two Figures (of any ſpecies whatever) have equal Circuit, i. e. if the meaſure of all the ſides, of each Figure, in one ſum, be the ſame, they have equal Areas; than which, nothing is more falſe, as will be made appear.

A Circle contains the greateſt Area of any plane Figure whatever, having an equal Circumference or Perimeter. If a Circle be depreſſed, though ever ſo little, it becomes elliptical; in which Caſe, it loſes of its dimenſions; and the more it is depreſſed, the leſs is its Area to that of the Circle. Conſequently, if a cylindrical Veſſel be bulged, or if its Sides are preſſed till it becomes elliptical, it loſes of its meaſure; and the more the Sides are preſſed together, the more it loſes; for it is evident, that, if they are preſſed quite flat, it loſes the whole internal Area.

So likewiſe, of right lined Figures; the more the Sides are multiplied, and the nearer it approaches, in figure, to a Circle, the greater Area it contains. Conſequently, regular Poligons, of a greater number of Sides, contain a greater Area, than thoſe of fewer Sides, having equal Perimeters.

Hence it is plain, that a Rectangle contains a greater Area than any other Quadrilateral whatever; if the meaſure of their Sides, in one Sum, be equal. But, a Square, which is the moſt perfect Rectangle, contains a leſs Area than a regular Pentagon; and a Pentagon contains leſs than a Hexagon; a Hexagon leſs than a Heptagon, and ſo on to a Circle, whoſe Perimeters are all equal; ſo that the Square and Circle, whoſe Perimeter and Circumference are equal, differ greatly in Area, as ſhall be illuſtrated hereafter, to demonſtration.

SINCE, in Menſuration, a Rectangle is the ſtandard by which the Areas of all Plane Figures, as well as of all other ſuperficial Contents, are aſcertained; I ſhall, in the firſt place, ſhew, and account for the methods of taking the dimenſions of various Figures; which meaſures, being multiplied into each other, will give the true Area; each Figure being equal to a Rectangle under thoſe Dimenſions.

1. THE Area of a SQUARE is obtained, by multiplying the meaſure of a Side into itſelf. e. g.

Let the Side of the Square ABCD be five feet; the meaſure of a Foot being repreſented by each ſmall diviſion on its Side, as 1, 2, 3, 4, 5. Now, if from theſe diviſions, lines are drawn, both ways, parallel to the Sides; it will be divided into as many leſs Squares, as the Side multiplied by itſelf produces; viz. 5 times 5, equal 25; which, is the number of ſmall Squares contained in the large Square, ABCD; each ſmall Square a, b, c, &c. being ſuppoſed one Foot, on each Side, is, therefore, called a ſquare Foot. So that, the Square AE, which has three feet on each Side, is a ſquare Yard; its Area is nine ſquare Feet.

But, when a Rectangle meaſures more, or is longer on one Side than the other; then, either Side multiplied by

the other (not oppoſite) gives its Area or ſuperficial Meaſure. Whereas, in a Square, the Sides are all equal; and conſequently, the meaſure of one is alſo the meaſure of any other; therefore, the Side is ſaid to be multiplied into itſelf, to produce its Area.

2. If the Rectangle ABCD meaſures 8 Feet on one Side (as AD) and 5 Feet on the other (AB) the oppoſite Sides are the ſame (15.1.); and if Lines are drawn through the diviſions (as in the Square) parallel to the Sides, it will be divided into 5 times 8, equal 40; the number of ſquare Feet, contained in the Rectangle.

Hence it is evident, that, in Menſuration, any two Numbers denoting Meaſure, in Inches, Feet, Yards, &c. being multiplied into each other, give the Area of a Rectangle under thoſe Dimenſions, in ſquare Inches, Feet, &c. and conſequently, all multiplication, of Meaſure, denotes the Figure, under ſuch dimenſions, to be right angled. Wherefore, the multiplication of Lines, in Geometry, implies a Rectangle conſtructed on the two Lines.

3. To any Point, E (in BC) let AE, DE be drawn, forming a Triangle, AED, on AD, its Baſe.

It is obvious, that the Triangle AED contains ſeveral whole and entire Squares, F, F, &c. and ſome Pentagons, as a, a; ſome Trapezia, as b, b; and ſome Triangles, c, c.

It would not be eaſy to aſcertain how many entire Squares all thoſe irregular Figures are equal to; for there are but 12, entire, 4 Pentagons, 3 Trapezia, and ſeven Triangles.

But, ſince (by Th. 19) we have demonſtration, that the Triangle AED is equal to half the Rectangle ABCD; and the Rectangle ABCD contains 40 ſmall Squares, conſequently, all thoſe irregular Figures are equal to 8 Squares; which, added to the 12 entire ones, make 20, the true Area of the Triangle AED.

4. NEXT, I will ſhew, that two Parallelograms may have the Sum of all their Sides equal, and differ greatly in Area.

The Rect. ABCD, and the Parallelogram AEFD, have their Sides, AD, BC, and EF, equal; for, AD is common to both; conſequently, BC and EF, being both equal to AD (15.1.) are equal to each other. - Ax. 3.

AB is equal to AE, and DC equal DF; and conſequently, AB+BC+CD+AD is equal to AE+EF+FD+AD.

But, the Area of the Par. AEFD is not equal to the Area of the Rectangle ABCD. For, if AE be multiplied into AD, it will produce an Area equal to the Rectangle ABCD, becauſe, AE is equal AB.

But, the Par. AEFD is equal to the Rect. AGHD only. For, it is demonſtrable, the Triangle AEG=DFH; 7.1. conſequently, if DHF be taken away, and its equal, AGE, be added, the Rectangle AGHD is equal to the Parallelogram AEFD.

Again; if AE be produced to I, and DF to K (in BC produced) then, the Parallelogram AIKD is equal to the Rectangle ABCD. For, the Triangle ABI=DCK; wherefore, by taking away DCK, and adding an equal, ABI, the thing is manifeſt.

Now, the Rectangle ABCD, being ſuppoſed depreſſed, or puſhed oblique, to AEFD, has loſt of its dimenſions, conſiderably; and if it be depreſſed lower, to ef, it ſtill loſes more; notwithſtanding the Perimeter, AefD, remains the ſame, equal ABCD. Conſequently, ABCD contains a greater Area than any other Parallelogram, having an equal Baſe, and equal Perimeter.

Therefore, if ABCD be depreſſed at all, i. e. if the Angles are not Right ones, it will contain a leſs Area; for, if AB and CD deviate ever ſo little from the Perpendicular, BC muſt neceſſarily fall lower. Conſequently, its Alti-

tude being leſs, and the Baſe remaining the ſame, its Area is manifeſtly leſs. - - - - Cor. 1. 6.

5. It may ſeem ſtrange, to ſome, that a Square ſhould contain a greater Area than any other Rectangle, having equal Perimeters.

Suppoſe the Rectangle ABCD to meaſure, on one Side (AD) 8 feet, and on the other Side (AB) 6 feet; its Area is $8\times6=48$; and its Perimeter is twice 8 + twice 6, $=28$.

Let AEFH be a Square, on AH, equal ſeven feet. Then, the meaſure of its Sides, in one ſum, is 7×4, or 4 times 7, equal 28, the ſame as the Rectangle.

Now, the Rectangle ABGH is common, both to the Square AEFH, and the Rectangle ABCD.

But the Rectangle, EG, which is the remaining part of the Square, contains ſeven ſmall ſquares; whereas, the Rectangle GD, the remaining part of the Rectangle, ABCD, contains but ſix.

The Area of the Square is 7×7, or 7 times 7, $=49$: whereas, the Area of the Rectangle is but 8×6, $=48$, one 49th part leſs, in its Area, than the Square; which is accounted for by the Figure; and from Theo. 11. Book V.

Hence it is plain, that a Square is the moſt perfect of Rectangles, or other Parallelograms; as a Circle, of all Plane Figures. And, ſince no other Figure, but Rectangles, can anſwer the purpoſes of Menſuration, in producing the true Area; it is certainly moſt conſiſtent, that all ſuperficial Meaſure ſhould be reduced to a ſtandard meaſure, by the moſt perfect of all Rectangles, a Square.

6. In the Parallelogram AEFD, let AE be divided into the ſame equal parts as AB or AD; and if, through thoſe diviſions, Lines are drawn parallel to the Sides, AD and AE, it will be divided into as many leſſer Parallelograms as the Rectangle, ABCD, contains Squares.

But, thoſe leſſer Parallelograms are Rhombuſes (Def. 36) each being in the ſame Ratio to a Square, of an equal Perimeter, as the whole Parallelogram AEFD to the Rectangle ABCD; conſequently they do not aſcertain its Area.

The Area, therefore, of a Parallelogram (except it be a Rectangle) is not obtained by the meaſure of its two Sides, AD, AE, or DF, but by the meaſure of one Side, as AD, only, and its perpendicular Altitude, DH; which, being multiplied together, produce the Area of a Rectangle AGHD; to which, the Parallelogram AEFD, is equal; as it is evident from the Figure.

The Par. AEFD=AGHD, and is deficient of the Rectangle ABCD, by the Rectangle GBCH, equal one ſixth part of ABCD (1. 6.) GB being one ſixth part of AB.

N. B. Either Side of a Parallelogram may be taken, in order to obtain its Area, but not both; unleſs it be a Rectangle. If DF be taken, then, the perpendicular diſtance, EI, between that Side and its oppoſite, AE, will alſo give the true Area.

7. THE Area of a regular Poligon is obtained, as follows.

The general Rule for meaſuring Poligons is, to multiply half the Perimeter, i. e. half the Sum of all the Sides, by a Perpendicular from the Center to any Side; which implies, that a Rectangle of thoſe Dimenſions is equal to the Poligon; the truth of which I ſhall make next to appear.

Let ABDEFGH be a regular Heptagon, whoſe Center is C. Draw AC, BC, &c. dividing it into ſeven equal, Iſoſceles Triangles.

Draw a Right line, a f, indefinite; in which, take a b, b d, &c. each equal to a Side of the Poligon; and, on the Baſes a b, b d, &c. conſtruct the Triangles, a c b, b l d, d k e, and e h f, equal and ſimilar to the Triangle ACB in the Poligon.

If a Right line, c h, be drawn through their Vertices, it will be parallel to a f (C. 18. 1.); and, there will be conſtructed three more Triangles c b l, &c. equal and ſimilar to the other; and conſequently, the Trapezium, a c h f, is equal to the Heptagon AGEB; for, it contains ſeven Triangles, equal to thoſe in the Poligon.

Draw c i perpendicular to a b; the Triangle a c b being Iſoſceles, the Baſe, a b, and conſequently, the Triangle, a c b, is biſected, by the Perpendicular c i. - C. 3. 9. 1.

Produce c h; and, make h g equal a i and draw f g, which will be parallel to c i, and alſo equal; - - 15. 1. wherefore, the Triangle f g h, is equal to the Tri. a c i. 7. 1. conſ. the Rectangle i c g f is equal to the Trap. a c h f; (which was made equal to the Poligon.) - - Ax. 6 and 7.

But, the Side i f, equal c g, of the Rectangle i c g f, is equal to IBDEF, half the Perimeter of the Poligon; and c i, equal g f, the other Side of the Rectangle, is equal to C I, a Perpendicular from the Center of the Poligon.

Conſequently, half the Sum of the Sides, AB, BD, &c. (equal i f) multiplied by the Perpendicular CI (equal c i.) gives the Area of the Heptagon ABDEFGH.

This Rule, holds good for all regular Poligons whatever.

8. From what has been advanced, concerning Poligons, we do with certainty infer, that a Triangle whoſe Baſe is equal to the Circumference of a Circle, and its Altitude equal to the Radius, is equal to that Circle; conſequently, half the Circumference multiplied by the Radius gives the Area of a Circle.

Every Poligon, circumſcribing a Circle, is equal to a Triangle, whoſe Baſe is the meaſure of all the Sides, in one Sum, and its Altitude equal to the Radius of the inſcribed Circle.

The Pentagon ABDEF, is leſs, than the circumſcribing Circle; by the five Segments AGB, BD, &c.

Biſect the Ark AGB and draw CG, which will be perpendicular to AB; and draw the Chords AG, GB, which are the Sides a of Decagon, inſcribed in the ſame Circle; and, if the Side AG be biſected, and CI drawn, it will be perpendicular to AG. - - - - Cor. 4. 9. 1.

The two Sides AG, GB, of the Decagon, are greater than the Side AB, of the Pentagon (13.1.); and, the Perpendicular CK, of the Decagon, is greater than the Perpendicular CH, of the Pentagon; for, CN is longer than CH (C. 2. 12. 1.). And CK is longer than CN; conſequently, the Area of the Decagon, whoſe Perimeter is ten times AG or GB, and its Perpendicular is CK, is greater than the Pentagon, whoſe Perimeter is five times AB, and its Perpendicular is CH. The difference is five Iſoſceles Triangles, equal ABG; whoſe Baſe is AB, the Side of the Pentagon, and the equal Legs or Sides, AG, and GB, are the Sides of the Decagon.

Again; the Area of the Decagon is leſs than the Circle, by ten Segments AIG, GB, &c. For, CK, produced, biſects the Ark AIG, at I; the Chords AI and IG, being drawn, are the Sides of a Poligon of twenty Sides, whoſe Perpendicular is CO.

This Poligon is greater than the Decagon, by ten Iſoſceles Triangles, equal AGI, in the Segment AG or GB; and it is ſtill leſs than the Circle by 20 Segments AI or IG.

From all which, it is clear, that every Poligon, inſcribed in a Circle, is leſs than the Circle, by as many Segments as the Poligon has Sides; and conſequently, every Poligon, circumſcribed, is greater than the Circle. But, the greater the number of Sides of the Poligon, the nearer is its Area to that of the Circle; and conſequently, it muſt at laſt end in the Circle; that is, the Perimeter of the Poligon, will be equal to the Circumference of the Circle, and the Perpendicular equal to the Radius.

For, if AI be again biſected in L; the Sides AL, LI, of a Triangle, in the Segment AI, are greater than the Chord AI, which is the Baſe of that Triangle; yet, they are leſs than the Arks of the Segments AL, LI; but they muſt, at laſt, by multiplying them, end in the Circumference, as it is evident.

So likewiſe, the Perpendicular CH, of the Pentagon, is leſs than CK, the Perpendicular of the Decagon, which is alſo leſs than CO, of the Vigintagon. And, if AL is again biſected, and a Perpendicular drawn to the Chord AL, it will be greater than CO; and will, at laſt, be equal to the Radius of the Circle.

Wherefore, the Circle may be conceived to be an aggregate of an infinite number of Iſoſceles Triangles, whoſe common Vertex is the Center, and whoſe Baſes are in the Circumference, and which are, altogether, equal to it. Conſequently, a Rectangle under the Semi-circumference and the Radius of the Circle; or, under half the Perimeter of the innumerable ſided Poligon (equal to the Circumference) and its Perpendicular (equal the Radius) is equal to the Circle; and conſequently, a Triangle whoſe Baſe is equal to the Circumference, and its Altitude equal to the Radius, is alſo equal to the Circle; for every Triangle is equal to half a Rectangle of the ſame Baſe and Altitude (17. 1.). Therefore, the Triangle is equal to a Rectangle under half its Baſe, and the Altitude.

HENCE, the Rule for meaſuring a Circle is, to multiply half the Circumference by the Radius; and, from hence it is clear, that, of all Figures which have an equal Perimeter or Circumference; i. e. whoſe Bounds are equal to the ſame Right line; a Circle contains the greateſt Area.

9. I HAVE already ſhewn, that a Square contains a greater Area than any other Rectangle, having an equal Peri-

meter, or the meaſure of all their Sides equal. I will next ſhew, how much it is leſs than a Circle, having equal circuit.

Suppoſe the Sides of the Square, ABCD, be each equal to a fourth part of the Circumference of the Circle, GH; then, the four Sides are equal to the whole Circumference. The Area of the Circle is greater than that of the Square.

From what has been ſaid, concerning Poligons and Circles, it is evident, that the Circle, GH, is equal to a Rectangle, two oppoſite Sides of which, are equal to the Circumference; the other are each equal to the Radius EG.

Let the Rectangle aegd be conſtructed ſo, that ae is equal twice AB, and ad equal EG; and draw cf, making ac and ef each equal EF.

The Square is equal to the Rectangle aefc; two oppoſite Sides being equal, each to two ſides of the Square, i. e. equal half the Circumference of the Circle, and the other two Sides, each equal to the Perpendicular, EF, only; which, being conſiderably leſs than the Radius EG; conſequently, the Rectangle aegd, whoſe Sides ae, dg, are each equal half the Circumference, GH (equal ae, cf, of the Rectangle aefc) and the Side eg (equal ad) is equal the Radius, EG (but, the Sides ac, ef, each equal the Perpendicular EF, only) will have a greater Area than the Rectangle aefc; and the difference is the Rectangle cfgd.

The Circumference of a Circle is to its Diameter, nearly, as 22 is to 7; which, is near enough, for meaſuring any Circle that can be formed: but if the Diameter be 113, the Circumference will be 355. In both, the Circumference is too much; the laſt, by little more than a five millionth part of the Diameter.

Suppoſe then (by the laſt) the Diameter, GH, to be 17, the Circumference will be 53,4, very near. The Side of the Square, ABCD, being a fourth part of the Circumference, is 13,35, the Perpendicular, EF, half the Side,

is 6,675; and the Radius of the Circle EG, is 8,5; the Area of each is as follows.

AB+BC, =26,7	Half the Circumference 26,7
× by EF, = 6,675	× by the Radius, EG, = 8,5
1335	1335
1869	2136
1602	
1602	Area of the Circle, 226,95
Area 178,2225, of the Square	178,2225
	Difference 48,7275

A regular Pentagon, having the ſame Perimeter as the Square, is greater than the Square, but it is leſs than the Circle; alſo a Hexagon, is ſtill greater than a Pentagon, of equal Perimeter. A Heptagon, is ſtill greater than a Hexagon; and an Octagon, greater than a Heptagon; ſo that, the more the Sides are multiplied, the nearer it approaches to the Circle; for, the Perimeter of the Poligon, by multiplying, will end, at laſt, in, or coincide with the Circumference of the Circle.

10. A CIRCLE, then, contains a greater Area than *any* right-lined Figure, having equal Perimeters; it alſo contains more ſpace than an Ellipſis, or any other curved Figure, whatever, whoſe Periphery is equal to the Circle's Circumference; for, as a Square, or other Rectangle, loſes of its Area, being depreſſed, or rendered acute angled (Art. 4.); ſo, a Circle, being preſſed and rendered elliptical, alſo loſes of its Area; and the more ſo, the flatter, or more excentric it is made.

A Circle, whoſe Diameter is a mean Proportional between the two Axes of an Ellipſis, contains an equal Area; by which, the Area of an Ellipſis is with certainty obtained. But ſuch a Circle will have a leſs Periphery than the Ellipſis, which, to demonſtrate is not ſo eaſy as right-lined

Figures; nor have I ſeen any Method, for aſcertaining the Periphery of an Ellipſis with certainty. I ſhall, therefore, ſuppoſe a Circle, whoſe Diameter is 7 Inches, depreſſed to an Ellipſis, till the conjugate Axe is but 6 Inches, the tranſverſe will be about 7,8; a mean Proportional between which is 6,8 or ſomewhat more; which is 2-tenths, nearly, leſs than 7. Again; ſuppoſe the ſame Circle depreſſed more, until the tranverſe Axe is 9 Inches; the conjugate will be nearly 4,5; the mean Proportional to which is, nearly, 6,364, which is near ,636 deficient of 7, the Diameter of the original Circle; from which it is manifeſt, that the more it is depreſſed, that is, the more excentric the Ellipſis is, having the ſame Periphery as the Circle, the leſs is its Area, in proportion to that of the Circle.

In Art. 8, it is determined, that a Circle is equal to a Triangle, whoſe Baſe is equal to its Circumference, and its Altitude equal to the Radius. But, an Ellipſis, having an equal Periphery with a Circle, is leſs than the Circle, in Area; for it is equal to a Circle, whoſe Diameter is a mean Proportional between the two Axes of an Ellipſis, which Circle has a leſs Periphery than the Ellipſis.

Let the tranſverſe Axe of an Ellipſis be 25 Feet, the conjugate 16; it will be equal to a Circle whoſe Diameter is 20 Feet. For, the Circumference of ſuch a Circle will be 62,832, near, half of which is 31,416; which multiplied by 10 (the Radius) gives 314,16, for the Area of the Circle, as by the 8th Article. Or, if the two of Axes of the Ellipſis be multiplied together, it will give the Area of the Rectangle ABCD; which Rectangle has that Area to the Ellipſis inſcribed, in the proportion of 1000 to 785. So that 314,000, the Product of $25 \times 16 \times 785$, being divided by 1000, gives 314 for the Area of the Ellipſis; the other is not $\frac{16}{100}$ parts of a Foot more.

From what has been advanced, it is manifeſt then, that the Theory of Menſuration, of Superficies, conſiſts in obtaining the Sides of a Rectangle, equal to the given Figure; ſince (as I have ſhewn) the Multiplication of any two meaſures, into each other, denotes a Rectangle of thoſe dimenſions.

11. HAVING ſhewn how the meaſures of Plane Figures, from a Square to a Circle, are taken, ſo as to produce their Areas; I ſhall, in the next place, account for the multiplication itſelf, by which, the Area is aſcertained.

Suppoſe, then, a Rectangle ABCD, whoſe Sides, AB and CD, meaſure, by ſome Scale of equal Parts, 5 and 9 twelfths, each; and the other Sides, AD and BC, each 8 and 6 twelfths.

Through the Diviſions 1, 2, 3, 4, &c. on the Sides AB and AD, which may repreſent the meaſure of a Foot, or 12 Inches, each (for, you may ſuppoſe any meaſure whatever, to be repreſented by them) if Right lines be drawn, parallel to the Sides of the Rectangle, as in the Figure, it will be divided into as many whole and entire Squares, as the whole numbers produce; viz the Rectangle AE, whoſe Sides, A 8, A 5, being multiplied, give 5 times 8, equal 40. The fractional parts, a, b, c, d, e, f, &c. produce 8 more Integers or Squares, and 7 eighths of another; for, the 8 ſmall Rectangles a, b, c, &c to E, are each 3 fourths of a Square; being the full meaſure on one Side, and 9 twelfths on the other; and are, altogether, equal to 6 entire Squares.

The 5 ſmaller Rectangles d, e, f, &c. to E, being but 6 twelfths (half the Integer) on one Side, are each equal to half a Square; and make, together, two and a half; and the ſmall Rectangle EC, being 3 fourths of one of theſe, or half one of the other, is equal 3 eighths of a Square.

The large Rectangle, AE, is equal to 40 Squares; to which, if the Rectangles BE, DE, and EC be added; BE=6, DE=$2\frac{1}{2}$, and EC=$\frac{3}{8}$ of a Square; together they are equal to 8, and 7 eighths; which, added to 40, make 48 and 7-eighths; the sum total of the Area of the Rectangle ABCD, in square measure, by that Scale.

AE	= 40
BE	= 6
DE	= 2 - 6
CE	= 0 - $4\frac{1}{2}$
ABCD	48 - $10\frac{1}{2}$

12. It may not be impertinent, nor superfluous to some to shew how the Product arises, by multiplying Feet and Inches, together; commonly called Cross Multiplication.

The measures of the two Sides of the Rectangle, AB and AD, 8 - 6, and 5 - 9, being placed one under the other ('tis not material which is first, or uppermost) as in the margin, we begin with the Feet, in the multiplier, and say, 5 times 6 is 30.

Now, it is evident, from the Figure, that this is Feet, one way, and Inches, the other; and is the Rectangle DE, 5 Feet long, E8, by 6 Inches, 8 D.

8 - 6
5 - 9
42 - 6
6 - 4 - 6
48 - 10 - 6

Feet multiplied by Inches, or Inches multiplied by Feet, produce Inches; i. e. the twelfth part of a square Foot; 30 Inches is, therefore, equal to 2 Feet and 6 Inches.

'Tis the same, if you begin with the 9 inches, first.

Set down 6 Inches, under Inches, and carry the two Feet forward, saying, 5 times 8 is 40; which being Feet both ways, it consequently produces square Feet, the large Rectangle, AE. The two Feet, arising from the Rectangle DE, being added, make 42 Feet; which, set down under the Feet.

Next, you take the Inches, in the multiplier, and say, 9 times 6 is 54; i. e. $9 \times 6 = 54$.

Now, theſe are Inches both ways (the ſmall Rectangle EC) and produce only Parts, or ſquare Inches, i. e. the 144th part of a ſquare Foot. For, if the Rectangle EC be divided, on each Side, into Inches (viz. 9 and 6) and lines be drawn through them, as in the large Rectangle AE, it will contain 54 ſquare Inches, which are called Parts, twelve of which are called an Inch. For, it muſt be obſerved, that what is meant by an Inch, here, is not a ſquare Inch, as by a Foot is underſtood a ſquare Foot; but 'tis the 12th part of a ſquare Foot, equal 12 ſquare Inches, or Parts; ſo that, the ſmall Rectangle EC, which contains 54 ſquare Inches, it is evident, is equal to 4 Inches and 6 Parts, only; ſince twelve of theſe Parts are but one Inch, and 4 times 12 is 48; 6 Parts, half an Inch, remaining, ſet down, a place to the right hand of Inches, and carry forward the Inches to the next Place.

Laſtly, ſay 9 times 8 is 72, which is the Rectangle BE, 8 Feet one way (5 E) by 9 Inches the other (5 B) and produces Inches; which, adding the 4 Inches from the laſt place, make 76 Inches, equal 6 Feet 4 Inches; ſet them down, as above, Feet under Feet, and Inches under Inches; which, added into one Sum, makes 48 Feet, 10 Inches, and 6 Parts (equal half an Inch) the Area of the whole Rectangle ABCD, in ſquare Feet, Inches, and Parts.

Thus, multiplying Feet and Inches together, the Area of a Rectangle is obtained, whoſe Sides are equal to ſuch Dimenſions. 'Tis needleſs to proceed further; as it is evident, if Parts were taken, they would be accounted for after the ſame manner.

13. MULTIPLICATION, by Feet and Inches, denotes the fraction of a Foot to be in twelfth Parts, which are called Inches; and theſe being again divided into twelve are called Parts, and ſo on to Seconds and Thirds; each of which is a twelfth part of the foregoing. Decimal Fractions implies the Integer, to be divided into

tenths, hundredths, and thouſandth Parts, and ſo on, by tens, *ad infinitum*, as the other by twelves; and are therefore called Duodecimals.

In the following Example, the Area of the Rectangle ABCD is produced, the ſame, by decimal Fractions.

The meaſures of the Sides of the Rectangle, in Decimals, are as in the margin. In Decimals, 75 hundredth parts of a Foot, is equal 9 Inches; and 5 tenths of a Foot, is equal 6 twelfths, half an Inch.

I place the leaſt Side (AB), firſt, becauſe it has more places of Decimals, and would occaſion nother Line, if placed otherwiſe, the Product would be the ſame.

5,75
8,5
2875
4600
48,875

In the operation, the Figures are multiplied together, as in common Arithmetic, without any regard to the places of Decimals or Integers; and, when added together, as many Places of Decimals as there are in both the Multiplicand and Multiplier, viz. three, are taken from the right hand of the Product; the remaining two are Integers, or Feet; which produces 48 Feet, and 875 thouſandth parts, equal 7 eighths, of a Foot; i. e. 10 Inches and a half. For, 500 is half a Thouſand, therefore equal to 6 Inches; and the remaining 375 is 3 eighth parts of a Thouſand, equal to 4 Inches and a half, which is 3 eighth parts of a Foot.

The value of a Decimal Fraction is obtained, as follows.

Let the Integer be whatever it may; multiply the Decimal by the number of parts of the next inferior denomination of the Integer; which, in this Caſe, is Inches, the twelfth part of a ſquare Foot, to which I would reduce the Decimal ,875.

Being multiplied by 12, take away from the	,875
Product as many Places as there were at firſt,	12
there is left 10 nches; the remaining Decimal	
(500) is now thouſandth parts of an Inch, equal	10,500
5 tenth; half an Inch, or 6 Parts, as a ſecond	12
operation evinces; Parts being the next inferior	
Denomination.	6,000

If the original ſtandard meaſure of a Foot was, inſtead of Inches, divided into tenth parts of a Foot; or, if a Foot were but ten Inches, inſtead of twelve, and the Inches ſubdivided into tenths, inſtead of twelfths or eighth parts, Menſuration would be greatly facilitated thereby; the meaſures being taken in decimal parts, it would be much readier, in all Caſes whatever.

It is rather to be lamented, that all Meaſures, almoſt whatever, are not divided by tens (as a Pound into Ounces, &c.); every Perſon, who has had experience, muſt give the preference to Decimals, as being greatly preferable to any other manner of dividing.

14. Having illuſtrated and explained the rationale of Menſuration of plane Superficies, and ſhewn, how the Dimenſions of all regular Figures are taken, in order to determine Rectangles equal to them, I ſhall give but one ſpecimen of irregular Figures; for this is a brief Theory only, ſhewing how it is founded in Geometry, not a Treatiſe on Menſuration; which, being well underſtood and digeſted, is eaſy to be applied in all Caſes whatever, that may occur, in Practice.

Menſuration of Superficies, applied to various kinds of Artificers Work, is ſtill the ſame in Theory, but much eaſier, in Practice, than Surveying Land; on account of the irregularity of its Figure, and uneveneſs of its Surface. The former depends, chiefly, on the mode, or cuſtom of taking their dimenſions, in order to reduce the

Work to ſquare meaſure; which is not ſo eaſy to deſcribe, but requires very little application under a Proficient. For, a Perſon ever ſo well verſed in Theory, being a ſtranger to Practice, would differ in his Eſtimate, for no other reaſon but taking his dimenſions different; in which they are not ſo very exact; inſomuch, that two Meaſurers, of long practice, taking their dimenſions ſeparately, would differ conſiderably in their Sums total.

ABCDEFG is an irregular Heptagon; having 7 Sides.

Draw the Diagonals BG, CG, GD and GE, or otherwiſe at diſcretion, dividing the given Figure into Triangles, as ABG, BCG, &c. in each Triangle, draw a Perpendicular, as AH, CI, &c. The meaſure of a Perpendicular and half the Baſe on which it falls, in Feet and Inches, or Decimal Parts, being multiplied into one another, give the Area of each Triangle ſeparately; which, added into one Sum, is the Contents of the whole Figure (Ax. 2. 1.)

Or it may be done ſomewhat readier, by taking any two contiguous Triangles, as ABG, BCG, which, together, make a Trapezium, ABCG; unleſs it ſhould happen to be a Parallelogram.

The Diagonal (BG) is then a common Baſe to the two Triangles, and the Area, of both, is obtained at once, by multiplying the Diagonal by half the Sum of the two Perpendiculars; or, the whole by half the Diagonal.

The Area of any other two contiguous Triangles, as GDE, GEF, forming a Trapezium, is got after the ſame manner; and if another Triangle remains, as GCD, its Area, being obtained, muſt be added to the Areas of the two Trapeziums, into one Sum; which is the true Area of the Heptagon ACDF.

In this Proceſs it may be obſerved, that not one Side of the Figure is meaſured, but only the Diagonals and Perpendiculars, all which fall within the Figure. The reaſon is obvious; becauſe the Sides are not at right angles with each other; or if ſome of them were, it would be of

no uſe, unleſs they formed a Rectangle, or other Parallelogram, with the adjacent Diagonal, which is ſeldom the caſe. Whereas, the Perpendiculars making right Angles with the Diagonals, it is eaſy to account for the true Area being obtained by them; from 19th of the 1ſt.

N. B. If the Figure to be meaſured be a large Field, the meaſures are taken in Chains and Links, &c.

15. In Surveying Land, it may ſo happen, that, on account of Bogs, Water, Trees, &c. it would be very difficult, if not impoſſible, to obtain a Perpendicular, within the Figure, when its Sides are acceſſible. In ſuch Caſe, the Perpendicular may be found arithmetically, deduced from the 13th of the 2nd Book; or, the Area may be had without it, by the following Rule.

From half the Sum of all the Sides of the Triangle, ſubtract each Side ſeverally; multiply the half Sum by the three Differences, continually, out of which Product, extract the Square Root; which is the Area of the Triangle, e. g.

Let the Sides of the Triangle ADB, be, AB=9, 8; AD=8, 5; and BD=6, 7. The half Sum of which is 12, 5; and the Differences, ſubtracting each Side, ſeverally, are, AB=2, 7; AD=4; and BD=5, 8.

See the whole Operation, as follows.

AB, 9,8	Half the Sides 12,5		783	(27,98
AD, 8,5	2,7		4	Area.
BD, 6,7	87,5	47)	383	
25,0	250		329	
12,5 half the Sum	33,75	549)	5400	
of the Sides.	4,		4941	
Differences. 2,7	135,	5588)	45900	
4,0	5,8		44704	
5,8	1080		1196	
	675			

The half Sum, 783, multiplied into the Differences.

In the Triangle ADB, biſect every Angle by Right lines, meeting at C; the Center of a Circle inſcribed. 4.4.

The Triangle ABD is thus divided into three Triangles ACB, BCD, and ACD; the Perpendiculars of all which, CE, CI, and CK, are equal; and conſequently, AE, is equal to AI; BE equal to BK, and DI to DK. C.2.16.3.

Hence it is manifeſt, that AE × CE is equal to the Trap. AICE. Alſo, BE × CE = BECK, and DI × CI = DICK.

Produce AB; and make BF equal to DI, equal DK; then will AF × CE be equal to the Triangle ADB; for it is equal to the three Triangles, ACB, ACD, and DCB;

I ſay, the ſquare Root, of AF × AE × EB × BF, is alſo equal to the Triangle ADB.

DEM Becauſe AE is equal to AI; EB = BK; and BF = DK, (Con.) AF is equal to half the ſum of the Sides, AB, BD, and AD. Alſo, AE, EB, and BF are, reſpectively, the Differences of thoſe Sides, ſubtracted from half of their Sum, as it is evident; for, AE = AF − BD; EB = AF − AD; and, BF = AF − AB.

Produce AC, and draw FG perpendicular to AF, cutting it at G. Make BH equal BF, and draw HG perpendicular to BD; alſo, BG perpendicular to BC; which will meet in the point G.

For, the Angles at H and K are right, and CBG is right (Con.); wherefore, GBK = KCB (10. 1.); conſ. the Triangles HGB, BCK are ſimilar (C. 2. 4. 6.). But, BCK is congruous with BCE, and HGB with BGF (7. and 11. 1.); for the Angles at H and F are right, and HBG = BGF (CBG being a Right angle, CBE + GBF = to one) and BH = BF; conſ. BCE, BGF, are ſimilar Triangles Alſo, becauſe GF is parallel to EC, the Triangles ACE, AGF are ſimilar; wherefore, CE : AE :: FG : AF; and alſo, CE : EB :: BF : FG. 4.6. and conſ. CE × CE × AF × FG = AE × EB × BF × FG.

Now, becauſe FG is on both Sides, let it be rejected, and ſubſtitute AF on both Sides; and conſequently, it will be CE × CE × AF × AF, i. e. CE□ × AF□, is equal to AE × EB × BF × AF; i. e. to AF, half the Sum of the Sides, multiplied into the three Differences.

But, AF multiplied by CE is equal to the Area of the Triangle; conſ. it is the ſquare Root of CE□ × AF□; which is equal to AF, multiplied into the three Differences, AE, EB, and BF. Q. E. D.

Or, AD being produced, and DL, DH, being made each equal BK; LG being drawn perpendicular to AL, HG to DB, and DG to DC, all meeting AC produced, at G, as before, it may be demonſtrated the ſame. The Triangles ACI, AGL, alſo, ICD, DGL being ſimilar.

MENSURATION of CURVED SURFACES.

16. THE Areas of the Superficies of regular curved Surfaces, are eaſily deduced from the 10th, 11th, 12th, the 15th and 16th Theorems, Book the 8th.

By the 10th, the curved Superficies of a Cylinder is equal to a Circle, whoſe Radius is a mean Proportional between its Side and Diameter. Or, it is obvious, from the 5th, that it is equal to a Rectangle, whoſe Sides are its Circumference and Height.

17. BY the 11th it appears, that the conical Superficies of a Right Cone, is equal to a Circle, whoſe Radius is a mean Proportional between its Side and the Radius of its Baſe. Or, it is equal to a Triangle, whoſe Baſe is equal to the Circumference of the Baſe of the Cone, and its Altitude, to the Side of the Cone.

By the 12th it is manifeſt, that the conical Superficies, of the Fruſtum of any Right Cone, is equal to a Circle,

whoſe Radius is a mean Proportional between the Side of the Fruſtum, and the Radius of the Baſe, added to the Radius of the oppoſite Circle. And every Circle is equal to a Triangle, whoſe Baſe is equal to its Circumference, and its Altitude to the Radius. Art. 8th.

Or, the conical Superficies is equal to a Rectangle, whoſe Sides are equal to the Side of the Fruſtum, and half the Sum of the Circumference of the Baſe, added to half the Circumference of the oppoſite Circle.

18. THE Area of the Surface of a Sphere is equal to a Circle, whoſe Radius is equal to the Diameter. 15. 8.

And the Area of any ſpherical Segment is equal to a Rectangle, one Side of which is equal to the Circumference of the Sphere, the other to the height of the Segment. For, it is equal to a cylindrical Surface of thoſe Dimenſions. - - - - - 16. 8.

The ſame holds true of any portion of the Surface of a Sphere, intercepted between parallel Planes.

The ſpherical Surface, intercepted between parallel Planes, is therefore equal to a Rectangle, whoſe Sides are the Circumference of a large Circle of the Sphere, and the perpendicular diſtance between the parallel Circles.

Theſe Rules, reſpecting curved Surfaces, being clearly underſtood, will be found extremely uſeful to Artificers, in meaſuring circular Halls, or Rotundas of any kind; alſo Domes, entire; or where a part is deficient by means of a Lanthorn, or otherwiſe, the Surface between any two parallel Circles being determined by the laſt.

MENSURATION OF SOLIDS.

AS, in Menſuration of Superficies, the whole Buſineſs is to find a Rectangle equal to the Figure propoſed; and to determine how many Squares of a certain Dimenſion,

the Figure is equal to; ſo, Menſuration of Solids conſiſts in determining how many cubical Feet, &c. are contained in the propoſed Solid.

In order to which, it is neceſſary to know the affinity and proportion between one Solid and another, as contained in the 7th and 8th Books of theſe Elements; and particularly Parallelopipeds, to which all other Solids muſt neceſſarily be reduced, in Menſuration; a right angled Parallelopiped being of the ſame importance, in reſpect of Solids, as the Rectangle amongſt Plane Figures. Alſo, as a Square is the criterion of ſuperficial meaſure, ſo a Cube, being the moſt perfect Parallelopiped, is the ſtandard, by which the quantities of Solids are eſtimated.

1. LET AGE be a Cube, whoſe Sides, AB, &c. are each equal to 3 feet; alſo, let acd be a Cube, whoſe Sides is one foot, or inch, &c.

Now, ſuppoſe the Cube acd to be the Unit of meaſure; it is required to know, how often acd, is contained in AGE. (For, the Definition of a Cube, ſee Def. 10. 7.)

It is evident, ſeeing the Meaſure of AB, BG, &c. are each equal to three times ab, bc, &c. that the Face ABCD, contains the Square bd, 9 times; every other Face, BGFC, or CDEF, the ſame; therefore, each ſmall Cube *abc*, *def*, *feg*, &c. made by Sections of Planes through *ag*, *gl*, &c. as in the Figure, are equal to acd. Wherefore, ſince each Surface contains 9 Squares, as BF; and if B*a*, C*g*, &c. be one foot in thickneſs, conſequently, the Solid *a*G*lg*, contains the ſmall Cube, acd, 9 times.

But, AB, is equal to three times ab; wherefore, the whole Cube AGE contains the ſmall one 27 times.

For, if it be ſuppoſed to be cut, by parallel Planes, through *agl* and *hik*, parallel to the Top and Baſe, the Parts *a*G*l*, *ail*, and *h*D*k* are equal, ſeeing that their Surfaces are equal; and being alſo of equal thickneſs, C*g*, *gi*, *i*D; conſequently, B*g*F, *ail*, and *h*D*k* are

equal (15. 7.). But, Bg FG is equal to 9 times acb; therefore, AGE is equal to three times 9, =27.

2. It is manifeſt, that if the Solid ACE (being a Parallelopiped) were longer, equal 5 times ab, having equal thickneſs, it would contain the ſmall Cube, a c *b*, 45 times, i. e. five times 9; and thus it will increaſe, as often as the meaſure ab is added in length.

Suppoſe half ab be added; it will contain half 9 Cubes, i. e. $4\frac{1}{2}$; ſeeing that the Surface of the End, or Section through CE, contains 9 Squares, and the Solid CDE has but half a Cube, i. e. half ab in thickneſs. Therefore, the whole Solid BDF contains the Cube acb $49\frac{1}{2}$ times.

Hence, the meaſure of a right-angled Parallelopiped is to multiply the Side AD by AB, which gives the Area of one Surface (the upper Face, or Baſe) and, that Product being multiplied by its height or thickneſs, AF, i. e. by the number of times it contains ab (as, in this caſe, three times) the Product of that multiplication is the ſolid Contents of the Parallelopiped.

But, all Parallelopipeds, having equal Baſes and Altitudes, are equal (17. 7.). Wherefore, whether it be right or acute angled, having obtained the Area of any one Face, in ſquare meaſure (Art. 12.) that Product being multiplied by the perpendicular Diſtance, between that Face and its oppoſite, gives the ſolid Contents of the Parallelopiped. e. g.

3. Suppoſe an acute angled Parallelopiped whoſe Baſe is 7 Feet, 9 Inches, on one Side; the other (being acute angled) is not material, but a Perpendicular is 5 Feet, 4 Inches; its Area in ſquare meaſure, is 41 Feet, 4 Inches.

The height of the Parallelopiped, being 3 Feet, 8 Inches, it is obvious, that the whole Solid contains 3 times 41-4, and $\frac{2}{3}$; i. e. as often as the meaſure of a Foot is contained in the height, ſo often the Solid contains cubical Feet, as its Baſe contains ſquare Feet.

7-9	
5-4	
38-9	
2-7	
41-4	
3-8	
124-0	
27-6-8	
151 6-8	

Wherefore, multiplying 41-4, the Area of the Baſe, by 3-8 (the Altitude) the Product is 151 Feet, 6 Inches, and 8 Parts; that is, half a Foot, and two thirds of an Inch, for the ſolid Contents of ſuch a Parallelopiped.

Note. It is immaterial how the Faces of a Parallelopiped are inclined to each other; for, the Area of any Face being obtained, that Area multiplied into its height (the diſtance between that Face and its oppoſite) give the ſolid Contents, in cubical Meaſure.

Note 2. What is here meant by an Inch, in Solid meaſure, is the twelfth part of a Cube Foot; viz. a Foot ſquare, and one Inch in thickneſs; or, that Quantity in any other Figure. And the Part, in Solid meaſure, is 12 cubical Inches; twelve Inches in length, and one Inch in breadth and thickneſs, or ſix by two, &c. a cubical Inch is, conſequently, a Second of a cubical Foot.

Hence it is manifeſt, that all Parallelopipeds, or Priſms whatever, (for Parallelopipeds are Priſms) having equal height, have that Proportion to each other, which is between their Baſes. And, having equal Baſes, they are conſequently, as their Altitudes.

The Rule, therefore, for meaſuring any Priſm whatever, is to multiply the ſuperficial Area of its Baſe, by its perpendicular height. And conſequently, the ſame Rule is applicable for Cylinders. - - - - 5. 8.

4. Every Pyramid is equal to the third part of a Priſm having equal Baſes and equal Altitudes. - Th. 4. 8.

Alſo, every Cone is equal to the third part of a Cylinder (5. 8.). And conſequently, Pyramids and Cones, having equal Baſes and Altitudes, are equal.

Hence the Contents of Pyramids and Cones are obtained; viz. by multiplying the Areas of their Baſes by one third of their height. Or, if multiplied by the whole height, it will give the Contents of a Priſm or Cylinder, of equal height; conſequently, a Pyramid, or Cone, is equal one third part of ſuch a Priſm, or Cylinder.

5. To find the ſolid Contents of a Sphere.

Having obtained its Diameter; find the Area of a Circle of that Diameter (Art. 8.); which being multiplied, by the Diameter, gives the Contents of a Cylinder, whoſe height and Diameter are equal.

The Sphere is equal to two thirds of ſuch a Cylinder. 9.8.

Or, having obtained the Area of a large Circle, of the Diameter of the Sphere; multiply the Product by two thirds of the Diameter.

Otherwiſe. Every Sphere is equal to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius. - - - - - - - - 17. 8.

Therefore, having obtained the Area of its Surface, multiply that Area by one ſixth part of the Diameter.

See an Example, in each, in the Margin.

Let the Diameter of a Sphere be 15; the Circumference of a Circle, of that Diameter, is 47,14, nearly; by the Ratio of 7 to 22.

Then, by Art. 8. the Semi-circumference (23,57) being multiplied by the Radius (7,5, half 15) gives the Area of a great Circle of that Sphere, equal 176,775; which multiplied by 10 (two thirds of the Diameter) gives 1767,75, or three fourths of the Integer.

Multiplying by 10, is only to give one place more of Integers; ſeeing that, a Cypher, being added, makes no difference in the Decimal; 750 thouſandth parts, being equal to 75 hundredth parts; i. e. equal to 3 fourths.

23,57
7,5
11785
16499
176,775
1767,75

Secondly. 15 being the Diameter of the Sphere, find the Area of its Surface (Art. 21.) which is equal to a Circle whoſe Radius is equal the Diameter of the Sphere.

The Circumference of every Circle has the ſame Ratio to its Diameter (Cor. 1. 14. 6. El.); conſequently, the Circumference of a Circle, whoſe Area is equal to the Surface of the Sphere, is double the Circumference of the Sphere; half of which, is equal to the whole of the other, equal 47,14.

Multiply 47,14 by 15, the Diameter of the Sphere, the Product is the Area of the Surface of the Sphere.	47,14 15,
Now, ſince the Contents of the Sphere is equal to a Cone, the Area of whoſe Baſe is 707,1; and its Altitude equal to the Radius, viz. 7,5; and a Cone is equal to one third of a Cylinder of the ſame Baſe and Altitude, therefore, 707,1, multiplied by 2,5, one third of 7,5, will produce the ſame Contents, as before; viz. - -	707,1 2,5 35355 14142 1767,75

6. The ſolid Contents of a Segment of a Sphere is obtained from the 17th of the 8th, Cor. 3rd.

The Contents of the Sphere being equal to a Cone, whoſe Baſe is equal to the Surface of the Sphere, and its Altitude to the Radius, as above; and conſequently, the Contents of any Sector of a Sphere, being equal to a Cone whoſe Baſe is equal to the ſpherical Surface, and its Altitude to the Radius (Cor. 1.); it is manifeſt, that a Segment of the Sphere having the ſame portion of the Sphere's Surface, as a Sector, is equal to the Difference between the Sector and a Right Cone on the Baſe of the Segment, and, its Side equal to the Radius of the Sphere.

Thus, having ſhewn how the ſolid Contents of regular geometrical Solids are obtained, the application of it, in complex ones, will not be difficult; being well verſed in the Theory, and having a ſolid foundation, for the whole, in Geometry.

7. In reſpect of Gauging; all large Veſſels are either Cylinders or Fruſtrums of Cones, and are meaſured on that Principle; allowing ſo many Gallons, Quarts, &c. to a cubical Foot, as it is known to contain.

Barrels, of all kinds, are meaſured as two Fruſtrums, whoſe common Baſe is the Diameter at the Bung; but being wholly curved, from one Head to the other, it will be truer to conſider it as four, or ſix Fruſtrums, two and two, of which, are equal; or the middle as a Cylinder.

8. The true Contents of irregular Solids cannot be obtained, otherwiſe than by covering them with Water, or other Fluid, in a regular Veſſel; when taken out, the Difference, in quantity of Water, is its Contents.

In meaſuring the Trunks of Trees, &c. which are, generally conical, it is uſual to take the Girt in the middle, in order to reduce them to right angled Parallelopipeds; which, if the intention be to produce the real ſolid Contents, is very erroneous. It is well known, that a Circle contains the greateſt Area of any plane Figure, having an equal Perimeter; (See Art. 10). Conſequently, a Cylinder is to a right-angled Parallelopiped, of equal length, having its Ends, Squares, whoſe Sides are, each, a fourth part of the Circumference of the Baſes of the Cylinder, as the Circle is to the Square.

If the Girt be taken in the middle, in order to reduce it to a Cylinder, it is alſo erroneous; for, it is obvious, that the Fruſtrum of a Cone is more than ſuch a Cylinder; the exceſs of the greater End being more than the deficiency of the other. But, as I do not intend to treat at large on thoſe matters, and having gone beyond my firſt deſign, I ſhall only obſerve, that, in this Caſe, the Fruſtrum of a Cone is equal to the Difference, between the whole Cone and the leſſer Cone, ſuppoſed to be cut off.

Some valuable Properties of ELLIPSES demonſtrated.

1. The Rectangle, under any two Abſciſſas, has the ſame Proportion to the Square of the Ordinate which divides them, as the Rectangle under any other two Abciſſas, has to the Square of the Ordinate dividing them.

In a Circle, the Rectangle, under the two Segments of a Diameter is equal to the Square of an Ordinate, at the point of ſection; conſequently they have all the ſame Ratio (Pr. 14.3.El.)

Let ACBD be an Ellipſis; and let there be deſcribed two Circles, the one circumſcribing it, the other, F CfD, inſcribed.

From any Point (E) in the outer Circle's Periphery, draw the Right line EF, parallel to the Conjugate Axe (CD) cuting the Periphery of the Ellipſis, in the Point G; from which Point, draw GH, parallel to the Tranſverſe, AB; alſo draw the Radius EI, which will cut the inſcribed Circle in the ſame Point, K.

For, becauſe KG is par. to IF, FE:FG::IE:IK, 2. 6. i. e. as IB:IC. And, EF par. to CD will be cut, in G, by the Periphery of the Ellipſis, as I*B* to IC. - 19. Em. C. Sec.

Now, GH is parallel to IB, and EF to CI - - - - - Con.
wherefore, the Angle HKI=EIF, and HIK=IEF 4. 1. El.
conſequently, the Triangles, IHK and IEF, are ſimilar;
wherefore, as IE:EF::IK:IH. - - - - - - - - - 4. 6.
But, AI=IE, IC=IK; and FG=IH; - - - - - - 15.1.
wh. AI:EF::IC:FG; conſ. AI:IC::EF:FG, alternately;
and conſequently, AI□:IC□::EF□:FG□. - - - - 15. 6.
But, AI×IB is the ſame as AI□; for AI=IB;
and, AF×FB is equal to EF□, - - - - - - - - - 14.3.
Therefore, as AI×IB:IC□::AF×FB:FG□.

The ſame holds true of any Abſciſſas, whatever, and their Ordinate. For, Af×fB:fg□::AI×IB:IC□
conſ. Af×fB:fg□::AF×FB:FG□.

2. The Latus Rectum (FG) is a third Proportional to the Tranſverſe and Conjugate Axes; AB and CD.

The common method of proving it, is to make FG to AB, as CE □ to AE □; i. e. as CD □ to AB □; which is, in reality, making FG a third Proportional to AB and CD; and then, proving it to be what it is by Conſtruction.

The diſtance of the Focus from the Center is equal the Square Root of the difference between AE (eq. CH) □ and CE □. 20. 1.

Now, IF+FH=AB=2EB, i. e. IF=2EB—FH; - Pr. 1.
wherefore, - IF □ =4EB □ +FH □ —4EB×FH. - - El. II.
But, - - IF □ =IH □ +FH □; - - - - - - 20. 1. El.
wherefore, - 4EB □ +FH □ —4EB×FH=IH □ +FH □;
conſequently, 4EB □ —4EB×FH=IH □ =4EH □, - Ax. 7.
and, conſ. - 4EB □ =4EB×FH+4EH □;
that is, - 4EB □ =4EB×FH+4CH □ —4CE □;
wherefore, - 4EB □ +4CE □ =4EB×FH+4CH □.
But, - - CH=EB; wherefore, 4CE □ =4EB×FH;
that is, - 4CE □ =2EB×2FH; i. e. CD □ =AB×FG;
conſ. FG is a third Proportional to AB and CD. - - 9. 6. El.

3. If from the Center (E) of an Ellipſis, EL (on the conjugate Axe) be made equal to the Semi-tranſverſe, AE, the Tangents, LF and LG, drawn from that Point, will paſs through the Extremes of the Latus Rectum.

4. If from the Point of Contact (K) of any Tangent (KN) to an Ellipſis, an Ordinate (as KI) be drawn to any Diameter, cuting it in I; half that Diameter (AE) is a mean Proportional between the Segment EI, made by the Ordinate, and the Diſtance of the Point N, from the Center, where the Tangent cuts the Diameter, AB, produced; i. e. EI:EA::EA:EN.

5. If a Tangent (MN) be drawn through the Extreme of any Diameter (as KL) cuting any other two conjugate Diameters, AB and CD, in M and N; the Rectangle under KM and KN, the Segments of that Tangent, will be equal to the ſquare of EF, half the Diameter FG, conjugate to the Diameter KL.

6. If from the Extremes, F and K, of any two conjugate Diameters (FG and KL) Ordinates are drawn to the tranſverſe Axe, cuting it in H and I; then is the Ordinate

FH to the ſemi-conjugate Axe, as the Segment EI, made by the other Ordinate, to the ſemi-tranſverſe Axe, i. e. FH:CE::EI:EA; and IK:CE::EH:EB.

7. Every Parallelogram formed by Tangents at the Extremes of any two conjugate Diameters, is equal to a Rectangle, under the tranſverſe and conjugate Axes.

As this is a curious and extraordinary Property, I ſhall give a brief Demonſtration of it, as follows.

Let, NOPQ be a Parallelogram, made by the Tangents, NO, OP, &c. at the extremes of the Diameter GF and its Conjugate KL.

Draw the Ordinates, FH and KI, to the tranſverſe Axe, cuting it in H and I; and draw EM perpendicular to NO; and conſequently, the Triangles EFH, ENM are ſimilar.

For, the Angles NME, EHF are Right; and MNE=FEH. 4. 1.
Wherefore, - EF:FH::EN:EM. - - - - - by 4. 6.

Now, - - FH:CE::EI:EA (6.) and EI:EA::EA:EN 4.
wherefore, - FH:CE::EA:EN; by equality of Ratios.
Then, ſince, EF:FH::EN:EM; and, FH:CE::EA:EN;
conſequently, EF:CE::EA:EM; by inordinate equality;
wherefore, - EF×EM=CE×EA - - - - - by 9. 6. El.
But, - - CE×EA=$\frac{1}{4}$ of the Rectangle under AB and CD.
And, - - EF×EM=$\frac{1}{4}$ of the Parallelogram NOPQ. 18. 1.
Therefore, the Par. NOPQ is equal to the Rect. RSTU. - Ax. 5.

FINIS.

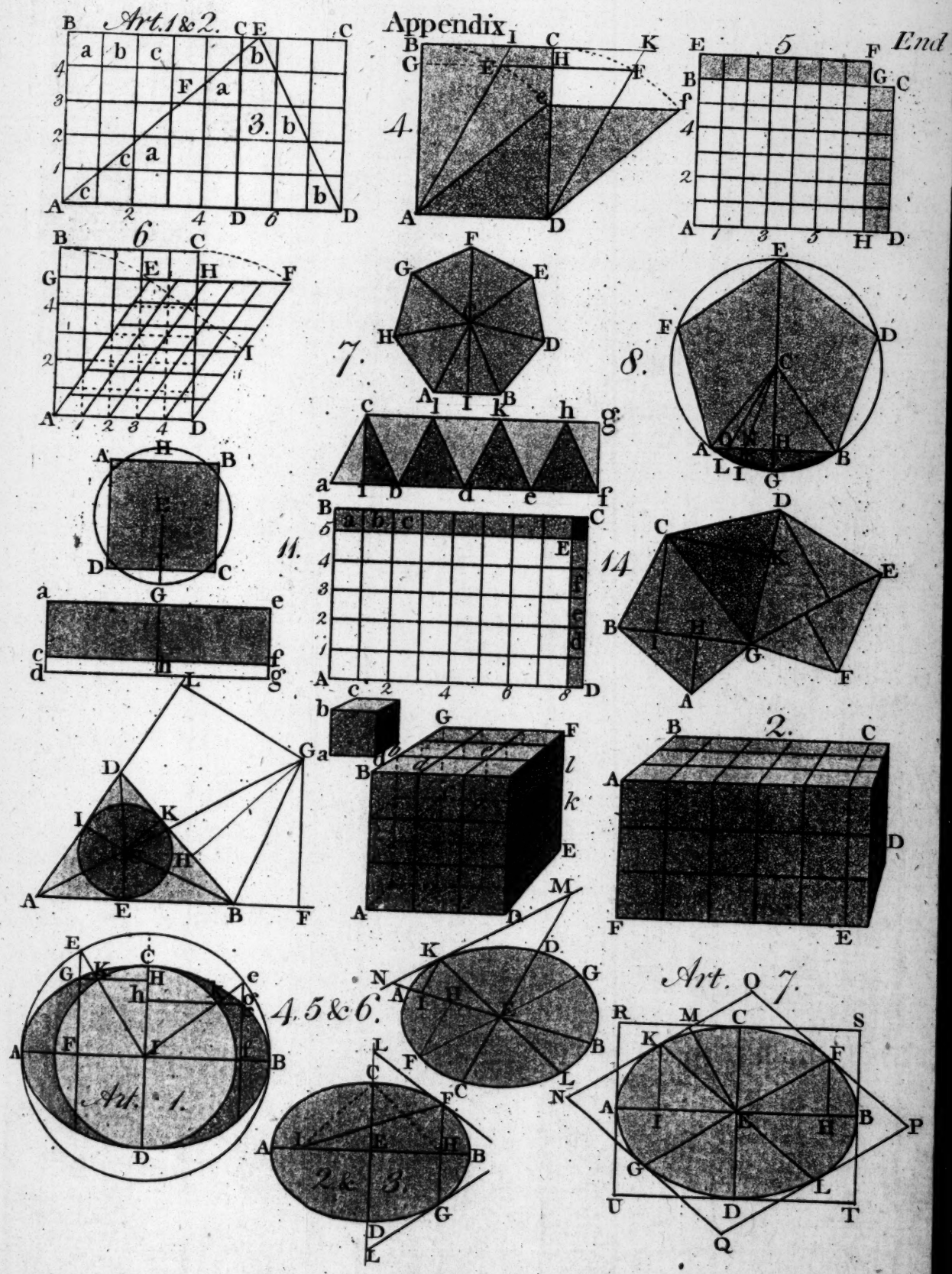
ERRATA.

I WAS in hopes that, from the Attention and Affiduity given to this Work, in the reviſal of it, in which I could have no Aſſiſtance, ſo many Errors could not have eſcaped; but, *Humanum eſt errare*; which will ever verify that well known Couplet, by the celebrated Critic, Mr. Pope, in theſe words;

> They who expect a perfect Work to ſee,
> Expect what neither is, nor was, nor e'er will be.

The following Errors being corrected, I flatter myſelf, that few more (literary Errors excepted) which are likely to miſlead, will be met with in this Work.

Page. Line.
54— 10 and 11, for EGD, read EGH.
64—12, for DG, read BG.
70— 5 and 9, after half Line, and half the Line, read, and part added.
5, Dem. for □ s, read □. Dele s.
75—11, for DE read DF.
80— 4, for + read —CB □.
95—14, for CD, read AD.
119— 5, Dem. for 11. 3. read Cor. 8. 10. 1.
122— 3, N. B. for 17, read 19. 1.
163— 1, Dem. 2nd; for 34, read 24.
Bottom Line; for Qualities, read Quantities.
188— 7, Bott. for DC, read DG.
243— 5, Bott. for BI, read DI.
256—laſt, for ac or *ac*, read ab or *ab*.
257— 6, for CABC, read CABD.
263— 2, Bott. for ABE, read ABF.
269—12, Bott. for akd, read akb.
278— 3, Dem. for AD+DE, read AD×DE.
297— 9, Bott. for 17, read 19, 1.
304—10, for half an Inch, read half a Foot.
308— 6, Bott. for BGF, read GBF.


Appendix
End
Art. 1 & 2.
3.
4.
5
6
7.
8.
11.
14
2.
4.5 & 6.
Art. 1.
2 & 3.
Art. 7.

THE NEW

ROYAL ROAD TO GEOMETRY,

AND

FAMILIAR INTRODUCTION TO THE MATHEMATICS.

PART II.

PRACTICAL GEOMETRY.

Containing all the Problems of Euclid's Elements, with other ſelect ones, from various Authors; which are, here, digeſted in more regular Order, than in any other Tract, extant.

By THOMAS MALTON, Sen.

To which is ſubjoined an Appendix, on the Conſtruction of Ellipſes, Proportional Scales, and the Line of Chords; with other intereſting Matters.

Alſo, as an Addenda, Strictures on a late extraordinary Publication, on Perſpective.

LONDON;

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AND MAY BE HAD OFF ROB. FAULDER, BOND STREET; ALSO, OFF F. WINGRAVE, BROADWAY, STRAND, LATE NOURSE; AND, OFF J. TAYLOR, OPPOSITE GREAT TURNSTILE, HOLBORN.

M DCC XCIII.

PREFACE.

IN the firſt Edition of this Work, the whole is comprized in one Volume, in two Parts, practical and theoretic; with nearly the ſame Appendix as in the firſt Part, of this, on the Theory of Menſuration. The Practice, conſiſting of 52 Problems only (excluſive of the Conſtruction, with ſome Properties of Ellipſes, demonſtrated) being firſt treated on; ſomewhat out of proper Order, I muſt allow, it being uſual, and more regular, in works of Science, to treat the Theory firſt, and then proceed to Practice, on the moſt ſolid Baſis poſſible.

Euclid has intermixed Practice with Theory, making each dependant on the other, alternately; as is, doubtleſs, the real caſe, in Geometry more than other Sciences. So that, thoſe who only want the practical Part are frequently at a loſs, to find the Problems; which is not only troubleſome, being diſperſed in various parts of the Work, but is alſo attended with loſs of Time, in ſelecting them, when interſperſed among theoretic Propoſitions that are not wanted; which renders it perplexing and diſcouraging to a young Student, who is anxious for, and eager in his purſuit of, the practical part, only.

This has induced ſome modern Geometricians to treat them ſeparately; ſome giving both in one Volume, ſome in two; others have treated on the practical part only, being, as they imagine, the only, and real uſeful part; not conſidering, that the Baſis of numerous other, extenſive Sciences, is founded on the Elements of Geometry, in which they have their Exiſtence. There are not, in Euclid, Problems ſufficient for a Volume of a tolerable ſize; but there is latitude enough, to make more of it; by treating them more familiarly, and ſhewing their

utility, in various mechanic Arts, and in delineating, in general; in ſhewing various methods, or ways of performing the ſame Problem, and, in deducing uſeful Leſſons from them, occaſionally.

SINCE the publication of the former Tract, I have ſelected from other Works, and collected otherwiſe, many valuable Problems, and have digeſted the whole into better order; for, in general, I found them very injudiciouſly diſpoſed. I have divided them into Sections, claſſing each under ſome general Head; ſo that, any Problem may be readily found, among others of the ſame kind; thereby rendering the Work more compleat, and the Subject more intereſting, as well as entertaining to the Student; the whole being diſpoſed in ſuch Order as I conceive to be the moſt ſimple and regular: Method being, in my opinion, as neceſſary, to render the acquiſition of Science pleaſant and facile, as Matter, to make it uſeful and intereſting.

ALTHOUGH I think it neceſſary to ſay thus much, by way of Preface to the ſecond Part, it would be ſuperfluous if not impertinent to add more, in a ſtudied Preamble; having given a copious Preface to the firſt, ſeting forth the value and utility of Geometry, as a neceſſary branch of Education, in almoſt every ſphere, and department in Life; and I do not apprehend that they will often be ſeparated, though they may ſometimes; as there are many who have real occaſion for practical Geometry, in their ſeveral Profeſſions, yet have not the leaſt inclination, or leiſure, perhaps, to ſtudy the Elements of it; and would freely diſpenſe with the Demonſtrations here given. For which reaſon, as they may and doubtleſs will ſometimes be parted; as this may fall accidentally into the hands of thoſe who have not the firſt Part, it

would be imperfect and almoſt uſeleſs, to ſome, without Definitions of the Terms made uſe of in it. I have, therefore, defined all that I conceive neceſſary to the Practitioner, merely as ſuch; which, with the Theory of plane Angles, make a proper, and neceſſary Introduction. I have alſo explained all ſuch Abbreviations and References as are uſed in the operations of the Problems; but, for the demonſtrative part, ſeeing it cannot be underſtood without being acquainted with Theory (the firſt Part) where every thing neceſſary is fully explained, it would therefore be ſuperfluous to explain them here.

THE whole is diſpoſed in the following Order, in nine Sections, excluſive of the Introduction, which I have not conſidered as one; for, although it may be, to ſome, and really is a neceſſary part of the Work, 'tis merely introductory; and, as it is in no wiſe problematical, might be diſpenſed with, by many.

The firſt Section contains the elementary Principles of the whole Practice; it treats of Lines and Angles; ſhewing how to meaſure, and diſpoſe Lines into the various Poſitions they may have to one another, forming Angles or otherwiſe, in parallel and perpendicular Poſitions. To divide Lines into two or three equal Parts; alſo to biſect Angles, and to triſect a Right angle; together with the formation of Triangles, the firſt ſpecies of right lined Figures; which are ſo ſimple in themſelves as not to require the aid of any foregoing Problem, ſave the firſt (which is the leading Principle of the whole) but are neceſſary and elementary in the conſtruction of almoſt all other right lined Figures, Parallelograms, only, excepted; alſo, of the conſtruction of Rectangles, in general.

The ſecond Section comprehends the conſtruction of regular Poligons; of all ſuch ordinate, right lined, Fi-

gures as have more than four Sides; by particular, and by general Rules.

The third reſpects and relates to Circles, only.

The fourth Section ſhews how to inſcribe and circumſcribe regular Figures, in general; and Triangles of any kind, in and by Circles; alſo various ordinate Figures, in, or about one another.

The fifth is on Proportionals, and dividing Right lines proportionally.

The ſixth ſhews how right lined Figures, of all kinds, may be transformed into other Figures; particularly into Rectangles, having the ſame Area as the original Figure; by which the Area is aſcertained.

The ſeventh Section treats of the diviſion of Figures; ſhewing how any right lined Figure may be divided into equal Parts, or in a given Ratio, by Right lines, in various caſes.

The eighth teaches how to reduce or enlarge right-lined Figures, at diſcretion.

The ninth, and laſt Section, is on the conſtruction of the five regular Solids, or Platonic Bodies; a very intereſting Subject, and properly connected with the reſt, in this Work.

An APPENDIX is added, on the conſtruction of Ellipſes, of Scales, the Line of Chords, &c. with other Matters; which could not, conſiſtently, be otherwiſe introduced. Alſo an ADDENDA, which is no otherwiſe connected with the Work, than, that it is founded on Geometry; it contains Strictures on a very extraordinary Performance, lately publiſhed, on Perſpective; which ought to be noticed, by all who have true reliſh for, and deſire to make a proficiency in, or to have juſt notions of, that moſt entertaining and very intereſting Science, and Art.

TO THE READER.

TO make an excuſe for the imperfections of any performance, literary or otherwiſe, may ſeem, to ſome, merely an apology for negligence, which ought not to be admitted; 'tis inexcuſable, in Works of this kind, particularly. I can only aſſert, in my own vindication, that I have been moſt aſſiduous, in the production of this Tract; nevertheleſs, I find, to my great grief, that it abounds with more errors than I could have imagined, owing to the new modeling of it; and having none to reviſe or correct it, but myſelf, as I could not obtain any aſſiſtance therein. In conſequence, the errors are more numerous than I could have imagined; yet there are but few of 'em that would miſlead, but would occaſion loſs of time to the Student, who is wholly unacquainted with the Subject. I have therefore noticed every one, that I have, on a ſtrict ſcrutiny, been able to diſcover; and preſume that few, if any, have eſcaped my obſervation.

The Reader would do well to correct them, before he proceeds in the ſtudy.

For diſpoſing the Plates, let the Binder obſerve that each Plate is numbered, and muſt face the Page anſwering thereto.

ERRATA.

Page. Line.

25— 9, for DB, read DC.
31— 2, 42 and 43. 6; 58. 10, E. 86. Pr. 2. for Pr. 8, read 10.
11, read Th. 10; omit 1.
32— 8. and 40. 9, for 11, read 12.
46— 8, Bott. for 10, read 7. 1.—50. 9, for; put,
52— 6, Dem. for BD, read AC, (the last, only)
53—17, read Cor. 1. 14. 6.—54. 1, read Cor. 2. 9. 3.
55— 2, for BC, read HC.
62— 5, after GI, add, cuting BE and BF.
64— 6, for 8. 5. read 9. 5.—65. 10, Pr. 21. after KN, add, and LM.
68— 4, for AEB, read ADB.
79— 8, Prob. 10, for AF, read GF.
80— 7, for BG, read BH. L. 3, Bott. change A3 : AE.
86— 4, Pr. 2. for 1. 5. read 5. 1.
93— 1, for and F, read at F. L. 5, for × put +.
5 and 7, (Fig. 3) omit F; in FEGB and FEDB.
95—14, for 6. 1. read 4. 1.
99— 4, for AFC, read AEC. L. 8, for ADC, read AEC.
103— 9, for BE, read BF.
105— 4, Dem. for BK, BF, read DK, DF.
108— 8, Dem. for BCD, read BGD.
112—13, dele equal ABC.—113. 5, Dem. for AH, read AB.
114— 9, for BEP, read BFP. L. 10. for AGP, read AB.
118— 7, change BIF & KIL.—119. 3. caret at *B* (Fig. 3) and draw *BB*
119—11, Bott. for EBG, read FBG.—L. 9. Bott. dele a.
122—13, for C, read P. L. 13. Bott. for foregoing, read Prob. 16,
123— 8, Prob. 1. for ABC, read ACB.
128— 7, Bott. for at E, read at F.
135— 3, Dem. for AD □, read AE □.
139— 7, Bott. for AB, read BE.
147— 6, Bott. for abf, read abc; and ABC, for ABF,
149— 7, Bott. for CD, read BD.
160— 6, Bott. for DE, read CE.—161. 2, for *or*, read *and* DI, each.
last, for aM, MC, read aL, LC.
163— 8, Bott. for AG, read BG.
192—17, for AD read, AE. L. 4, Bott. for 3. 6. read 2. 6.
193— 1, for CA, CB, CD, read CE, CF, and CG.
195— 1 and 5, for AB, read FB. L. 7. for AG, read AD.
18, for abD, read Ccd.
196— 5, for CP, read CB,—197. last, for OT, read PT.
211— 7, Par. 2. after QZ, put a , for ;
Last. 9, for KH, read FH. L. 11, for FH, read both.

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INTRODUCTION.

DEFINITIONS.

TO define a POINT, ſatisfactorily, is a matter of difficulty, ſeeing it is imagined to be without Dimenſions; and yet, it ſeems to be ſo well underſtood, in general, as to render a Definition of it unneceſſary. As the Subject of this Tract is not merely ſpeculative, but practical or operative, I inform the Student, that nothing more is meant by a Point, here, than a Mark on the Paper, or Board, made by a fine Pen, or ſharp pointed Utenſil of any kind; as the Point of a pair of Dividers, &c.

N. B. The interſection (E) of two Lines (AB and CD, ſtreight or crooked) is conſidered as a Point, the Lines being conſidered as having no breadth; alſo, the junction of two Lines which tend towards each other, in the ſame Plane, meeting at B, as AB and CB; which termination or junction, being the common extreme of both Lines, is to be conſidered as a Point, ſimply, and inconteſtibly.

DEF. A. A LINE is a Magnitude of one Dimenſion, only, viz. Length, without breadth or thickneſs; for, it is the boundary of a Surface, only, but no part of it.

DEF. B. A *Right line*, or *ſtreight* Line, is the ſhorteſt that can be drawn between two fix'd Points; A and B.

If a Line be not ſtreight, it is neceſſarily crooked, that is curved; all ſuch are, in general, called *Curve lines*.

DEF. C. A *Surface* is the boundary of whatever has ſubſtance, or body of any kind, ſolid or fluid; wherefore, it has Length and Breadth, only; for it is no part of the Body, and therefore it has no thickneſs.

DEF. D. A PLANE is a perfectly even or flat Surface; uniformly ſtreight, throughout, in every direction.

Such Surfaces as are externally round, or protuberant, are called Convex; an internally round Surface, hollow, or diſhed inwardly, is ſaid to be Concave.

DEF. E. Lines, or Surfaces, which are equidiſtant in every part, as well as at their Extremes, ſo that, being extended, they would neither approach nearer together, nor ſeparate farther aſunder, are ſaid to be *Parallel.*

DEF. F. An ANGLE is formed by the junction, or meeting of two Lines in a Point; which is called the *Vertex* of the Angle.

When both Lines are in the ſame Plane, it is called a *Plane angle*, ſimply; and they are either right lined, curved, or mix'd Angles. As A, B, and C.

N. B. The length of the *Sides*, or Lines which form and conſtitute an Angle, is not conſidered in comparing Angles, in reſpect of their magnitude; for, being produced, or extended infinitely, the Angle remains the ſame. Magnitude regards only their Inclination; for, the more they are inclined to each other; that is, the nearer they approach to paralleliſm, the leſs is the Angle; and the contrary.

If one Line meet another, at any diſtance from its Extremes, it conſtitutes two Angles, as A and B; and, if two Lines interſect, four Angles are formed, having a common Vertex, in the Point of their mutual interſection, C.

N. B. 2. Angles are equal, when their Sides (the Lines which form them) are equally inclined, in both Angles.

DEF. G. One Right line (as AB) meeting or cuting another, between its Extremes, which does not incline

to the other on either Side, but makes equal Angles with it (ABC, ABD) is said to be *perpendicular*; and the Line AB, so situated, is called a PERPENDICULAR.

DEF. H. The Angles ABC, ABD, which are made, when one Line is perpendicular to another, are called RIGHT ANGLES.

But, if the Line incline on either Side, as BE, making unequal Angles, that which is less than a Right angle, as EBC, is called *Acute*; the other (EBD) being greater than a Right angle is an *obtuse Angle*.

DEF. I. A PLANE FIGURE is a Space bounded, wholly, by one or more Lines, which are all in the same Plane.

If the boundary Lines be all Right lines, it is called, simply, a *Right lined Figure*; but if both right and curved, *mixed Figures*.

DEF. K. A CIRCLE is the first and simplest of Plane Figures; being bounded by only one, uniformly curved, Line, which has neither begining nor end; for, in its formation, the circumscribing Line falls into itself, without forming an Angle; which boundary Line is called the *Circumference*, and any portion of it an *Ark*, or *Arch*; the Point on which it is described is its *Center*.

DEF. L. Every Right line (as AB) passing through the Center of a Circle dividing it into two equal Parts, and bounded by the Circumference, is a *Diameter*; half of which (from the Center to the Circumference) is its *Radius*, or Semi-diameter, by which the Circle is generated, or described, on its Center, C.

N. B. Every Diameter cuts the Circle into two equal Parts (X and Z); and each Part (being so divided) is called a *Semicircle*.

DEF. M. Any portion of a Circle, cut off by a Right line, is called a *Segment of a Circle*; and the cuting Line, which is the Baſe of the Segment, is called a *Chord Line*, or *Subtenſe*.

If a Segment be greater than a Semicircle; i. e. if it includes more Space than half the Circle, it is called a greater, and if leſs, 'tis called, ſimply, a leſſer Segment.

DEF. N. All Plane Figures, bounded by three Lines, have the general Appellation of TRIANGLES, which are the firſt of right lined Figures; as not leſs than three Right lines can include a Space, and form a Figure.

When all the three *Sides* of a Triangle are equal, each to the other, it is called an *Equilateral Triangle* (as A); when only two are equal, it is *Iſoſceles* (B); and, when they are all unequal, it is called a *Scalene* Triangle.

If a Triangle has a Right angle, it is called, ſimply a *right angled Triangle* (as D) and the Side, EF, oppoſite the Right angle is called the *Hypothenuſe*. And, if an Angle be obtuſe, 'tis an *obtuſe angled Triangle*; yet they may be either Iſoſceles or Scalene.

DEF. O. A PARALLELOGRAM is a four ſided Figure, whoſe oppoſite Sides are parallel, and they are alſo equal; by Theo. 15. 1.

DEF. P. When all the Angles of a Parallelogram are right, it is called a RECTANGLE; and, when all the Sides are alſo equal, a SQUARE.

DEF. Q. Every Quadrilateral, or four ſided Figure, which is not a Parallelogram, is a *Trapezium*; which may have two of its Sides parallel, but not the other, alſo.

DEF. R. All Figures having more than four Sides are called POLIGONS; which have particular Names from the number of their Sides.

The firſt is a *Pentagon*, having five Sides; A *Hexagon* is one that has ſix Sides; a *Heptagon* has ſeven; an *Octagon* has eight, a *Nonagon* nine, and a *Decagon* ten Sides. Ordinate or *regular* Poligons are equal ſided and equiangled.

The Sum, or meaſure of all the Sides of a Poligon, is its *Perimeter*.

DEF. S. A Right line drawn between two Angles, i. e. from one Angle to another, within a Figure, is called a *Diagonal*.

A Diagonal cannot be drawn in a Triangle.

DEF. T. BASE of a Figure is the Side on which it is ſuppoſed to ſtand; the Plane of the Figure being vertical, upright.

The height of the uppermoſt Side or Angle, by a Line perpendicular to its Baſe, is the *Altitude* of the Figure.

DEF. U. *Equal Figures* are ſuch as contain equal Spaces, included within the boundary Lines; the meaſure of which is called the *Area* of the Figure, or its Capacity.

DEF. W. *Similar Figures* are ſuch as have all their Angles reſpectively equal; and, the Sides which lie between equal Angles, in each Figure, reſpectively, are proportional, each to each.

Figures which are ſimilar and alſo equal, are *Congruous*.

DEF. X. PROPORTIONALS are ſuch Lines, or *Quantities* of any kind, between two and two of which, there is the ſame Ratio or Proportion *; and, when the

* Ratio and Proportion are ſynonimous Terms.

ſame Proportion is continued between three or more Quantities, they are then in *continued Proportion*, or geometrical Progreſſion.

E. g. An Inch has the ſame Proportion to a Foot, as 3 Inches has to a Yard (3 Feet); conſequently, 1, 3, 12, and 36 are proportional Numbers; and, taken thus, or alternately, by changing the two middle Terms (as 1 to 12, ſo is 3 to 36) 'tis ſtill the ſame; for, being Proportionals, the Product of the Extremes (the firſt and laſt) is always equal to that of the two middle Terms, change them as you pleaſe; as 12 to 1 ſo is 36 to 3; or, 12 to 36 as 1 to 3.

But 3 has not the ſame Ratio to 12, as 1 to 3, wherefore, the Proportion is not continued; for, as 1 to 3, ſo is 3 to 9, and 9 to 27, &c. and, as 1 to 4, ſo is 4 to 16, and 16 to 64, &c. in geometrical Progreſſion.

DEF. Y. When a Line, or other Quantity, is ſo divided, that the leſſer portion is to the greater as the greater is to the whole Line, or Quantity, it is ſaid to be divided in *Extreme* and *Mean proportion*; conſequently, the two parts, and the whole, are in geometrical Progreſſion.

The whole Quantity and the leſſer Part are the *Extremes*; and the greater Part, being the middle Term, is called a *Mean proportional*; and, either of the Extremes is a *Third proportional*, in reſpect of the other Extreme and the Mean.

DEF. Z. When four Quantities are *Proportionals*, whether they are progreſſively proportional, or diſcretely, only (as in the firſt Caſe); any one of them is a *Fourth proportional*, in reſpect of the other three, taken in a certain order, which may be various.

E. g. If 1, 3, and 12 be given, or 1, 12, 3, which are not proportional in themſelves, *any* how taken; but, a fourth Term being added (36) which has the ſame Ratio

to 12 or 3, as 3 or 12 has to 1, they are *then* Proportionals; either Extreme, (1, or 36) being a fourth Proportional, in reſpect of the other three, in that Order. But, taken otherwiſe, 3, 1, 12, or 3, 12, 1, then the fourth Term will be but 4; which has the ſame Ratio to 12 or 1, as 1 or 12 has to 3; ſo that a fourth Proportional is various, according to the arrangment of any three Terms given.

To *biſect* is to cut a Line, or any other Quantity, into two equal parts. To *triſect* is to divide any Quantity, into three equal Parts. To *produce* a Line is to extend or lengthen it, at pleaſure. To *deſcribe* or *conſtruct* a Figure is to draw the Figure required. To *inſcribe* is to draw one Figure within another, the largeſt poſſible of the kind; and, to *circumſcribe* is to draw a Circle, or other Figure, ſo as to paſs through every Angle of the given Figure.

Theſe are all the Terms, which I conceive neceſſary to be defined, for the Practitioner.

THE THEORY OF PLANE ANGLES.

BEFORE we proceed further, I think it proper, firſt, to explain the Theory of PLANE ANGLES, as being a neceſſary Introduction to practical Geometry; inſomuch that, without it, a practical Treatiſe is imperfect; all that I have ſeen are deficient in this reſpect.

In order to have a clear Idea of Plane Angles, and to determine their Quantity by a certain ſtandard or meaſure, imagine the Circumference of a Circle divided into a certain number of equal parts, called Degrees; ſuppoſe 360 (the number of Degrees on the Equator)*; it is evident that thoſe diviſions will be leſs in a ſmall Circle than in a larger.

1. If from any two Points in the Circumference of a Circle, as E and F, Lines are drawn to the Center, as EC and FC, there is made an Angle at the Center, C; which is greater or leſs, according to the number of Degrees on the Ark EDF, but 'twill be the ſame in a ſmall Circle as in a large one, i. e. the lines will have the ſame inclination to each other. e. g.

2. ADB is a Semicircle, whoſe Center is C; the Arch AEFG contains 180 degrees, half 360, the whole Cir-

* This Diviſion has long been eſtabliſhed, and applied to aſtronomical Purpoſes; but any other Diviſion (not in too large Parts) as 320, or 300, would do equally as well, for meaſuring Angles.

cumference. Consequently, if from the middle point, D, of the Arch ADB, which is 90 degrees each way, from A and B, CD be drawn, it will be perpendicular to AB; for ACD and DCB, are Right angles, at the Center (Def. 11.) the Arks AD, DB, being each a fourth part of the whole Circumference, or half the Semi-circumference.

Hence, a Right angle is said to be of 90 Degrees.

3. If the Ark AD be bisected in E, and EC be drawn, the Angles ACE, ECD, will be each of 45 deg. half ACD a Right angle, or 90 Degrees. And, if the Ark DB be trisected, at F and G, and FC be drawn, the Ark DF containing 30, FGB 60 degs. the Angle DCF is said to be an Angle of 30, and FCB an Angle of 60 degs.

4. On the same Center, C, with any other Radius, as Ca, let the Arch aefb be drawn, which is also a Semicircle.

It is very obvious, that it is also divided into the same number of parts, and in the same proportion, as the Arch AEFB; for it is bisected in d, and ad is again bisected in e, and db is also trisected at f and g; wherefore, AD, ad are each a fourth; ED, ed an eighth; BF, bf, a sixth; and DF, df, a twelfth part of their respective Circumferences and the Angles ACD, aCd; ECD, eCd, &c. are the same in both.

From which, it is clear, that, Angles may be formed, or measured by an Ark or Circle of any Radius. And also, that equal Arks of the same, or of equal Circles, or an equal number of degrees in a Circle of any Radius, will form equal Angles at the Center.

5. If you would have an Angle of 60 degrees at the point, C, of the line AB.

With any Radius, at pleaſure, on C, deſcribe the Ark BD, cuting the Line CB in B; with the ſame Radius, on the Center B, cut the Ark BD at F, and draw CF.

It is very clear, that the Angle BCF would be the ſame, if a leſs Radius had been taken, as Cb.

For, draw the Chord lines, FB and fb, each will be equal to the Radius of its reſpective Circle; and, the Triangles CFB, Cfb are equilateral; whoſe Angles are of 60 degrees each (Cor. 1. 9. 1.); for the Arks BF and bf are, each, a ſixth part of the whole Circumference of their reſpective Circles, of which, C is the common Center (ſee Prop. 11. 4.); and conſequently, each contains 60 degrees on the circumference of that Circle, of which it is a Part; whoſe Radius is CB or Cb.

A deſcription of the Inſtrument called a PROTRACTOR, with the application of it, in meaſuring and making Angles, of any known quantity or meaſure, may not be improper in this place; which is of eſpecial uſe in Surveying, and in drawing Plans of Ground for building on, &c.

The PROTRACTOR is a Semicircle, divided on its Limb or Semi-circumference, ADB, into 180 equal parts; having a ſmall Notch at C, the Center. See the Figure.

Some Protractors have a Scale, added to the Semicircle, which are the beſt and readieſt in uſe.

In meaſuring an Angle, apply the Diameter, i. e. the Edge or Right line AB, to either Side of the Angle ACE, with the Vertex, C, of the Angle, at the Center of the Protractor; and, wherever the Side CE, cuts the Limb or circular edge of the Inſtrument, obſerve how many Degrees there are from A to E, the Ark intercepted between the Sides AC and CE, of the Angle ACE.

If it contains 45 or 50, or whatever number of Degrees it happens to be, as (45 by the Figure) the Angle ACE is of so many Degrees. If it had cut the Arch at D, as ACD, it is a Right angle; and if beyond D, as ACF, it is obtuse; the Ark FB, subtracted from the Arch of the Semicircle, ADB, 180 degrees, gives the quantity of the Angle ACF: Or, DF, 30, added to AED, 90, make 120.

2. If it be required to lay down or make an Angle of some known Quantity (as 45 deg.) at the Point C, of the Line AB.

Apply the edge of the Protractor, as above, with the Center, C, at the Point given; make a Mark or Point at E; take away the Instrument, and draw EC.

Thus, may any Angle whatever be laid down on Paper.

N. B. No Angle can be 180 Degrees, consequently not more; for, ACB is a Right line, making no Angle at C; so that, on which ever side of B a Line is drawn, from C (as CG) however near to B, 'tis evident, that the Angle made with AC (as ACG) must be less than 180 Degrees.

A Scale of equal Parts is nothing more than a Right line divided into any number of equal Parts, at pleasure.

12

INSTRUCTIONS

FOR

YOUNG STUDENTS.

I ADVISE the young Practitioner to draw every Figure as he proceeds, carefully remarking what things, as Points, Lines, Angles, &c. are given; which are, in general, ſtronger marked than the operative Lines, they being either dotted or finer Lines; the given Lines, &c. are, by that means, diſtinguiſhable from the other.

In Geometry, as in Arithmetic, there is always ſome Data, or thing premiſed; from which, in Theory, certain Properties are deduced, as a neceſſary conſequence. In Practice, ſomewhat is required to be done, or performed, from what is given.

Let the Practitioner ſelect the given things, and mark them down, firſt, in the poſition given in the Premiſes; but, with as much variation as it will admit of; i. e. he need not put them exactly as in the Figure, only obſerve that they are as required.

E. g. In Prob. 4. an Angle is required to be made at the extremity of a given Line; but the poſition of that Line is not determined; alſo, the Angle may be made at either extreme, and either on one ſide or the other of the given Line.

In the 6th Problem, the Perpendicular may be drawn, and the Point, in the 7th, given on either Side of the Line; for, let it be obſerved and carefully remembered, that, by the Term Perpendicular, nothing more is meant than the Poſition one Line has to another; which Poſition, is when they make a Right angle or Right angles with each other; no regard being had to the poſition, or ſituation, of either, ſeparately.

Theſe things being premiſed, and the given Lines, &c. deſcribed on Paper; carefully obſerve the Directions given in the operation, and proceed accordingly, ſtep by ſtep, drawing every Line, Angle, Ark, &c. as the Problem directs.

A pair of Compaſſes or Dividers, a Drawing Pen, and a ſtreight Ruler, are all the Utenſils requiſite, in Plane Geometry.

Note. The Board or Paper, on which we draw any Figure, is ſuppoſed to be a geometrical Plane.

N. B. That DEM. means Demonſtration; and COR. Corollary. APPL. Application or Uſe. *Schol.* Remark, or Leſſon.
Th Pl. ANG. refers to the Theory of Plane Angles.

Any ſingle Number, as 3, 6, &c. refers to the Problem of that Number, the 3rd, or 6th of *that* Section. But when there are two Numbers; as 2, or 5. 1; it refers to the 2nd or the 5th Problem, of the firſt Section. Or, 12. 5, or 6; to the 12th of the 5th or 6th. Q. E. F. Signifies, which was to be done.

PRACTICAL GEOMETRY.

SECTION I.

On Lines *and* Angles; *of the Conſtruction of* Triangles, *and* Rectangles.

PROBLEM I.

To deſcribe a Circle of any given Radius, and on a given Center.

AB is the Radius given, and C the given Center.

FIX one Point of a pair of Compaſſes in either extreme of the given Line, AB, and extend the other Point to the other extreme; i. e. open the Compaſſes equal to the given line.

Then, fix one Point of the Compaſſes in C, the Center given, in ſome Plane; and revolve the other Point around, on the Plane; which, by its revolution, will deſcribe the Circumference DEF. - - - - Poſt. 4.

D E F contains the Circle required (Def. K.)

For, if a Right line be drawn from the Center to the Circumference, as CD, it is equal to the given Line AB, by Conſtruction; no other Demonſtration is neceſſary; the diſtance between the Points of the Compaſſes, during the revolution, being equal to AB.

PROBLEM II. 2 nd. I. Euclid.

To draw a Right line from a Point given, as C, which ſhall be equal to a given Line; AB.

Extend or open the Compaſſes equal to the Line given; fix one of their Points in C, the given Point, and, with the other, deſcribe a Circle, or portion of a Circle only, at D. - - - - - - Pr. 1.

Apply a ſtreight Ruler cloſe to the Point C, and croſſing the Ark at D, (in whatever poſition you require the Line to be) and, with the point of a Pencil or a Drawing Pen (applied firſt to the Point C, and drawn, along the edge of the Ruler, to the Ark at D) deſcribe the Right line CD (Poſt. 2.); which is equal to the given Line AB, by Conſtruction; and by N. B. Def. 20. El.

Thus, the geneſis of a Right line (Def. B.) is conceived to be by the direct motion of a Point.

N. B. In the practice of Geometry, it is often required to draw or to make a Line equal to another Line given; which is done by drawing an Ark of a Circle, as AB, from the given Line, AC, till it cuts the other; if the two Lines, AC and CB, meet at the Point given, C, which is made the Center.

But, if they do not meet, or cut, the Line given is taken for Radius (as in the Problem) and an Ark drawn where it is required; for equal Circles have equal Radii, as well as all Radii of the ſame Circle are equal; which needs no other Demonſtration than the geneſis of a Circle, in N.B. Def. 20. El.

So that, hereafter, when two Lines are obſerved to be Radii of the ſame Circle, it is ſufficient Demonſtration that the Lines are equal; and alſo, when they are made Radii of equal Circles.

APPL. The Application of this Problem, in deſigning, is to delineate or draw, on Paper, &c. a Right line equal to ſome known meaſure, as AB, by a Scale of equal Parts.

PROBLEM III. 3. I. Euclid.

A Right line being given, indefinite,* to cut off a a portion, or ſegment, equal to another given Line, or known meaſure.

AB is the firſt given Line, and CD the meaſure of the Segment required to be cut off.

Extend the Compaſſes from C to D, i. e. with the Radius CD, ſeting one Point of the Compaſſes in either extreme of A B, (from which the given Segment is required to be cut off) as A, with the other Point, cut or mark the Line AB, at E. Q. E. F.

2. After the ſame manner, any portion of an Ark of a Circle may be cut off. As Ae, from AeB.

APPL. In delineating, the given Line AB may be ſuppoſed to be an indefinite Line already drawn; or, it may repreſent a certain meaſure, by ſome Scale, ſuppoſe 5 Feet, being made equal to 5 Diviſions on the Scale; and it is required to cut off 3 Feet from the extreme Point A, of the Line AB, or two Feet from the other extreme, B.

By which means, a Right line may be divided, mechanically, in any Proportion required.

PROBLEM IV. 23. I. Euclid.

To make an Angle, equal to any right-lined Plane Angle given.

ABC is the given Angle, and DE a Line given.

It is required to make, at the point E, an Angle, with the Line DE, equal ABC.

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* By indefinite Line is meant an unlimited, or undetermined line, in reſpect of any meaſure.

With any Radius, at diſcretion, on the Vertex of the given Angle, B, deſcribe an Ark, ab, cuting the two Sides AB and BC in the Points a and b.

With the ſame Radius, on the Point E, draw the Ark DF; take the meaſure of the Ark ab in your Compaſſes; make DF equal ab. - - - - 2. Pr. 3.
Draw EF through the Point F; and it is done; Q. E. F.

The Angle DEF is equal to the given Angle ABC; i. e. FE inclines the ſame to ED, as AB to BC.

This is evident, from the Theory of Plane Angles, Art. 3. ab and DF being equal portions of equal Circles.

Or, by drawing the Chord lines ab, and DF, the Triangles Bab, EDF, are congruous. - - - Con.
And the Angle DEF is equal ABC. - - - P. 7. 1.

APPL. From this Problem we learn to delineate, i. e. to lay down or draw, on Paper, any right-lined Angle, of a piece of Ground or Building, &c. which we have meaſured, to form a Plan of it.

The Angle may be taken in the manner following.

If it be an internal Angle, as CBD; ſet off equal meaſure, from B, to C and D, and draw, or meaſure only, the Diagonal, or Chord CD.

Suppoſe you have made BC and BD each equal 5 feet, and the Chord CD meaſures 6 feet 6 inches, or as it happens; then, by a Scale of equal Parts, having drawn BD, at diſcretion, take 5 diviſions off the Scale; which may be either inches, half inches, or any other meaſure, provided they are equal.

With that Radius, on E, draw the Ark CD, cuting BD in D.

Take $6\frac{1}{2}$ of the ſame diviſions in your Compaſſes; and, feting one Point in D, cut the Ark DC at C, with the other; draw BC.

So ſhall CBD be an Angle laid down, equal to the Angle which was meaſured.

For (by N. B. Def. F.) the length of the Lines or Sides makes no difference in the Angle; wherefore, if the Lines BC and BD were produced, equal to the Originals; i. e. equal 5 feet each;

then would CD measure 6 feet 6 inches, and the Angle CBD would remain the same; which is obvious, if Bc and Bd be taken a half or a third part, or any other portion of BC or BD; for then, cd will be the same portion of CD; viz. a half or third, &c.

But, if ABC be the external Angle of a Building, so that, by reason of the obstruction of the Walls, &c. we cannot obtain the measure of the Chord line, AC, it must then be got by its Complement of two Right angles; i. e. by producing one Side; applying a long, streight Ruler to AB (the Wall) extended beyond the Corner, towards D; mark the Line BD, which is a continuation of AB, being produced or continued. Or, having fixed a small Cord close to the Wall, at a proper distance from the Corner (as at A) stretch it so (at D) as to touch the Corner, B; mark the Line BD, on the Ground, as the Cord directs, and proceed as above.

Then, having assumed the Point B in a Right line, AD, and made the Angle CBD equal to the external measured Angle; the remaining Angle, ABC, is the inaccessible Angle required.

PROBLEM V. 31. I. Euclid.

To draw a Line through a given Point, which shall be parallel to a Right line given.

AB is the given Line, and C is the Point given.

Through C draw, at pleasure, a Right line, as CB, cuting the given Line AB, in B.

Make the Angle BCD equal ABC - - - - 4.

With any Radius, on B, describe the Ark Aa; and, on C, the Ark bD, with the same Radius.

Make bD equal to Aa, - - - - - - - - - Pr. 3

and through the Points C and D, draw a Right line; which will be parallel to the given Line AB. Q. E. F.

For, the Alternate angles, ABC, BCD, are equal. Con.

Therefore, CD is parallel to AB. - - P. 4. 1. El.

Otherwiſe, thus.

With any Radius, at diſcretion, ſeting one Point of the Compaſſes in the given Point, C, fix the other, at pleaſure, in E, either in the Line AB or out of it; on which Center deſcribe a Circle, paſſing through C, and cuting AB, in A and B; or draw the Arks AC and BD only.

Make BD equal AC, and draw CD;

Then is CD parallel to the given Line AB. Q. E. F.

For, CB being drawn, the Angle ABC=BCD. Cor. 10. 3.
Therefore AB is parallel to CD. - - - P. 4. 1. El.

Or, when the Lines are very long, the following method is the moſt eligible.

Take the diſtance of the given Line AB, from the given Point C, in the Compaſſes; as CA; and, aſſuming any Point, B, towards the other end of the Line, deſcribe an Ark, at D, with that Radius.

Apply a Ruler to the Point C and the Ark at D, and draw CD; which will be parallel to AB. - - Def. E.

For, the Perpendiculars, AC and BD, being drawn, are equal, by Conſtruction.

PROBLEM VI. 11. I. Euclid.

To draw a Perpendicular, from any Point, in a given Right line.

AB is the given Line, and, let C be the Point given.

With any Radius, on the given Point C, deſcribe a Semicircle, cuting the given Line in two Points, A and B; i. e. make CB equal AC.

Then, with any Radius, greater than AC, on A and B, deſcribe two Arks interſecting at D, and draw CD.

The Line CD will be perpendicular to AB. Q. E. F.

For, draw DA and DB.

DEM. Then, the Triangles ADC, CDB, are congruous. Wherefore, the Angles are reſpectively equal, and ACD is equal DCB. - - - - - - P. 7. 1. Therefore, CD is perpendicular to AB. - Def. G.

After the ſame manner, a Perpendicular may be erected on the Ark of a Circle, AEB, at the point E.

PROBLEM VII.

To make a Right angle; or, to draw a Perpendicular at the extreme Point of a Right line, AB.

It is required to make a Right angle, at the extreme, B.

Set one point of the Compaſſes in the point B; and, with any Radius, at diſcretion, fix the other Point, at pleaſure, in C (on that ſide, you require the Perpendicular) and draw the Semi-circumference ABD, through the Point B, and interſecting the Line AB in A.

Draw a Right line through the Interſection, A, and the Center, C, cuting the Ark on the oppoſite Side, at D. Draw BD, which will be perpendicular to AB. Q. E. F.

DEM. For the Angle ABD, being in a Semicircle, is a Right angle. - - - - - - - P. 12. 3.

OR, it is frequently, and not inelegantly, performed thus.

It is required to draw a Perpendicular, to AB, at the extreme Point A.

Fix one point of the Compaſſes at A, and with any Radius, deſcribe the Ark BCE, cuting AB in B.

On B, deſcribe the Ark AC, cuting the former in C; on which Center deſcribe a Circle, or an Ark AED only; and on E, where it cuts the Ark BCE, draw ACD; or cut the other Ark, only, at D.
Draw AD, which will be perpendicular to AB. Q.E.F.

Or thus, without the Point E.

Deſcribe the Arks BC and AC, interſecting at C.
Through B and C draw the Right line BCD, indefinite.
Make CD equal CB, and draw AD; and it is done.

By producing BA to F (making AF equal AB) and FD being drawn, it is demonſtrated as the 6th.

The uſe of Right angles, perpendicular and parallel Right lines, are ſo well known, that it would be impertinent to point them out; particularly to thoſe concerned in Building, and various other mechanic Arts. They give beauty, ſtrength, and conveniency to a Building and its ſeveral Appurtenances; alſo the execution, of the ſeveral parts thereof, depends on them wholly.

PROBLEM VIII. 12. I. Euclid.

To draw a Perpendicular to a Right line, from a given Point which is out of the Line.

Let C be the given Point, from which it is required to draw a Perpendicular to AB.

On C, the Point given, with any Radius greater than CE, deſcribe an Ark, cuting AB, in A and B.
With the ſame or any other Radius, on A and B, deſcribe two Arks, interſecting at D. Apply a Ruler to the Points C and D, and draw CE. Then will CE be perpendicular to AB. Q.E.F.

For, draw AC and AD, BC and BD.

DEM. The Triangles ACD, DCB are congruous. Con.
Wherefore, the Angle ACE is equal to ECB. - P. 7. 1.
Conſ. the Triangles ACE, ECB are alſo equal. - 8. 1.
and the Angles AEC, CEB, being equal, are conſequently Right angles. - - - - C. 2. 1. 1.
Therefore, CE is perpendicular to AB. - Def. H.

In the ſame manner, a Perpendicular may be drawn to the Ark of a Circle, AFB.

PROBLEM IX. 11 & 12. XI. Euclid.

From any given Point,* to draw a Right line perpendicular to a Plane, in which that Point is not ſituated. Alſo, from a Point in the Plane.

Let A be the given Point.

In the Plane BEC, draw, at pleaſure, the Right line BC, and, from A, draw the Right line AD, perpendicular to BC. 8.

If AD be perpendicular to the Plane BEC, the thing is already done; but, if it be not, proceed as follows.

From the Point D, draw DE, in the Plane BEC indefinite, alſo perpendicular to BC - - - Prob. 6.
and, from A, draw AE perpendicular to DE. - - 8.
I ſay, that AE is perpendicular to the Plane BEC.

Through the Point E, draw FG, parallel to BC; Pr. 5.
conſequently, it is in the ſame Plane with BC; - Ax. 2. 7.
becauſe the Point E is in the Plane.

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* Profeſſor Simſon ſays, from a Point given *above* the Plane.
I am rather ſuprized at the expreſſion, from one of his extraordinary Sagacity; becauſe, the ſame thing holds true however the Plane be ſituated, or the Point in reſpect of the Plane, provided it be not in the Plane.

DEM. Becauſe AD and DE are both perpendicular to BC, BC is perp. to a Plane paſſing thro' AD and DE. 2. 7. El.
But, FG is parallel to BC; by Conſtruction;
wherefore, FG is alſo perpendicular to the Plane ADE.
Now, ſince AE is perpendicular to FG, and, alſo to DE, it is perp. to the Plane in which thoſe Lines are ſituated. 2.
Therefore, AE is perpendicular to the Plane BEC. Q.E.F.

Secondly. Let A be the given Point, in the Plane AC.

Aſſume any Point, B, out of the Plane; and, from that Point, draw BC perpendicular to the Plane; by the foregoing.
Then, from the Point A, draw AD, parallel to BC.
AD is perpendicular to the Plane AC.

DEM. Becauſe BC is perpendicular to the Plane, by the foregoing; and AD is parallel to BC, by Conſtruction;
AD is perpendicular to the Plane AC. - - P. 3. 7.

I believe, that the manual operation, of this Problem, would be eaſier performed by two Right angles; thus.

Let A be the given Point, at which a Perpendicular is equired, to the Plane ACD.

Apply a Right angle, BAC, to the Point A, at pleaſure, on AC; and, in any Angle, apply another R. angle, BAD. i. e. raiſe up the two right angled Triangles BAC, BAD, on AC and AD, till the Points, B, coincide.
Then, AB is perpendicular to the Plane CAD. - P. 2. 7.

COR. Hence it is manifeſt, that there cannot be drawn two Lines, from the ſame Point, perpendicular to a Plane, and on the ſame Side.

PROBLEM X. 10. I. Euclid.

To biſect a given Right line, AB.

With any Radius, more than half the given Line, on each extreme, A and B, deſcribe two Arks, cuting each other, at C and D.

Draw the Right line CD; which will biſect, or divide into two equal Parts, the given Line AB, in the Point of ſection, E. Q.E.F.

Draw AC and AD, BC and BD.

DEM. Now, ACBD is a Parallelogram; by Con. 15. 1.
And, the two Diameters AB and CD biſect each other, in its Center, E. - - - - - - 16. 1.

This Problem is deducible from the foregoing.

After the ſame manner an Ark of a Circle may be biſected; as AFB in the Point F.

N. B. It is not neceſſary to draw the whole Arks from C to D, but only the Interſections at C and D; nor to draw CD; only, applying a Ruler to C and D, cut or mark AB, at E.

PROBLEM XI.

To triſect a given Right line, AB.

With any Radius, at diſcretion, on A and B, deſcribe two Arks, at C and D; as in the foregoing.

Join AC and BC, which biſect, in E and F. - - - 10.

Draw DE, and DF, cuting AB in e and f, which will be triſected in thoſe Points.

DEM. Having joined BD, the Triangles, AeE, eDB, are ſimilar (2. 6.); for, AE is parallel to BD (15. 1.) and equal half BD, equal AC. Conſequently, Ae is equal half eB; i. e. a third of AB.

Therefore, AB is triſected, in e and f. Q. E. F.

PROBLEM XII. 9. I. Euclid.

To biſect a right-lined Plane angle, A B C.

On the Vertex, B, of the given Angle, deſcribe an Ark, a b, at diſcretion, cuting both Sides, in a and b; and, on the interſection a, with the ſame Radius, draw the Ark CD. Make CD equal a b (Pr. 3.) and draw BD.

Then will BD biſect the given Angle ABC.

Compleat the Semicircle CDbB, and draw a D.

DEM. Then DaC is an Angle at the Center, and DBC is at the Circumference of that Circle.

Th. DaC, (eq. ABC, Con.) is equal twice DBC. P. 9. 3.

And conſequently, ABC is double of DBC. Q. E. D.

OR, an Angle may be readily biſected in this manner.

On the Vertex B deſcribe an Ark (AC) as before.

On the two Interſections, A and C, with the ſame, or any other Radius, deſcribe two Arks, cuting each other, at D.

Draw BD; which biſects the Angle ABC.

Draw DA and DC.

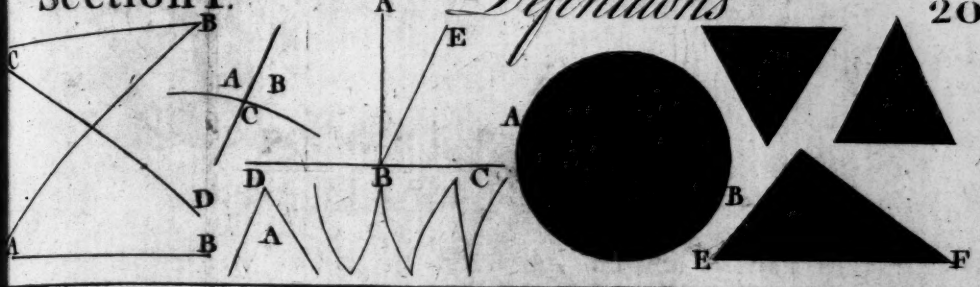
DEM. Then, the Triangles ADB and DBC are congruous, by Conſtruction.

Therefore, the Angle ABD is equal DBC. - P. 7. 1.

Angles may be thus divided into four, eight, or ſixteen equal Parts, by biſecting again and again; but there is no geometrical method, by which, Angles, or curved Lines, may be divided into any equal Parts, at pleaſure, as a Right line may be divided; otherwiſe than by dividing the Ark with Compaſſes.

APPL. By this Problem, Carpenters and Joiners, &c. find, what they call, the Mitre of any Angle (whether it be Right, Acute or Obtuſe) with eaſe and expedition.

In returning, or breaking Mouldings at an Angle, whether external or internal, the Angle biſected is the Mitre; in which the Mouldings will exactly fit each other.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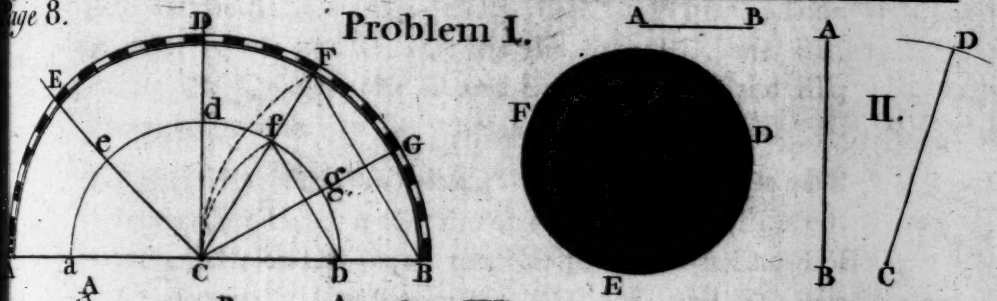
Problem I.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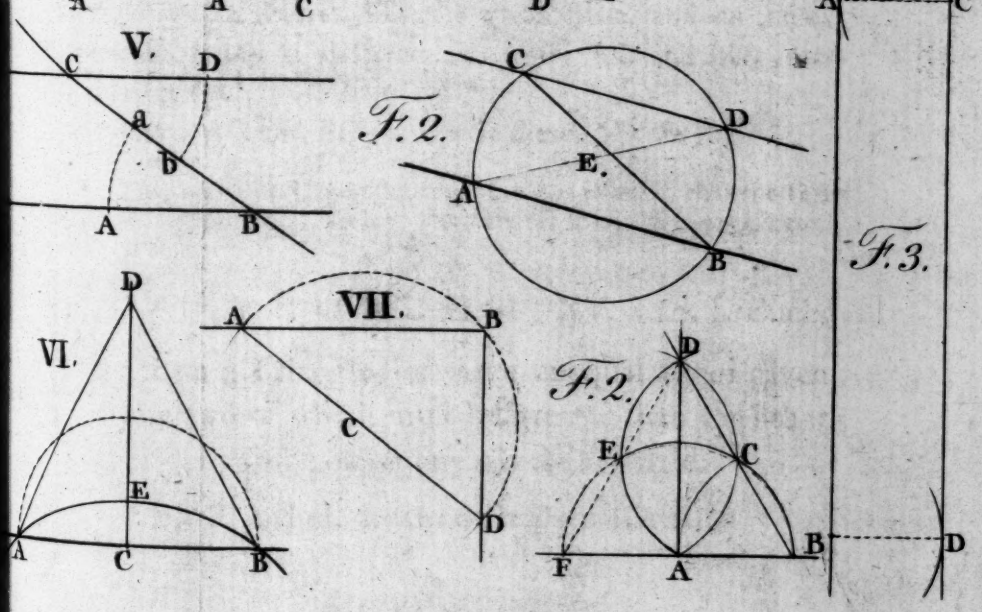
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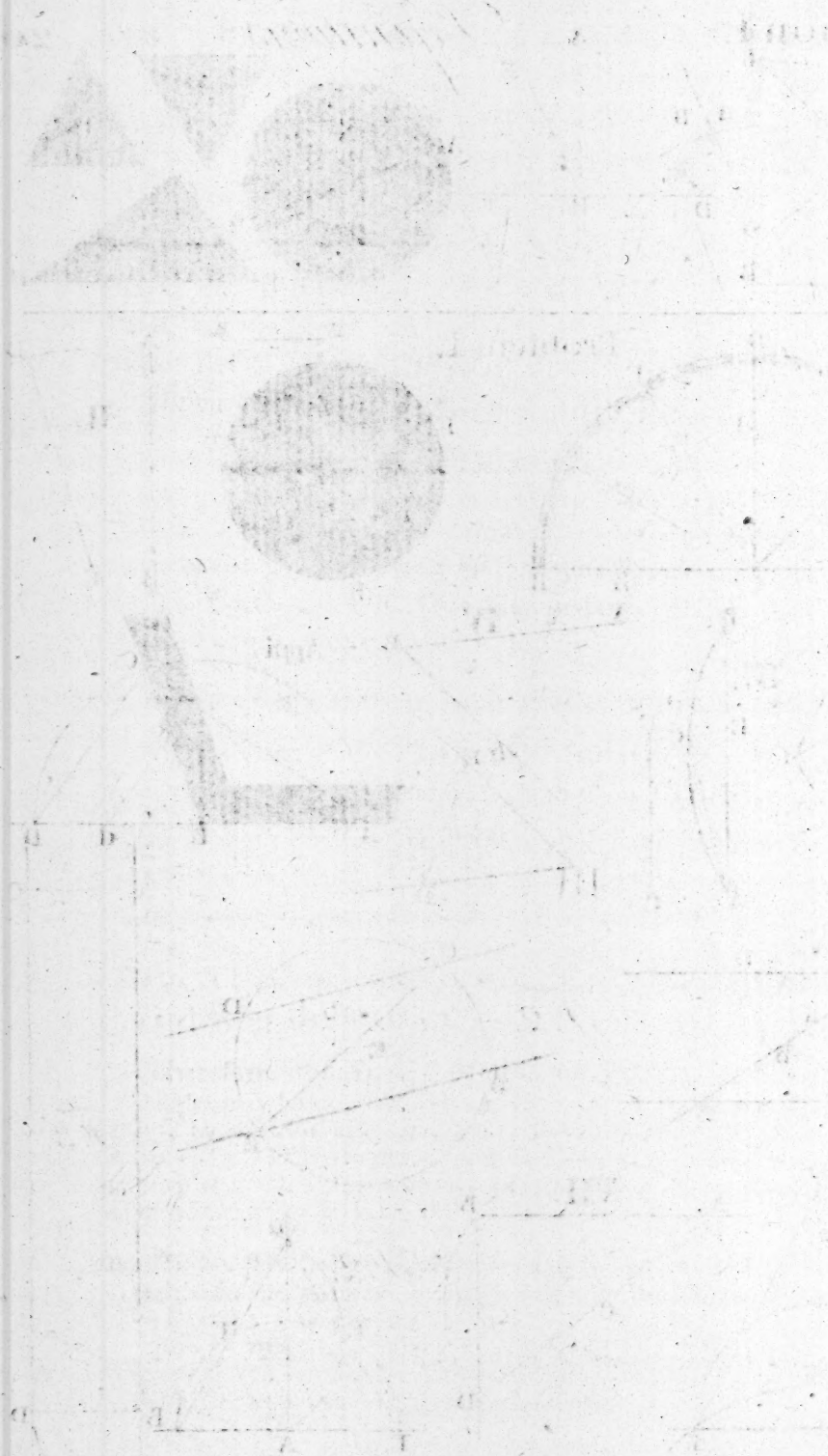


PROBLEM XIII.

To triſect a Right angle, ABC.

With any Radius, at diſcretion, on the Vertex B, deſcribe the Arch, or Quadrant, ADEC. With the ſame Radius, on A and C, deſcribe the Arks BD, and BE, cuting the Quadrant at D and E. Draw BD and BE, which will triſect the Right angle, ABC. Q. E. F.

DEM. The Arch ADEC, being a Quadrant (the Angle ABC being right. Hyp.) it contains 90 degs. Art. 2. Th. Pl. ANG. But, AE, or DB (equal the Radius AB) is a Chord of 60 degrees. Conſequently, AD, DE, and EC, are each 30 degrees; and, the Angles ABD, DBE, and EBC are all equal; therefore ABC is triſected.

PROBLEM XIV. 1. I. Euclid.

To form an Equilateral Triangle, on a Line given, AB.

With the Radius AB, the given Line, and on the extremes A and B, deſcribe two Arks, AC and BC, interſecting at C.

Draw AC and BC; and it is done. Q. E. F.

This needs no Demonſtration; for it is evident, that the three Sides are all equal, ſeeing, they are all Radii of equal Circles.

PROBLEM XV. 22. I. Euclid.

To form a Triangle, of three unequal Lines given; any two of which muſt be greater than the other; or, congruous with any given Triangle.

F, G, and H, are the three given Lines.

Let them be placed at the ends of each other in one Right line, in what order you pleaſe, i. e. having drawn the Right line AD, indefinite, make the Segments, AB, BC, and CD, reſpectively equal to the given Lines, F, G, and H. - - - - - - - Pr. 3.

With the Radius AB, on B, deſcribe a Circle, or the Ark AE only; and on C, deſcribe the Ark DE, with the Radius CD, cuting the other Ark in E; draw BE and EC, and it is done. Q. E. F.

This Problem and the laſt need no Demonſtration; every thing being as required, by Conſtruction.

N. B. The Triangle BEC would be the ſame, if either of the other two Lines, F or H had been made the Baſe, by placing it in the middle; only the poſition of the Triangle would be varied. This follows from Prop. 7. 1. of Elements.

2. There is no neceſſity for placing them, at all, in this manner; only, with the different Radii of two of the Lines, draw Arks, on the extremes of the other Line, cuting each other; taking any one for the Baſe, and applying the other at either extreme.

Two Lines only, being given, an Iſoſceles Triangle may be thus conſtructed.

AB and CD are two given Lines; of which, let AB be the Baſe.

Take the meaſure of CD in your Compaſſes, and with that Radius, on the extremes of the other Line, A and B, deſcribe two Arks, interſecting at E; draw AE and EB, and it is done. Q. E. F.

If the other Line (CD) had been required for the Baſe, the operation would be the ſame; taking AB for Radius, and drawing the Arks, interſecting at F; draw CF and FD.

AEB and CFD are Iſoſceles Triangles (ſee Def. N.) for AE and EB, alſo CF and FD are equal, by Conſtruction.

PROBLEM XVI. 46. I. Euclid.

To conſtruct a Square on a given Line. AB.

On either extreme of AB, make a Right Angle; as ABC (7.). Make BC equal AB (Pr.2.). Then, with the Radius AB, on A and C, deſcribe two Arks, interſecting at D, and draw AD and DC.

The Quadrilateral ABCD is a Square.

DEM. For it is equilateral, by Conſtruction;
conſequently it is a Parallelogram. - - P. 15. 1.
Wh. the Angle D is equal B; conſ. D is a Right angle.
But, the Angles A and C are alſo equal; conſ. Right.
Therefore, ABCD is a Square. - - Def. P.

PROBLEM XVII.

To conſtruct a Rectangle, two Lines being given.

On AB make a Right angle; ABC (Pr. 7.). Make BC equal to the other given Line, E; Draw AD parallel to BC, and CD to AB (5.). ABCD is the Rectangle required.

For, ABCD is a Parallelogram, - - Con.
and the Angles are all Right; - - - P. 4. 1.
therefore it is a Rectangle; - - - - Def P.

After the ſame manner, a Parallelogram, under any two given Lines, and any Angle may be conſtructed; making ABC, or BAD, equal to the given Angle.

A right angled Triangle having two Lines given for the Sides, containing the Right angle, is thus conſtructed; making ABC or ADC a Right angle. and AB, BC, or AD, equal, reſpectively, to the given Lines, and drawing the Hypothenuſe AC.

The Square, Rectangle, and right angled Triangle, are of great uſe in mechanic Arts; as moſt regular Figures are right angled.

In Menſuration, the Area of every Figure is reduced to a ſtandard meaſure; by the Square, or other Rectangle.

PRACTICAL GEOMETRY.

SECTION II.

On the Conſtruction of POLIGONS, *in general.*

FIGURES having more than four Sides are, in general, called Poligons, ſignifying many Sides (ſee Def. R); the ſubject of this Book is confined to ſuch as are regular, only, having equal Sides and Angles. Being irregular, every Side and Angle muſt be given, in proper order; in which caſe, 'tis to make one Figure ſimilar to another, equal, greater, or leſs, as required; for which, the firſt Problem, Sect. 8. is adapted to any right lined Figure whatever. The equilateral Triangle (Prob. 14.) and the Square (16. Sect. 1.) are ordinate, regular Figures, of three and four Sides; but a Pentagon is the firſt which comes under the denomination of a Poligon.

PROBLEM I.

On a given Line (AB) to conſtruct a Pentagon.

On A and B, with the Radius AB, deſcribe two Circles, cuting each other in C and D, and draw CD.

On D, with the ſame Radius, deſcribe the Ark EABG, cuting the two Circles and the Right line CD, in E, F, and G. Draw the Right lines EF and GF, and produce them till they cut the Circumferences, in H and I.

Then, on H and I, with the Radius AB, deſcribe two Arks, interſecting at K, and join the Points AI, IK, KH, and BH, which compleats the Pentagon, AIKHB. Q.E.F.

Of this conſtruction of a Pentagon, no Demonſtration is given.

ANY regular Poligon, from a Pentagon to a Duodecagon, may be conſtructed by the following Rule.

The number of Sides of the Poligon being determined, the proportion an Angle has to a Right angle, and the difference between them is given, as follows.

A Right angle is to the Angle of a

	Ratio	Diff.		Ratio	Diff.
Pentagon	as 5 to 6	— 1	Nonagon	as 9 to 14	- 5
Hexagon	as 3 to 4	— 1	Decagon	as 5 to 8	- 3
Heptagon	as 7 to 10	— 3	Undecagon	as 11 to 18	- 7
Octagon	as 2 to 3	— 1	Duodecagon	as 3 to 5	- 2

Note; the Hexagon, Octagon, &c. having an equal number of Sides, the Ratios are reduced to the loweſt Denomination; otherwiſe, the Right angle is always ſuppoſed to be divided into the ſame number of Parts as the Poligon has Sides; which will then be, for a Hexagon, 6 to 8, difference 2; in an Octagon, as 8 to 12, difference 4; and ſo of the others.

To conſtruct a Pentagon, on the Line AB.

Make a Right angle (ABF) at the extreme Point, B.

With the Radius AB (or any other) on B deſcribe an Ark from A to C.

Divide the Ark AGF (of the Right angle) into five equal Parts, as in the Figure; each being 18 degrees, the Angle of the Pentagon is ſix of thoſe Parts (as 6 to 5); add the difference, 1, from F to C, and draw BC.

The Angle ABC is the Angle of a Pentagon, containing ſix fifths of a Right angle, on the Ark AGC.

On A, with theſame Radius, AB, draw the Ark BGE, cuting the other at G; make GE equal GC, and draw AE.

With the ſame Radius, on E and C, deſcribe two Arks, interſecting at D, and join CD and DE; which compleats the Pentagon, ADB, on the given Line AB.

DEM. The Sides AB, BC, &c. are equal, by Conſtruction.
And the Angles are alſo equal, being ſubtended by equal Arks, AGC, BGE, &c. of equal Circles. - C. 2. 9. 3.
Therefore, ABCDE is a regular Pentagon.

THE moſt elegant and geometrical, and perhaps the moſt correct, method is the following.

Divide AB in extreme and mean Proportion, in the Point C. - - - - - - - Pr. 3. Sect. 5.

Produce AB; and make BD equal to BC, the greater Segment; then, AD is to AB, as AB to BC, and as BC to AC.*

On A, with the Radius AD, deſcribe the Ark DEF; and, on B, deſcribe the Ark GF, interſecting at F.

On F, with the Radius AB, deſcribe the Ark EG; i. e. make FE and FG each equal AB, and join the Points A and G, B and E; alſo EF and FG, which compleats the Pentagon. Q. E. F.

The Demonſtration of this Conſtruction of a Pentagon may be obtained, from the 8th and 9th Prop. of the 4th of Elements.

For, AED is an Iſoſceles Triangle, having its Angles at the Baſe, ED, each double the Angle EAD, at the Vertex; AFB is the ſame, which may be conſidered as inſcribed.

And, by 34 of the 6th, the Diagonal of a Pentagon, AF or AE (equal AD) has that Ratio to the Side, AB, BE, &c. as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion.

OR it may be conſtructed thus.

Having found the Point D, as above, and drawn the Ark DEF, on the Center A.

On B, with the Radius AB, deſcribe the Ark AE, cuting the other at E, and draw AE and DE.

Draw BF parallel, to DE; and alſo AG, indefinite.

Draw FG parallel to AE, cuting AG in G, and join FE and BE, which compleats the Figure.

PROBLEM II.

To conſtruct a Heptagon, on a given Line, AB.

Make a Right angle ABD; and divide the Ark of it into ſeven equal Parts, as many as the Figure has Sides. The Angle of a Heptagon is ten of thoſe Parts, difference three.

* See N. B. 2. Prop. 35. 6th of Elements.

Set off three Parts from the Perpend. BD to E, & draw BE. Make BE equal AB; biſect the two Sides AB& BE; Pr. 8. and draw the Perpendiculars, FC and GC, interſecting at C. On C, the Center, and Radius CA, or CB, deſcribe a Circle; which will contain the given Line, AB, ſeven times in its Circumference, at A, B, E, H, I, K, L; which Points, joined by Right lines, will compleat the Heptagon required. Q. E. F.

DEM It is equilateral by Conſtruction; and it is alſo equiangular, becauſe inſcribed in a Circle; for, equal Segments contain equal Angles. - - - Th. 1. 10. 3.

N. B It is plain that the Angle ABE, which is an Angle of the Heptagon, is equal to the Right angle ABD added to the Angle DBE; the Right angle containing ſeven parts, the Acute angle, DBE, three; and conſequently, the Angle ABE contains ten; therefore, the Ratio is as 10 to 7.

After the ſame manner a Circle may be found, which ſhall contain a given Line any number of times, to twelve, applied to the Circumference, by the above Rule; it is therefore unneceſſary to give more Examples of this method, of conſtructing Poligons.

PROBLEM III.

To deſcribe a Hexagon, on a given Line, AB.

In the conſtruction of a Hexagon, you need not regard the Rule, nor proceed after that method; ſeeing that, the Side of a Hexagon is always equal to the Radius of a circumſcribing Circle. - - - - - - - Th. 11. 4. El.

Therefore, with the Radius AB, deſcribe two Arks, on A and B, interſecting at C, and compleat the equilateral Triangle ACB. On the Center C, with the ſame Radius, deſcribe a Circle. Produce AC to E, and BC to F; and, through the Center, C, draw GD parallel to AB, and join the Points A and G, G and F, F and E, &c.

Or, having found C, the Center, and deſcribed a Circle, apply the given Side, AB (equal to the Radius, AC) ſix times round the Circumference, from B to D, E, F, G, and join the Points as before.

AGFEDB is a regular Hexagon. Q. E. F.

The circumference of a Circle contains the Radius inſcribed, as a Chord line, ſix times, - Th. Pl. Ang. Art 5. conſ. it is equal to the Side of a Hexagon inſcribed. 11.4. El.

This needs no other Demonſtration; for, the Sides are all equal by Conſtruction; and the Triangles ACB, BCD, DCC, &c. are equilateral, whoſe Angles are alſo equal (C. 1. 9. 1.). And, each Angle of the Hexagon, contains two Angles of a Triangle, (ABD, equal ABC added to CBD, &c. which are therefore equal) and, conſequently, have that proportion to a Right angle as 4 to 3; as the Perpendicular BH (biſecting the Angle CBD) indicates.

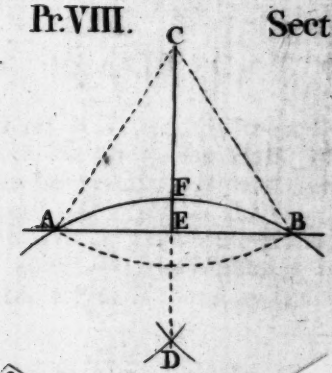
PROBLEM IV.

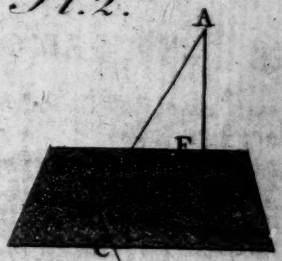
To deſcribe an Octagon, on a given Line, AB.

On the extremes of the given Line, AB, draw the Perpendiculars AG and BH, indefinite.

Produce AB, both ways, and biſect the external Angles, IAG and HBK, by the lines AC and BD; which make equal, each to AB, and draw CD; make LN equal LM; and, thro' N, draw EF paral. to CD. Draw CE and DF, parallel to AG and BH. Make NH equal NF, and draw GH parallel to EF; join EG and FH, which compleats the Octagon. Q. E. F.

N. B. It is obvious, in this Figure, that its Angles, as ABD, have the proportion to a Right angle, ABL, of 3 to 2; for, the Angle LBD is half the Right angle LBK; and LBD added to ABL, equal ABD, is the Angle of the Octagon.





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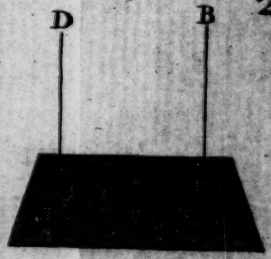

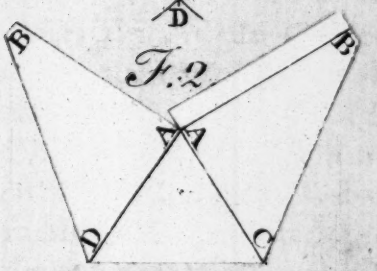

Fig. 2.

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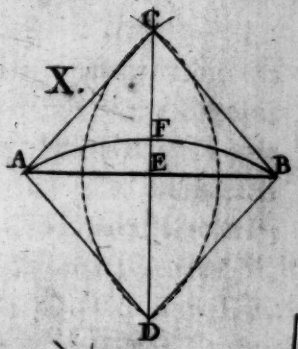

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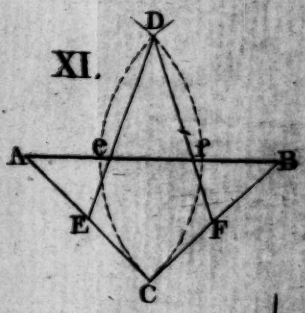

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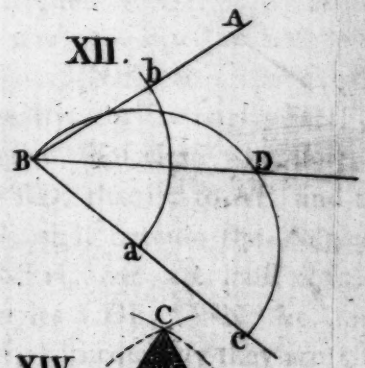

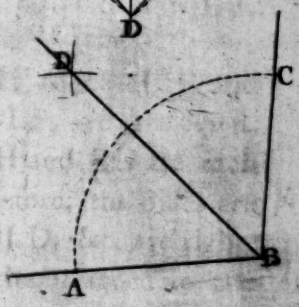

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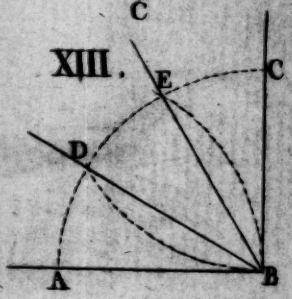

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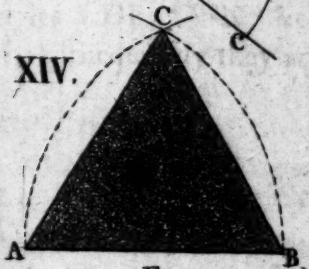

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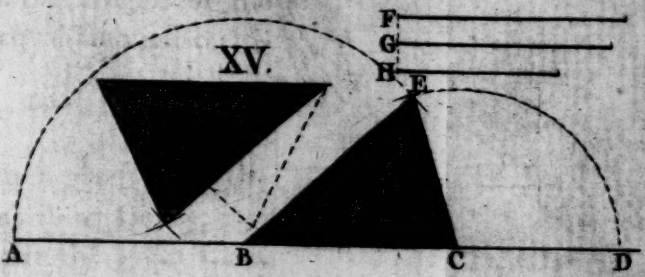
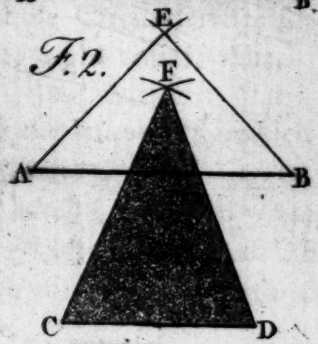


Fig. 2.



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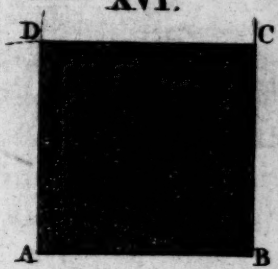

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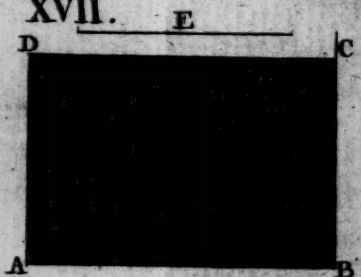
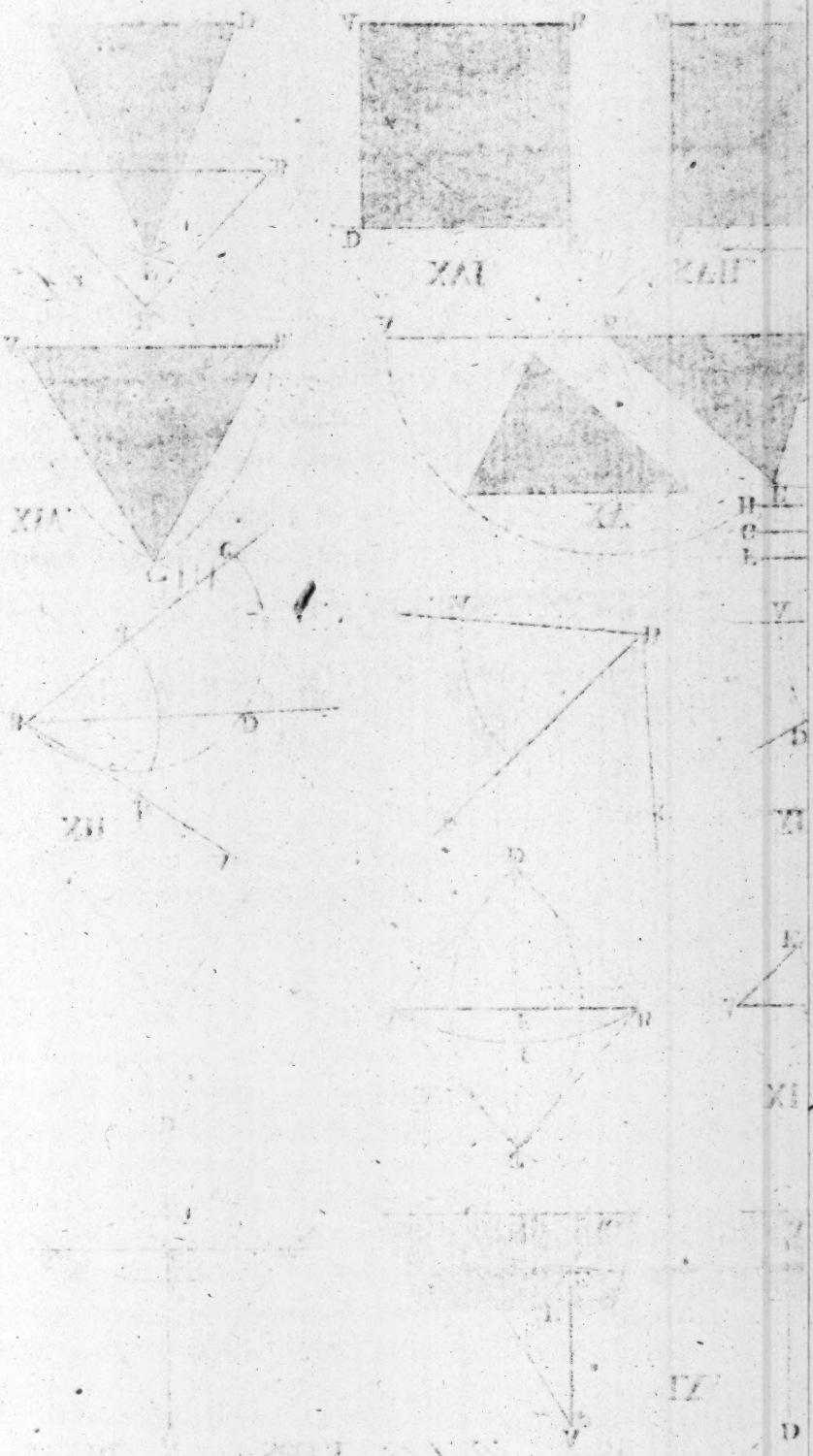




The Octagon is, perhaps, the moſt uſeful of all Poligons; particularly in Building, the Hall, or Saloon, is often, and the Veſtibules frequently, octagonal; and, as well as Rooms, are moſtly regular Octagons. Of which, a Side being determined (as AB, above) I believe no readier method of conſtructing it has been given. If the Diameter, or width, CD, be given, or determined, a Side is found by the following Problem.

That AEHD is a regular Octagon is evident, from its Conſtruction, and obvious from inſpection of the Figure; for the Angle ABD, &c. are each equal to a Right angle and half, by Conſtruction; and, as AC, BD, were each made equal to AB, it is manifeſt, that CD is parallel to AB; wherefore MONL is a Square, on AB, or its equal LM, and conſequently, GH is equal AB; alſo CE and DF, for LN was made equal to LM.

Moreover, NH was made equal NF (equal LD) and conſequently, the Triangles HNF, DLB, &c. are equal, right angled, Iſoſceles; wherefore, FH and EG are each equal to BD, that is, to AB, and therefore, the Sides are all equal; and, becauſe the Angles BLD, &c. are right, LDB, NFH, &c. are half right, which added to the Right angles LDF, DFN, &c. form the Angles of the Octagon; conſequently they are all equal amongſt themſelves.

AN OCTAGON may be conſtructed on a given Line, by determining the Center of a circumſcribing Circle; that is, whoſe Circumference ſhall contain the given Line, applied eight times, thus.

AB is the Line given, which biſect, at C, and draw the Perpendicular CE, indefinite. Deſcribe a Semicircle on AB, cuting CE at D; on which, deſcribe the Circle AEB, with the Radius AD, equal DB; and, on E, where it cut the Perpendicular CE, with the Radius AE or EB, de-

ſcribe the Circle AFB, which will contain AB eight times, applied to its Circumference, as in the Figure; conſequently, A1, &c. to B, is a regular Octagon.

DEM. The Angle ADB is a right one; - - 12. 3. El.
conſequenty, AEB is half right. - - - 9. 3.

But, a Right angle is 90 Degrees, and conſequently, its half is 45; - - - - 2. Th. Pl. ANG. which is the Angle at the Center of an Octagon, or a circumſcribing Circle, ſubtended by a Side.

It is evident, that the Circle AEB will circumſcribe a Square, on AB; for ADB is a Right angle, at its Center; wherefore, ADB contains a Square, of which, AB is the Diagonal.

PROBLEM V.

To determine the Side of an Octagon, from a given Diameter or width, AB.

Deſcribe a Square on AB. (16. 1.) Draw the two Diagonals AC and BD, which give the Center, E. 16. 1. El.

With the Radius of half the Diagonal, AE, on every Angle of the Square, deſcribe an Ark or Quadrant, bEg, aEd, &c. i. e. make Ab, Ag, Ba, Bd, &c. each equal half the Diagonal; and, joining the Points ah, bc, &c. you will have a regular Octagon abcdefgh. Q. E. F.

The Side of the Octagon, it is obvious, is equal twice the Difference between half the Side of the Square and half the Diagonal; conſequently, it is equal to the Difference between the width given, and the Diagonal of a Square whoſe Side is equal to the width. So that, multiplying the given width into itſelf, the Square Root of

double that Sum is the Diagonal; from which ſubtract the given width, the remainder is a Side of the Poligon.

APPL. The uſes of theſe Problems, to various Mechanics, are very obvious. By the 4th we learn how to form an octagonal Building; as a Temple, Library, &c. an Alcove, or Bow Window, &c. (which are frequently half a regular Octagon or Hexagon) of a given meaſure, for a Side of the Building, &c.

By this Problem, we find the meaſure of a Side, when the width, or Diameter, is firſt determined.

Theſe are the moſt neceſſary Poligons for mechanical uſes. By the Rules already given, any Poligon, to twelve Sides, may be readily conſtructed.

To enumerate all the uſes of Poligons, would be impertinent and foreign to the purpoſe; my deſign being to ſhew how to conſtruct them, in the eaſieſt and readieſt manner, the application of them will readily occur, as occaſions require.

ANY regular Poligon may be conſtructed, on a Line given, by the following general Rule; dividing 360 by the number of Sides of the Poligon required, and ſubtracting the Quotient from 180, the remainder is the Angle of the Poligon.

PROBLEM VI.

To conſtruct a Nonagon, on a given Line, AB.

360 divided by 9 gives 40, the meaſure of the Angle, in Degrees, which the Side of a Nonagon ſubtends at the Center of a circumſcribing Circle, equal to its external Angles. See *Theo.* 2. 10. 1. El.

AB being produced both ways, towards D and E, at the extremes, A and B, make the Angles DAF, EBG, each equal 40 Degrees; make AF and BG each equal AB, which biſect in H and I; draw HC and IC perpendicular to AF and BG, interſecting in C.

A Circle deſcribed on C, with the Radius AC or BC, will contain AB nine times applied to its Circumference.

It is evident, that the external Angle DAF (equal EBG) is equal to the Angle ACB at the Center.

For, ACB, BCG, and ACF, are congruous Iſoſceles Triangles; whoſe Angles at the Baſe, AB and BG, are 70 Degrees each; two, of which, form an internal Angle of the Nonagon; as ABC+CBG=ABG; and, the external Angle, EBG, is the Compliment of two Right angles; conſequently, it is equal to the remaining Angle ACB, of the Triangle ACB, or BCG.

So that, an Iſoſceles Triangle, ABC, being conſtructed on the given Line AB, whoſe Angles, at the Baſe, A and B, are each of 70 Degrees; the Angle at the Vertex (C) will be 40. C is the Center of a Circle which will circumſcribe a Nonagon, the meaſure of whoſe Sides is AB.

The Angle ACB, of 40 Degrees, being repeated 9 times, as in the Figure; will include all the ſpace of the Circle, equal four Right angles; and, each Angle will be ſubtended by a Side of the Poligon; which is evident from the Figure.

The three Angles of every Triangle are, together, equal to two Right angles, i. e. equal to 180 Degrees (10. 1.El.). And, ſince the Angle C (at the Center) of the Triangle ACB, is equal to 40 Degrees, conſequently, the two remaining Angles, CAB, CBA, at the Baſe, being equal (9. 1.) they are, each, equal to half the difference between the Angle ACB, and two Right angles; i. e. of 70 Degrees, each, as by Conſtruction.

Hence it is evident, that, in any Right line, as DE, cuting a Circle, if from either Point, A or B, of the part AB, within the Circle, another Chord, AF or BG, be drawn, equal AB; the external Angle, DAF or EBG, made by that Chord and the Line DE, is equal to the Angle at the Center of the Circle ſubtended by the Chord, AF or BG, equal AB.

The external Angle of every regular Polygon is equal to an Angle at the Center, ſubtended by a Side. For, all

the external Angles of every Right-lined Figure are equal to four Right angles (Th. 2. 10. 1.); and so are all the Angles about the Center of a Circle. - - - C 2.2.1.

Wherefore, an ordinate or regular Polygon, having all its internal Angles equal, has likewise, all its external Angles equal; and, being equal in number to the Angles at the Center, subtended by the Sides, each external Angle is, consequently, equal to an Angle at the Center, subtended by a Side, e. g.

If the Poligon has eight Sides (an Octagon) the external Angle is the 8th part of 360, i. e. 45 Degrees; if a Nonagon, it is 40 Degrees; if a Decagon 36; each being equal to the Angle at the Center, subtended by a Side.

From hence is deduced the Rule (Pr. 2.) for constructing Poligons; by dividing the Ark of a Quadrant, or Right angle, into as many equal parts as the Poligon has Sides.

For, the Ark of a Right angle being divided into five equal parts, each will be 18 Degrees, wherefore, the Angle of a Pentagon being six of those parts, or one added to a Right angle, i. e. $18+90=108$; which being subtracted from two Right Angles, or 180 Degrees, the Difference is 72, equal four times 18; which is the Compliment of two Right angles; consequently, the external Angle of a Pentagon is equal to an Angle at the Center, subtended by a Side.

For a Nonagon, the Ark of a Right angle is divided into 9 equal Parts, each equal 10 Degrees.

Now, the external Angle being 40 Degrees, equal to the Angle at the Center, the Angle of the Nonagon is $50+90$ or five ninth parts added to a Right angle; for, the internal and external Angle are, together, equal to two Right angles - 1. 1. El.

Therefore a Right angle is to the Angle of a Nonagon as 9 to 14, difference 5, as in the Table.

THE reason of this is so very obvious, 'tis needless to say more about it; seeing it is manifest, that all the external Angles

of any right-lined Figure, whatever, are, together, equal to all the Angles about a Point, ſubtended by the Sides. i. e. equal to four Right angles. And, all the internal Angles of any right-lined Figure are equal to twice as many Right angles as the Figure has Sides, wanting four, (Th. 1. 10. 1. El.) conſequently, the external Angles being equal to thoſe four (Th. 2. of the ſame) are equal to all the Angles at the Center; and, being equal, in number, they are alſo equal in quantity.

HENCE, the Angle of any regular Poligon, whatever, may be readily obtained.

For, in a Pentagon, all the internal Angles, together, are equal to ſix Right angles; conſequently, each Angle is 6 fifths of a Right angle, i. e. it exceeds a Right angle one fifth part; ſeeing, there are five Angles in the Figure, and they are, altogether, equal to ſix Right angles.

The Angles of a Hexagon are, altogether, equal to eight Right angles; conſequently, each Angle is equal to eight ſixths of a Right angle, or four thirds; i. e. equal to one Right angle and one third part of a Right angle.

A Heptagon has all its Angles, together, equal to ten Right angles; wherefore, each Angle is equal to ten ſevenths of a Right angle, i. e. equal to a Right angle and three ſeventh parts of a Right angle.

All the Angles of an Octagon being equal to twelve Right angles, it is evident that each Angle is equal to one Right angle and a half; i. e. to 12 eights, ſix fourths, or three ſeconds, i. e. one and a half.

To particularize more would be unneceſſary; it is eaſy, from what I have ſaid, to calculate the quantity of the Angles of any Poligon whatever, by the Proportion it bears to a Right angle, or to two Right angles; which muſt ever be more than the Angle of any Poligon whatever.

For Poligons which have an even number of Sides, let it be obſerved, that the diviſion of a Right angle, may be reduced to a lower Denomination; as for a Duodecagon, for inſtance; all

its Angles, together, being equal to 20 twelfths, i. e. 10 ſixths, or 5 third parts, i. e. to one Right angle and two thirds; the Right angle need be divided into three equal Parts, only, inſtead of twelve.

Whereas, thoſe which have an odd number of Sides cannot be reduced lower; but, in forming them, a Right angle muſt neceſſarily be divided into as many parts as the Figure has Sides.

PROBLEM VII.

To determine the Radius of a Circle, which ſhall contain a given Line, any number of times applied to its Circumference, for the conſtruction of Poligons, from ſix to twelve Sides.

Biſect the given Line AB, and, from the middle, draw a Perpendicular (6. 1.); and, with the Radius AB, on B, deſcribe the Ark AC, which divide into ſix equal Parts.

On C, with the Radius AC, deſcribe a Circle; which will contain the Line AB (equal AC) ſix times; applied to its Circumference. - - Cor. to 12. 4. El.

With the Radius C1, C2, &c. deſcribe the Arks 1D, 2E, &c. cuting the Perpendicular, at D, E, F, &c. on which Centers, and with the ſeveral Radii AD, AE, &c. deſcribe Circles, as in the Figure.

The outer Circle, deſcribed on I, with the Radius AI, will contain the given Line, twelve times; that on D ſeven times, on E eight, and ſo of the reſt, in regular order.

If you would continue it to 24, the Ark AC muſt be divided into 12 equal Parts; each of which, being tranſferred to the Perpendicular, as before, and thoſe tranſferred again, twelve more Centers will be obtained, on the Perpendicular, on which the Circles muſt be deſcribed; the Circles, already deſcribed, give ſix,

PROBLEM VIII.

To divide the Circumference of a Circle, in any proportion required; or to determine the Side of any Poligon, to be inſcribed.

Draw a Diameter, AB, on which conſtruct an equilateral Triangle ACB; or draw the two Arks only, interſecting at C.

(This preparation is the ſame for any Poligon whatever.)

Then, divide the Diameter into as many equal Parts as the Poligon, required, has Sides (Pr. 11. 5.); and, through the ſecond diviſion, from either extreme, draw a Right line, from C to the oppoſite Side, of the concave Circumference. e. g.

For a Pentagon, the Diameter muſt be divided into five equal Parts; through the ſecond Diviſion, from A, or B, draw CD; then, AD is the ſide of a Pentagon; i. e. a fifth part of the whole Circumference, or two fifths of the Semicircumference AFB.

The Side of a Hexagon is equal to the Radius; and the Side of an equilateral Triangle is the Diagonal of two ſides of a Hexagon; yet the ſame Rule holds equally true in all.

If a Right line, be drawn, from C, through the Center, E, to F, it divides the Circle equally into four, and, 'tis evident, that AF, or FB, is the Side of a Square.

For, the Diameter being divided into four equal Parts, the Center being equally diſtant from each extreme of the Diameter, AE is, conſequently, two of thoſe Parts.

One of thoſe Parts, GB, equal half the Radius EB, is two eighths of the Diameter; wherefore, if CG be drawn and produced, it will cut the concave Circumference in the Point H; and BH, or FH, being joined, is the Side of an Octagon; for it will biſect the Ark, FHB, of the Side of a Square. And, AI, being a fifth part of the Diameter, is equal to two tenths; wherefore, if CI be drawn, to I, it will biſect the Ark AID; and AI or ID, being joined, is the ſide of a Decagon, inſcribed; for, it is equal to two tenths of the concave Circumference AFB.

THUS may the Side of any Poligon whatever, contained in a Circle, be obtained; by obſerving the Rules given above. And it is truly worthy of notice; that, any Right line, drawn from C, cuting the Diameter and the concave Circumference, will cut them both in the ſame Proportion; or, in whatever Ratio one of them is divided, a Right line being drawn, from C, through the point of diviſion, will cut the other in the ſame Ratio.

Of this equal diviſion of the Diameter and the Circumference, no Demonſtration has been given. On the contrary, Mr. Clarke, of Mancheſter, has demonſtrated it negatively; ſo that 'tis an approximation only, and not ſtrictly true.

For the inſcription and circumſcription of Poligons, generally, ſee the 4th Section.

PRACTICAL GEOMETRY.

SECTION III.

On the Circle *and its Attributes.*

PROBLEM I. 1. III. Euclid.

To find the Center of a given Circle; and through a given Point, to deſcribe a Circle parallel to the given one, and concentric with it.

AEB is the Circle given; and G the given Point.

DRAW a Chord Line, AB, at pleaſure. Biſect AB (Pr. 8.) and through D, the point of biſection, draw EF perpendicular to AB (6.) cuting the Circumference in E and F. Biſect EF, in C, which is the Center of the Circle AEB. Q. E. F.

For EF is a Diameter, and conſequently paſſes through the Center of the Circle. - - - P. 1. 3. El.

2nd. Having (as above) found the Center (C) of the given Circle; join CG. With the Radius CG, and on the Center C, deſcribe a Circle GH, and you have done. Q.E.F.

SCHOL. *All parallel Circles, in the ſame Plane, have the ſame Center, and are called* concentric; *but, if they are in parallel Planes, as on a Cylinder, Cone or Sphere; a Right line perpendicular to thoſe Planes, if it paſſes through the Center of one Circle, it will paſs through the Centers of them all; which Line is the Axis of the Cylinder, Cone, or Sphere.*

PROBLEM II. 25 Prop. III. Euclid.

From a given Ark or Segment of a Circle, to perfect or complete the Circle.

ABD is the Ark given.

Draw, at pleaſure, two Chord Lines; AB and BD.

Biſect the two Chords, at E and F; from which, draw the Perpendiculars EC and FC, interſecting at C. Pr. 8. and 6.

C is the Center of a Circle; on which, with the Radius CA, CB, or CD, the Circle ABDG may be completed.

As, in the foregoing, the Line EF, biſecting the Chord AB, paſſed through the Center; conſequently, the biſecting Lines, EC and FC, of both, determines the Center.

N. B. By this method, may be found the Center of a perfect Circle, as readily as in the foregoing; nor is it neceſſary to draw the Chords; only, aſſuming, at pleaſure, three points, A, B, and D.

WITH any Radius, at diſcretion, on B, as a Center, deſcribe the Ark HIKL; and, with the ſame Radius, on A and D, deſcribe two Arks, HI and KL, cuting the former in H, I, K and L; through which Points, draw Right lines, HI and LK, interſecting at C, the Center ſought.

HENCE, through three given Points (not lying in a Right line) as A, B, and D, the Circumference of a Circle may be deſcribed; whoſe Center (found, as above) is C.

The Center of the Circle may be found arithmetically, by the meaſure of the Chord, or Subtenſe of the Segment, and the height of the Ark it ſubtends.

ADB is the Segment given.

Divide the Square of half the Baſe by the perpendicular height, and add the Quotient to the Diviſor. See Caſe 2. 14. 3.

e. g. Let the Baſe AB be 14 and the Perpendicular CD, 4.

The Square of AC or CB, 7 multiplied by 7 is equal 49; which, divided by 4, gives the Quotient 12,25, equal CE; to which add the diviſor, 4, the height of the Ark, CD, it gives 16,25, for the Diameter.

For a Rectangle under CE and CD is equal to the Square of AC. - - - - - - - - - P. 14. 3. El.

Suppoſing the Integers Feet, the Diameter, DE, is 16 Feet 3 In.

PROBLEM III.

To draw a Line touching a Circle, through a given Point in the Circumference. And, to determine the Point in which a Tangent touches the Circle.

Firſt; B is the Point given; through which a Right line, a Tangent, is required to be drawn.

Having found A, the center of the Circle (by the foregoing) join the point B, and the center of the Circle by a Right line, AB. At the Point B, make a Right angle, ABC, and produce CB, towards D.

The Right line CD will touch the Circle, in the Point B. - - - - - - - - - P. 8. 3. El.

2nd. CD is a Line touching a Circle; it is required to determine the Point of Contact.

Draw a Perpendicular (AB) to the Line, from the Center of the Circle; cuting CD in B; - - - Pr. 7.

B is the Point of Contact, in which the Line CD touches the Circle. - - - - Cor. 3. 8. 3. El.

PROBLEM IV. 17. III. Euclid.

To draw a Tangent to a given Circle, from a Point given without the Circle; i. e. to determine the Point, in which a Right line drawn from the given Point ſhall touch the Circle.

From the given Point, A, draw AC; to the Center of the Circle. Biſect AC; and, with the Radius AB, deſcribe a Semicircle; cuting the Circumference of the given Circle in D, the Point ſought.

DEM. Having drawn AD and CD, the Right line AD will touch the Circle at D. For, the Angle ADC is a Right one (12.3.). Wherefore, AD touches the Circle in D; as is manifeſt, from P. 8. 3, Cor. 2.

OTHERWISE.

Join the Point A, and the Center C, as before, cuting the Circumference in B.

With the Radius CA deſcribe the Ark AD. Draw the Perpendicular BD, cuting AD at D. Laſtly, draw CD, cuting the given Circle at E, the Point ſought. Draw AE.

The Right line, AE will touch the Circle, at E.

DEM. The Triangles AEC, BDC are congruous.

For, the Sides AC, CE are equal to DC and CB, reſpectively; and the Angle C is common to both; therefore, AE is equal to BD; the Angle at A is equal D, and the Angle AEC equal DBC. - - - P. 8. 1. El.

But CBD is a Right angle (Con.) wh. AEC is a R. angle.

Therefore, AE touches the Circle at E. - C. 2. 8. 3.

PROBLEM V. 34. III. Euclid.

To cut off a Segment of a Circle; which ſhall contain an Angle equal to a given one.

DEFC is the Circle given, and X the given Angle.

Draw AB touching the Circle (Pr. 3.). At the Point of Contact, C, make the Angle ACD equal to X, the Angle given; and DEFC is the Segment required. Q. E. F.

DEM. For, if from any Point in the Circumference, as E or F, Lines are drawn to the extremes of the Chord CD; the Angle DEC, or DFC, is equal to ACD, (equal X, by Con.) - - - - - P. 13. 3. El.

PROBLEM VI. 33. III. Euclid.

On a given Line, to deſcribe a Segment of a Circle, which ſhall contain an Angle equal to a given one.

AB is the given Line, and X the given Angle.

Make an Angle, BAD, equal to the given Angle, X. 4.1. Draw AC perpendicular to AD (10.) and, on the other Extreme, B, make the Angle ABC equal BAC.

Or, having biſected AB, draw EC perpendicular to AB, cuting AC in C; on which, with the Radius CA or CB, deſcribe the Ark AFB, the Segment required.

DEM. From any Point, in the Circumference, as F, draw FA and FB. The Angle AFB=BAD, equal to the given Angle X. - - - - - 13. 3.

OR, without drawing AD, on the given Line AB, make an Iſoſceles Triangle, whoſe Angles at the Baſe, BAC, ABC, are each equal to the Complement of the given Angle, to a Right angle.

The Vertex C will be the Center, and CA, or CB, the Radius of the Circle required.

PROBLEM VII.

A Circle being given, to cut off a Segment, and, on a given Line, to conſtruct a Segment, ſimilar to a given one.

Firſt. It is required to cut off, from the given Circle FHI, a Segment ſimilar to X.

Find the Center, C, of the given Segment, X, Pr. 2. and compleat the Semicircle ABD.

Through the Center, E, draw FG, and make the Angle GFH equal DAB; - - - - - - Pr. 4. 1.

The Segment, Y, cut off by the Chord FH, will be ſimilar to the given Segment.

DEM. For, the Angle GFH=BAD (Con.); And (having joined BD and HG) the Angle FHG=ABD, Ax. 9. For they are Right angles; - - - 12. 3. El. conſequently, FGH=ADB; - - C. 5. 10. 1. wh. the Triangles ABD, FHG are ſimilar. C. 2. 4. 6. Therefore, as FG:AD::FH:AB. - - 4. 6. El. And, by taking away the Triangles ABD, FHG, from the Semicircles, there is left the Segment Y ſimilar to X.

N. B. If the Segment given had been greater than a Semicircle, the operation is the ſame; the Triangle GFH being added to the Semicircle FIG, inſtead of taking it away from FHG.

N. B. 2. The Angles, A and F, &c. of ſimilar Segments are equal; and they, alſo, contain equal Angles, at K; viz. AKB equal FKH; by which, ſimilar Segments of Circles may be conſtructed, but not with that preciſion, as by the method ſhewn, here.

PROBLEM VIII.

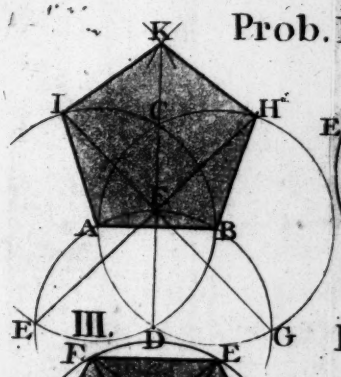
The Angles being given, under which, each two contiguous Objects, of three, ſituate in a Right line, are ſeen, and their Diſtances from each other known; to determine the Station from which they are ſeen.

A, B, and C are the ſtations of the three Objects. Draw CD and AD, making the Angles ACD, CAD, alternately, equal to the given Angles, i. e. make ACD equal to the Angle under which A and B are ſeen, and CAD equal to the other; let them interſect at D. Deſcribe a Circle through the two extreme Objects, A and C, and the Angle D (Pr. 2.); Draw DB, and produce it, till it cuts the oppoſite Circumference, at E, the Point ſought.

DEM. For (having drawn AE and CE) the Angle AEB=ACD, and the Angle CEB=CAD. • P. 10. 3.

The elegance of this Problem, and the conciſe Demonſtration of it, more than its real utility, induced me to give it a place, in this Work; nor could I claſs it, with the reſt, in any other Section, with more propriety than in this; which will, I preſume, be ſome Apology, with thoſe who may be of a contrary opinion.

Prob. I.



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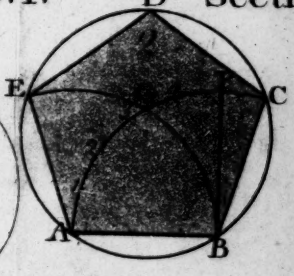

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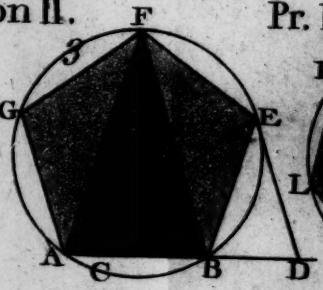

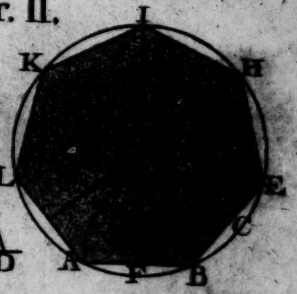

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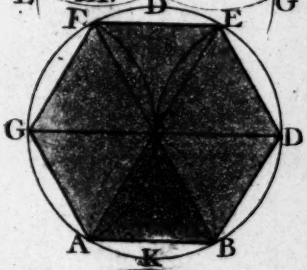

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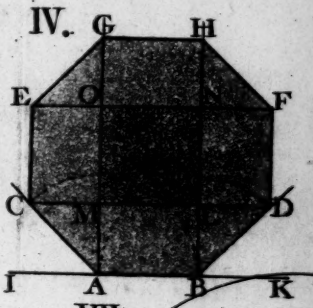

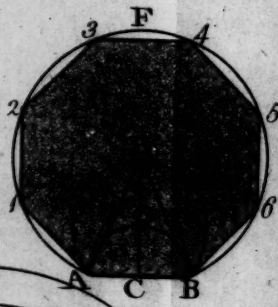

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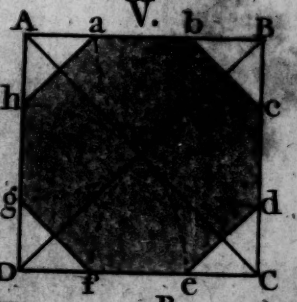
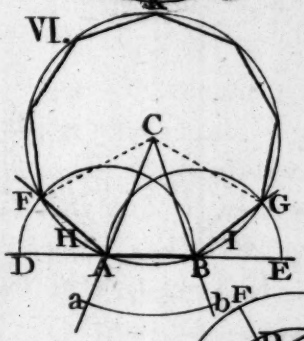


VI.



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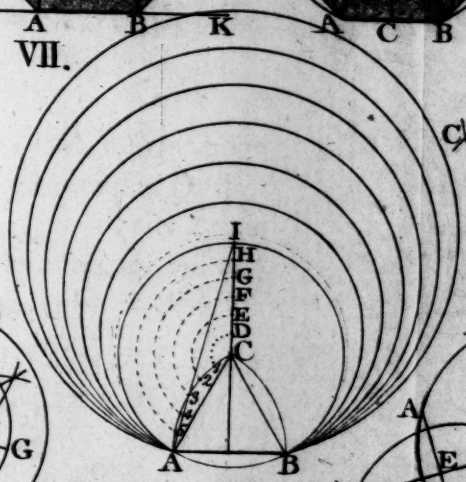

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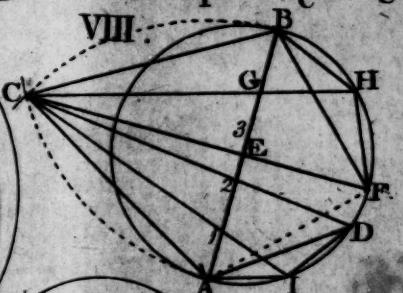

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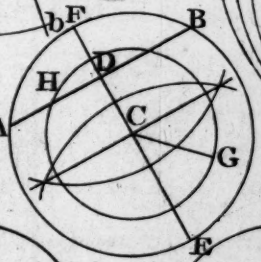

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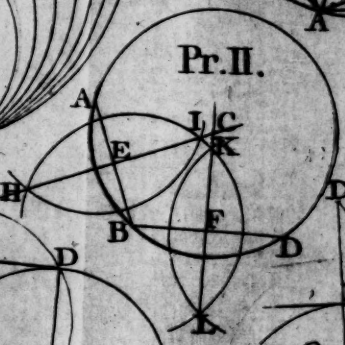

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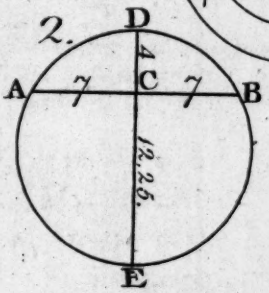

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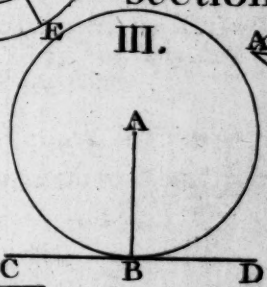

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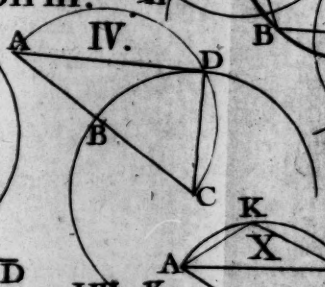

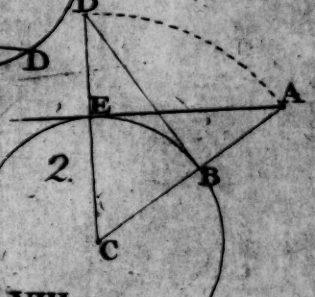

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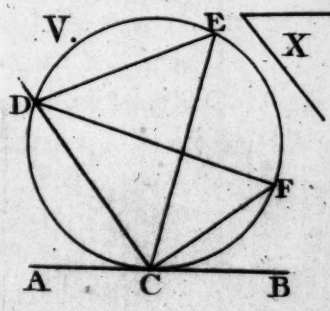

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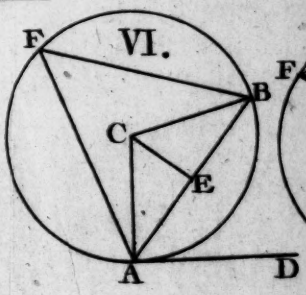

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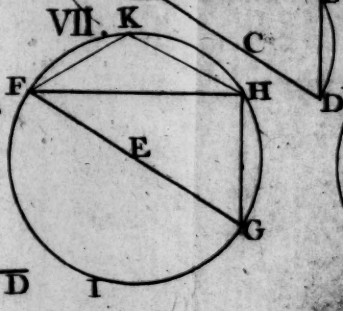

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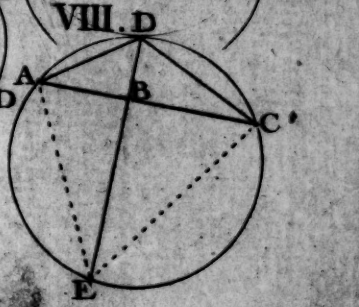
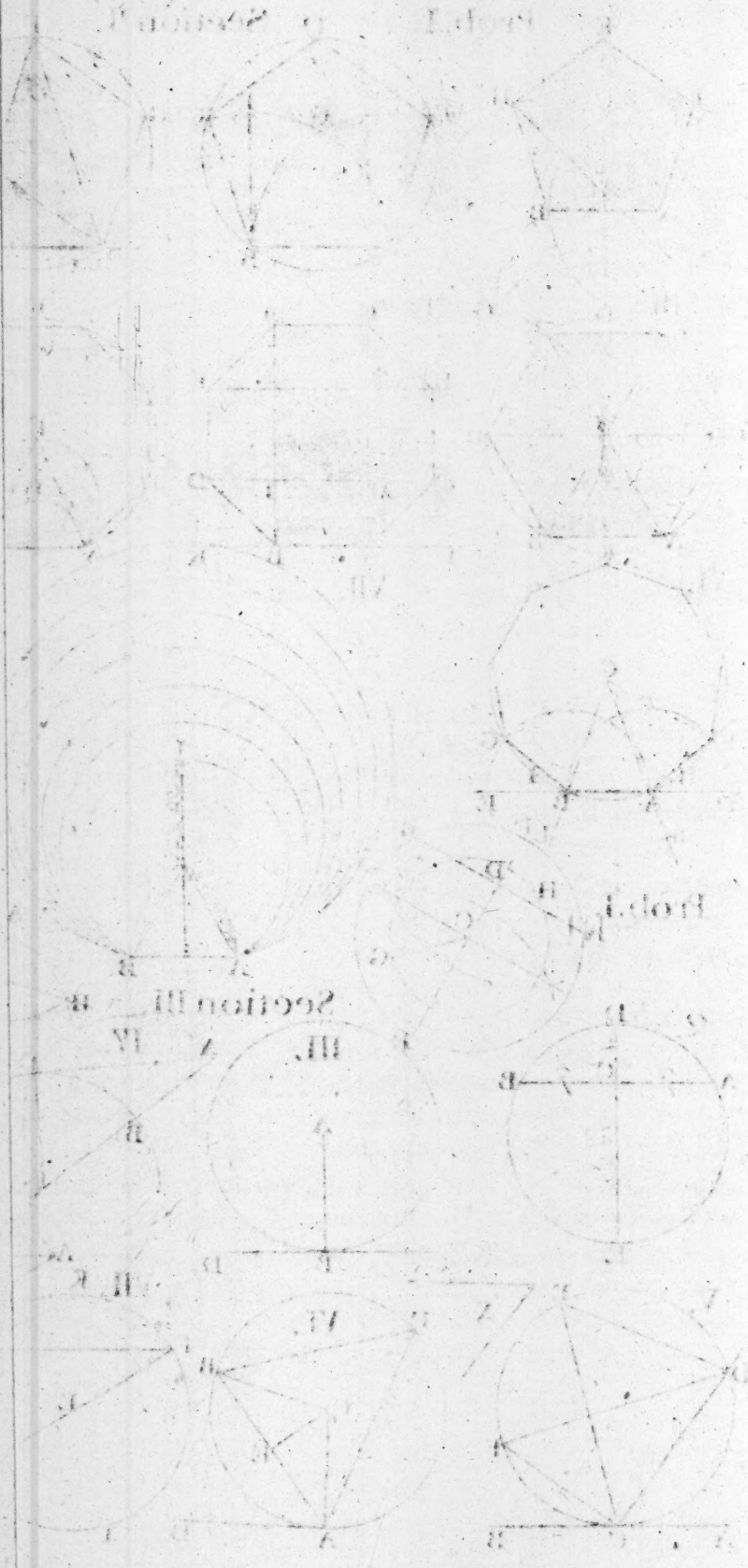




PRACTICAL GEOMETRY.

SECTION IV.

On Inſcription *and* Circumſcription *of* Figures, *in general.*

PROBLEM I.

To inſcribe a Triangle in a given Circle, equiangular to a given Triangle.

Let ABC be the given Triangle.

DRAW, at pleaſure, the Right line GH, touching the Circle in any Point of its Circumference; as D. 3.3. Make the Angle GDF equal to A; i. e. to any Angle of the Triangle; and HDE equal to another, B, and join EF. The Triangle DEF is equiangular with ABC.

DEM. The Angle GDF=DEF, and HDE=DFE. 13.3. Wherefore, the Angle DEF, being eq. A, and DFE eq. B, the remaining Angle, EDF, muſt neceſſarily be equal to C. Th. the Triangle DEF is equiangular to ABC. 10.1.El.

N. B. By taking the Angles in this order, the Triangle inſcribed will be poſited as the given one; but, any two being taken, the Triangle, inſcribed, will be the ſame.

OR, draw, at pleaſure, a d, cuting the Circumference in a and d. Make the Angle, a d c, equal to B, and join a c. Make, c a b, equal to the Angle A, and join b c. The Triangle a b c is equiangular to ABC.

For, the Angle a b c=a d c (10. 3.) equal B; and, the Angle c a b was made equal to A; conſ. a c b is equal to the remaining Angle C. - - - - 10. 1. El. Therefore, the Triangle, a b c, is equiangular to ABC.

PROBLEM II.

To circumſcribe, that is, to deſcribe or draw, a Triangle, about a given Circle (touching it on every Side) equiangular to a given one.

Let ABC be the given Triangle, and DFG the Circle given.

Produce any Side of the Triangle, as AC, both ways; and draw a Radius, DE.

Make the Angle DEF, equal to the external Angle A; and FEG equal C. Through the Points D, F, and G; draw Tangents to the Circle, cuting each other in H, I and K. Then is HIK the Triangle required.

Dem. Becauſe HI, &c. touches the Circle, and DE, EF and GE are drawn to the Center, EDI, IFE, &c. are Right angles. - - - - - 8. 3. El.

And, becauſe the Angles of every Quadrilateral, are equal to four Right angles (11. 3) I+DEF=to two Right ones. But, DEF=A (Con.) conſ. DIF=BAC. Ax. 7 The Angle K may be proved eq. to ACB; and H, to ABC. Therefore, the Triangle HIK is equiangular to ABC.

PROBLEM III.

To inſcribe a Circle in a given Triangle (ABC) touching every Side of the Triangle.

Biſect any two Angles of the Triangle, ABC, and CAB, by the Right lines BD, and AD, cuting each other in D; from which Point, draw a Perpendicular (DE) to any Side (AB) of the Triangle.

With the Radius DE, on the Center D, deſcribe the Circle EFG, which will touch every Side of the Triangle, ABC.

Draw DF and DG, perpend. to the Sides, AC and BC.

DEM. AF=AE (C.2.16.3.) and, the Angle EAD=DAF; wherefore DE is equal to DF. - - P. 8. 1. El. By the ſame, DG is equal to DE, and alſo to DF; and conſequently, the Circle, EFG, will touch every Side of the Triangle ABC.

PROBDEM IV.

To circumſcribe a given Triangle with a Circle; or, to deſcribe a Circle about a given Triangle.

ABC is the Triangle given.

Biſect any two Sides of the Triangle, as AB and BC, in E and F. Draw ED and FD, perpendicular to AB and BC, reſpectively, cuting each other in D, the Center of the circumſcribing Circle.

If the Triangle has a Right angle, (*Fig.* 2.) the Hypothenuſe biſected gives the Center.

If an Angle be obtuſe, the Center of a circumſcribing Circle falls without the Triangle; the Operation is the ſame.

Join the Points AD, BD, and CD.

DEM. Becauſe AB is biſected in E, and DE is perpendicular to AB, the Sides AE, ED, of the Triangle AED, are equal, reſpectively, to BE, ED, of the Triangle BED; and the Angle AED=BED; - - Def. G. wherefore, AD=BD. (9. 1.) In the ſame manner, CD may be proved equal AD, or BD; conſequently D, is the Center of the Circle, ABC.

PROBLEM V.

To inſcribe and circumſcribe Squares, in and about Circles.

Draw two Diameters, AC and BD, at Right angles. Join the Extremes, A, B, C, D, and a Square will be inſcribed.

2nd. Draw Tangents through the extreme Points of the Diameters, meeting in E, F, G, and H; or draw Lines, touching the Circle, parallel to each Side of the inſcribed Square, and EFGH is a Square, circumſcribing the Circle.

DEM. Becauſe AI, IC, IB, and ID, are equal, and contain equal Angles, AB, BC, CD and AD, are equal. P. 8. 1. And, the Angles ABC, BCD, &c. being in a Semicircle, are Right angles. Therefore ABCD is a Square.

2ndly. Becauſe EF and HG are perpendicular to the Diameter BD; and EH, FG to BD (8. 3.) they are parallel; and for the ſame reaſon, EH is parallel to FG;

But, AC is perpendicular to BD; wherefore, EH is perpendicular to HG, and EF to FG and EH;

But EF, FG, EH, and HG, are each equal to the Diameter.

Th. EFGH is equilateral; conſ. it is a Square - Def. P.

N. B. The Square, EFGH, circumſcribing a Circle, is double the Square, ABCD, inſcribed in the ſame Circle.

For the Triangle ABC is equal half the Rect. AEFC. - 19. 1. And ADC is equal to half the Rectangle AG; conſequently, the Square EFGH is double ABCD.

PROBLEM VI.

To inſcribe and circumſcribe Circles, in and about Squares.

Let ABCD be the given Square.

Draw the Diagonals AC and BD cuting each other in E. From the Center E, draw EF perp. to AB. With the Radius EF, on the Center E, deſcribe a Circle, which will touch every Side of the Square ABCD.

2nd. With the Radius EA or EB, on E, deſcribe the Circle ABCD; which will paſs through every Angle of the Square.

DEM. The Diags. AC and BD biſect each other, in E; 16.1. and ſo do the Diameters GI and HF. - Cor. to 16. Conſequently the Circle FGHI touches every Side; and ABCD paſſes through every Angle of the Square.

N. B. A Circle circumſcribing a Square, is double of a Circle inſcribed.

For, the Square of AC = AB ▫ + BC ▫; conſequently, AC ▫ is double BC, equal FH, ▫.

But Circles are, to each other, as the Squares of their Diameters; th. the Circle ABCD is double FGHI. C. 1. 14. 4.

PROBLEM VII.

To inſcribe a regular Pentagon in a Circle.

ABC is the given Circle.

Inſcribe an Iſoſceles Triangle, ABC, whoſe Angles, BAC, ACB, are, each, double the Angle ABC. Pr. 10. 4. El. Biſect each of the Angles BAC, ACB, by the Right lines AD and CE, cuting the Circumference in D and E.

Join the Points, A and E, &c. by the Right lines AE, EB, BD, and DC. AEBDC is a regular Pentagon.

DEM. For, becauſe each Angle, BAC, ACB, is double the Angle ABC (Con.) and thoſe Angles are biſected, the Angles, ABC, ACE, ECB, BAD, and DAC, are all equal; conſequently, the Pentagon is equiangular.

But, equal Angles ſtand on equal Arks, - C.29.3.
and, equal Arks have equal Chords or Subtenſes, 2.3.3.
wherefore, AC, AE, EB, BD, and DC are all equal;
and conſequently, the Pentagon, AEBDC, is equilateral.

Cor. The Diagonals of a Pentagon are parallel to their oppoſite Sides.

For, the Triangles AEC, ADC are equal; P. 8. 1. El.
Therefore, ED is parallel to AC. - - Cor. 18. 1.

It is evident, that to inſcribe a Poligon of any kind, in a Circle, is but to find the Chord of the Ark, ſubtended by a Side of the Poligon, of whatever denomination; which, for a Pentagon, is thus determinable.

Draw two Diameters, AB and CD, interſecting at right angles. Biſect the Semi-diameter CE, at F; on which, as a Center, and with the Radius AF, deſcribe the Ark AG; and, on A, the Ark GH.

AH, being drawn, is a Side of a Pentagon, inſcribed.

The Ark AH, being biſected, gives the Side of a Decagon.

PROBLEM VIII.

To deſcribe a regular Pentagon about a Circle.

Let ABCDE be the angular Points of a Pentagon, inſcribed in a Circle, through which draw the Tangents FG, GH, HI, &c. cuting each other, in F, G, H, I, and K. i. e. having, made the Arks AB, BC, &c. equal, (viz. each equal 72 Degrees) through the Points A, B, C, &c. draw the Tangents, as before.

The Pentagon FGHIK is equilateral and equiangular.

DEM. For (L being the Center of the Circle) AL, BL, &c are equal (Ax. 1. 3.) AG=GB, and BH=BC (16.3.); and, Angle ALB=BLC (Con.); wherefore, the Triangles AGL, GLB, &c. are congruous; and AF, AG, GB, &c. are all equal (7. 1.); (for AL, LB, LC, are perpendicular to FG, GH, and HI, reſpectively; 8. 3.) and conſequently, FG is equal to GH, equal HI, &c. Alſo, the Angles AGL, LGB, LHB, LHC are equal; therefore, the Angle FGH is equal GHI, &c. Ax. 5. 1.

Note. If the Circumference of a Circle be divided in five equal Parts, and the Points joined by Right lines; then, if Tangents to the Circle be drawn, parallel to every Chord, a regular Pentagon will be circumſcribed.

See this demonſtrated at large, in Prop. 10. 4. El.

PROBLEM IX.

To inſcribe a Circle in a regular Pentagon; and, to deſcribe a Circle about a Pentagon.

Biſect any two Sides, AB, and BC, of the Pentagon ABCD, by the Perpendiculars EF and FG, interſecting in F; on which, with the Radius EF, or FG, deſcribe a Circle; which will touch every Side of the Pentagon.

2nd. With the Radius AF or FB (the Point F being found as before, or by biſecting two Angles, ABC and BCD) deſcribe a Circle, which will paſs through every Angle of the Pentagon.

See this demonſtrated in the 11th Prop. Book 4.

But 'tis obvious from inſpection of the Figure only; F, being the Center of the Pentagon, is equally diſtant from every Side, and alſo from every Angle; conſequently, a Circle inſcribed will touch every Side, and alſo paſs through every Angle, being circumſcribed.

PROBLEM X.

To inſcribe a regular Hexagon in a given Circle.

In the given Circle, AGF, draw a Diameter, AB.

With the Radius of the Circle, AC, on the Centers A and B, deſcribe two Arks, DCE, FCG, cuting the Circumference in the Points D and E, F and G.

Or, having drawn the Ark DCE, only, cuting the Circle in D and E, draw DC and EC, and produce them to F and G.

Join the Points A and D, AE, EF, &c. by Right lines, which compleats the Hexagon, ADGBFE. Q. E. F.

I ſay, it is both equilateral and equiangular.

The Side of a Hexagon inſcribed in a Circle is equal to the Radius, being the Chord of 60 Degrees.

For which, ſee the Demonſtration of Prop. 12. 4.

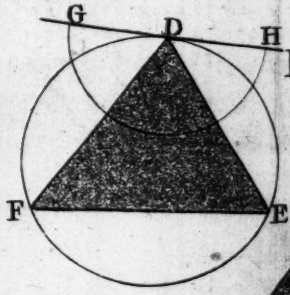
Cor. A Duodecagon may be inſcribed in a Circle, by biſecting each Ark on the Side of a Hexagon inſcribed. i. e. The Radius being applied ſix times in the Circumference, each Ark biſected gives the Sides of a Decagon.

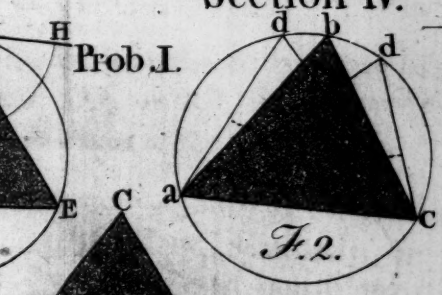
PROBLEM XI.

To determine the Side of a Heptagon, in a given Circle.

Draw a Radius, at diſcretion, as AC; which, biſect at D, and draw EF perpendicular to AC; i. e. with the Radius AC, on A, cut the Circumference at E and F, and draw EF.

Prob. I.





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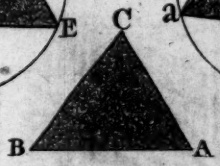

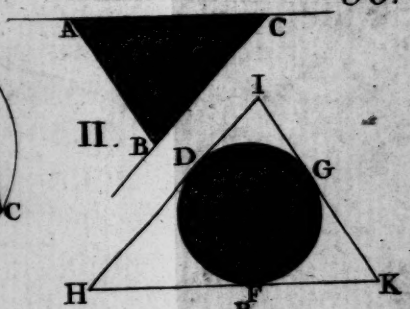

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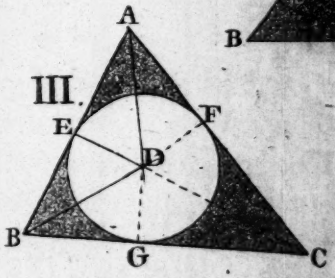

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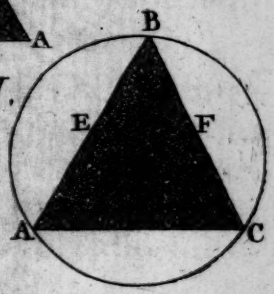

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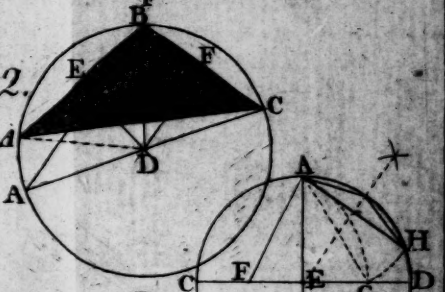

Fig. 2

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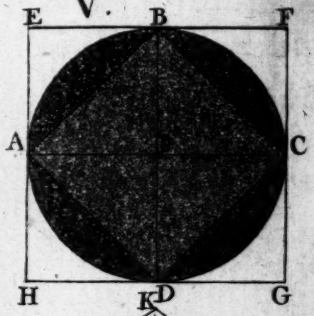

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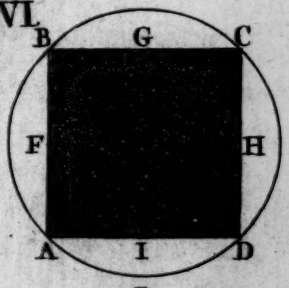

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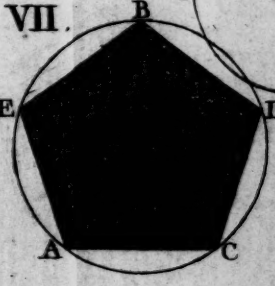

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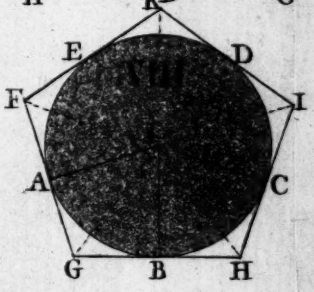

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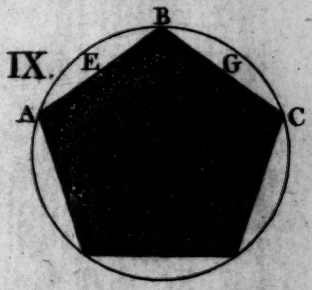

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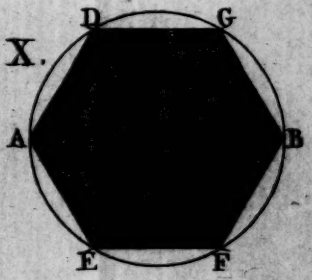

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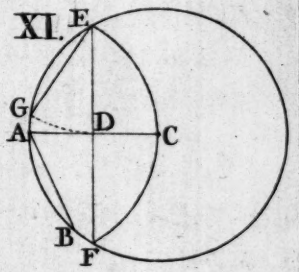
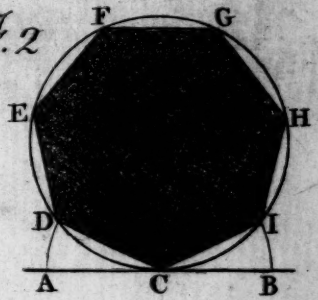


Fig. 2



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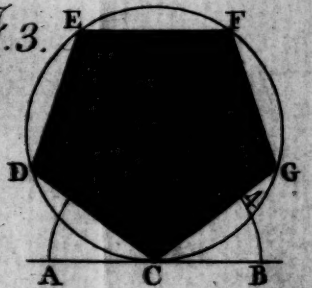
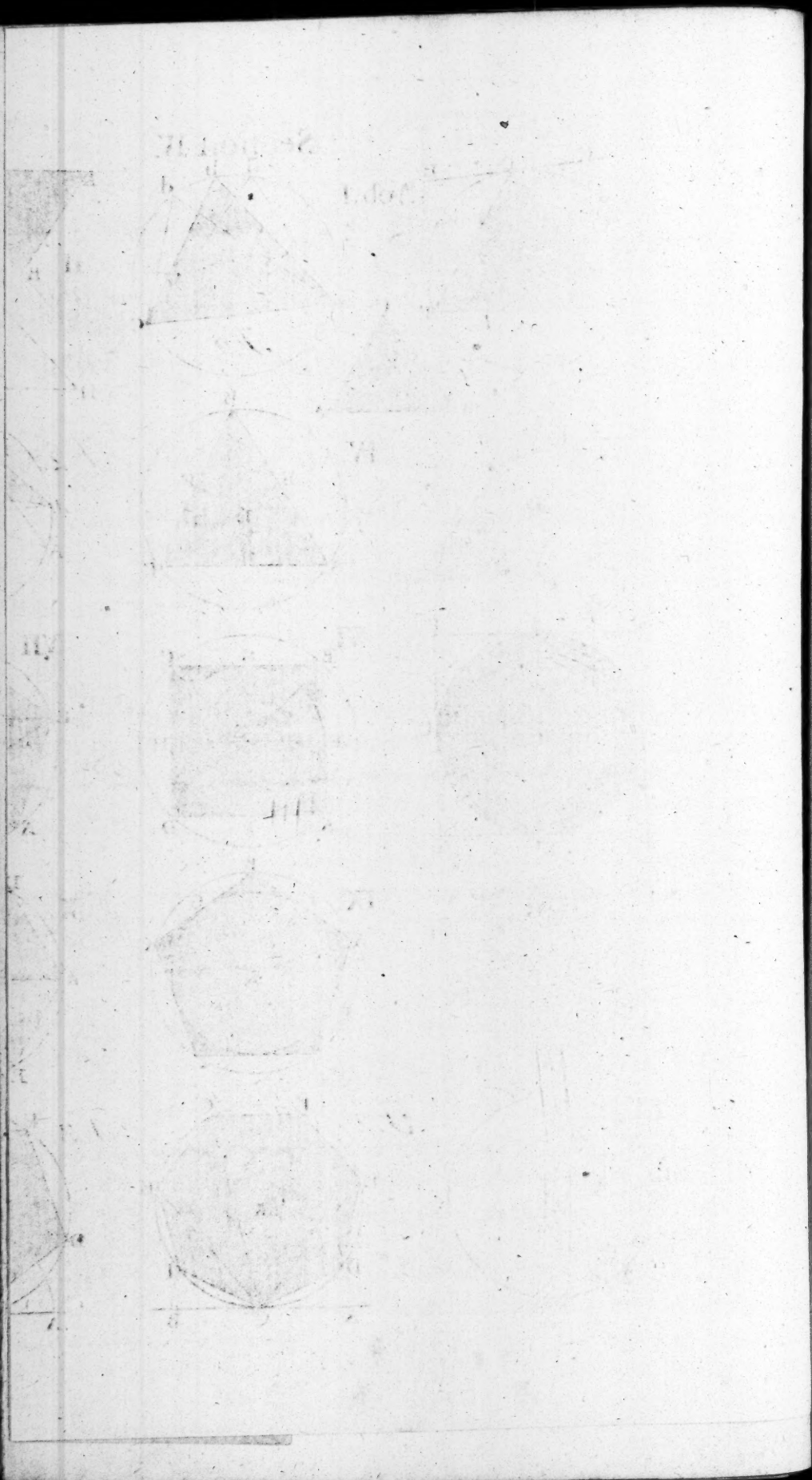




Make AB equal to DE; or, transfer the meaſure of DE to the Circumference; EG, or AB, is very nearly a ſeventh part of the Circumference of the given Circle.

A Heptagon, or any other Poligon, may be inſcribed in a Circle, by dividing the Arch of a Semicircle, deſcribed on a Tangent to the given Circle, into as many equal parts as the Poligon has Sides; as follows.

Divide the Ark of the Semicircle AKB, into ſeven equal Parts, through every Diviſion, C1, C2, &c. Lines being drawn, cuting the given Circle in D, E, F, &c. will divide the Circumference into as many equal Parts, in thoſe Points; which being joined, by Right lines, compleats the Heptagon CDEFGHI.

See an Example for a Pentagon, by the ſame Rule.

A Semicircle, AHB, being conſtructed on AB (as a Tangent to the Circle at C) of any Radius, at pleaſure, and divided into five equal parts (as before into ſeven) draw C1, C2, &c. and produce them till they cut the other's Circumference, at D, E, &c. which, joined by Right lines, compleats the Pentagon.

The Demonſtration of this Method is as follows (granting the Arch of the Semicircle to be truly divided):

DEM. Becauſe the Angs. DCE, ECF, and FCG are equal, the Arks, and conſ. the Chords DE, EF, & FG, are equal. Alſo, the Angle ACD (eq. DCE by Con.) = DEC. 13.3, wh. the Triangle CDE is Iſoſceles; and CD = DE. 9. 1.

After the ſame manner CG may be proved equal to FG; conſequently, DC, DE, EF, FG, and GC are all equal. Therefore, the Pentagon, DEFGC, is equilateral; and, being inſcribed in a Circle, it is neceſſarily equiangular, each Angle, CDE, DEF, &c. ſtanding on equal Parts of the Circumference, CGFE = DCGF.

It may be obſerved, that, becauſe the equal Angles ACD, DCE, ECF, &c. are at the Center of the Circle AKB, or AHB, and at the Circumference of the other, and, becauſe the Angle at the Center of a Circle, is double of that at the Circumference, on the ſame or equal Arks (C. 1. 9. 3.); therefore, the Semi-circumference, of one, and the whole Circumference, of the other, are divided into the ſame number of equal Parts, by the Right lines, CD, CE, &c.

If AC be taken equal to the Radius of the given Circle, it may appear more manifeſt.

Euclid's method (for a Pentagon) of inſcribing a Triangle ſimilar to another, (which muſt firſt be formed) whoſe Angles at the Baſe are double of the Vertex, is very ingenious and perfectly geometrical; but it is liable to great error in the Conſtruction; inſomuch, that it is extremely difficult to do it, with accuracy, and tedious in the Operation; whereas, this is both more expeditious and more to be depended on.

FOR inſcribing an Octagon in a Circle, nothing more is required than to draw two Diameters at right angles, and to biſect the Arks of the Quadrants made by the Diameters.

PROBLEM XII.

To inſcribe a Nonagon in a given Circle.

C being the Center of the given Circle, draw the Radius AC; and, on A, deſcribe the Ark DCE, and draw DE, which biſects AC, in B. - - - Pr. 8. S. 1.

Produce DE, making BF equal to the Radius, AC; on which, deſcribe an equilateral Triangle; - 14. 1. or, draw the two Arks, BG and FG, only.

Draw CG, cuting the Circumference at H; EH being drawn, is the Side of a Nonagon, as required.

As, in Prob. 6th of the 2nd Section, any Poligon may be conſtructed, by a general Rule, on a given Line, for the Side, ſo (by the ſame Rule) any Poligon may be inſcribed in a given Circle; by making an Angle at the Cen-

ter, of the number of Degrees, which is the Product, ariſing from the Diviſion of 360 by the Denomination of the Poligon. e. g. 360 divided by 5 (for a Pentagon) gives 72; for a Hexagon, 60; for an Octagon, 45; for a Nonagon, 40; for a Decagon, 36. A Heptagon does not divide the Number of Degrees exactly; but it is nearly, $51\frac{1}{2}$, or more nearly, 51,43.

Wherefore, having (by means of a Protractor, or Line of Chords) made an Angle, ACB, at the Center of a Circle, equal 40 Degrees, the Chord AB, of the Ark it ſtands on, is the Side of a Nonagon inſcribed in the Circle; i. e. AB is one ninth Part of the whole Circumference.

For, let the Angle ACB be conſtructed or meaſured on any Ark, as ab, either of a greater or leſs Radius, the portion AB, of the Circle AKB, will be the ſame; for AB and ab are each one ninth part of their reſpective Circumferences. See the Figure of Prob. 6, Sect. 11.

PROBLEM XIII.

To inſcribe an Undecagon in a given Circle.

Draw a Radius, AC, which biſect, at B.

On A and B deſcribe the Arks BE, and AD, with the Radius AB; and on E (with the Radius DE) the Ark DF, and draw BF; alſo, on F, the Ark BG.

BF, or FG is the Side of an Undecagon, to be inſcribed; for, the Chord of BF, being applied eleven times to the Circumference, meaſures it very nearly.

PROBLEM XIV.

In a given Circle, to inſcribe whatever Poligon you pleaſe.

Deſcribe a Circle, AGB, which will contain the Diameter (AB) of the given Circle, as often as the Poligon required has Sides; ſuppoſe nine; - - by Pr. 7. S. 2.

Draw the Diameter CD, parallel to AB; and, through the extremes, C and A, or D and B, draw CE or DF, cuting the Circumference of the given Circle, in E or F. EF, being drawn parallel to AB, will be the Side of a Nonagon, to be inſcribed.

Or, by drawing CB, and AF parallel to CB, and EF to AB, the ſame is effected. Q. E. F.

DEM. Becauſe AB is parallel to CD, and CB to AF, the Angle BCD=BAF; - - - P. 6. 1. El. and, becauſe DF cuts AB and CD, the Angle CDB=ABF (4. 1.); conſ. the Triangles DCB and BAF, are ſimilar; wherefore, CD:AB::CB:AF - 4. 6. El. And in the ſimilar Tris. ACB, EAF, CB:AF::AB:EF; and therefore, CD:AB::AB:EF, i. e. the Diameter of one is to the Diameter of the other Circle, as the Chord of one to the Chord of the other. - - - 13. 6. El.

PROBLEM XV.

To inſcribe a Square in a given Triangle, ACB; and to circumſcribe a Square by a Triangle ſimilar to another.

On either extreme of the Baſe, AB, draw a Perpendicular, which make equal to the Baſe; AD equal AB. 7. 1.

Draw the Perpendicular CE; alſo draw DE, cuting the Side AC at F, and draw FG, parallel to AB; alſo, GH, and FI, parallel to AD.

FGHI is a Square, inſcribed in the Triangle ACB.

The ſame would be effected if BK were made perpendicular and equal to AB, and KE drawn, cuting BC, the other Side, at G.

From both, the Demonſtration is evident, as follows.

DEM. AD and BK are, each equal to AB, and, FI and GH are parallel to AD, or BK, - by Conſtruction. Wherefore, by the ſimilar Triangles ADE, IFE, AD: FI :: AE : IE; alſo, EGH is ſimilar to EKB; conſequently, BK : GH :: EB : EH. - P. 4. 6. El.

But, BK = AD; conſequently, GH = FI.

And, AE + EB = AB (eq. AD); alſo, IE + EH = IH; wherefore, AB (eq. AD) : IH :: AD : FI; conſ. IH = FI; therefore, FGHI is a Square.

2 nd. To circumſcribe a Square by a Triangle, ſimilar to a given one.

Conſtruct a Triangle, on any Side, AC, ſimilar to X, the given one, as ABC, by making the Angles, at A and C, equal, reſpectively, to any two, of the Triangle X, and producing its Sides, BA and BC, until they cut the Baſe of the Square produced, at D and E.

Then will BDE circumſcribe the Square given, and be ſimilar to ABC; i. e. to the given Triangle X.

For, ABC is ſimilar to X, by Conſtruction; and, becauſe DE is parallel to AC, BDE is ſimilar to ABC, and conſequently to X. - - - - 2, 6. El.

PROBLEM XVI.

To circumſcribe an equilateral Triangle, by a Square.

ABC is the Triangle given, to be circumſcribed.

Produce AC, both ways, and draw the Perpendicular BD. Make DE and DF each equal to BD; and, on D, with the Radius AD, deſcribe a Semicircle, cuting the Perpendicular, produced, at G; and, laſtly, thro' A and C, draw GH and GI.

BIGH is the Square required, circumſcribing the Triangle ABC.

DEM. Becauſe DE and DF were made equal to BD, and BD perpendicular to EF, the Angle EBF is right; EBD, and DBF, being half right.

And, for the ſame reaſon, the Angle AGC is a right one, and BG biſects them both; conſequently, the Angles GAC, ACG are half right, equal EAH, equal ICF; wherefore AHE, CIF, and conſ. BHG, BIG are Right angles; and AH=AI; therefore BIGH is a Square.

PROBLEM XVII.

To inſcribe an equilateral Triangle in a Square.

Draw the two Diagonals; and, on the Center, deſcribe a Circle, circumſcribing the Square, ABCD.

With the Radius of the Circle, on the Angle C, deſcribe the Ark FEG, cuting the Circumference; and draw AF and AG, cuting the Side of the Square at H and I. Draw HI, which compleats the Triangle, AHI, as required.

DEM. CF and CG were made equal, each to the Radius of the Circle; wherefore, FG is the Side of an equilateral Triangle, inſcribed.

For, the Angles FAC, CAG, at the Circumference, on the Arks CF, and CG of 60 Degrees each, are each 30 Degrees; conſequently, FAG is 60 Degrees, which is the Angle of an equilateral Triangle. - 9. 3. El.

But, HI is parallel to FG, therefore, the Triangle AHI is ſimilar to AFG; i. e. equilateral. - 2. 6. El.

PROBLEM XVIII.

To inſcribe an equilateral Triangle in a Pentagon.

Circumſcribe a Circle about the Pentagon ABCDE; 9.4. and, on A, with the Radius of the Circle, deſcribe the Ark FG, cuting the Perpendicular at F.

Biſect FG, and, through H, draw AI.

With the Radius AI, draw the Ark IK; join IK.

AIK is an equilateral Triangle, as required.

For, AG and FG being drawn, FAG is an equilateral Triangle; the Angle of which is biſected by the Line AI; and the Ark IK is biſected by a Perpendicular, from the Angle A; wherefore, the Angle KAL is equal LAI, and conſequently, the Angle KAI is 60 Degrees; and therefore KI is equal to AK, equal AI.

This Problem may be performed as the former, by deſcribing an Ark, on O, with the Radius of the circumſcribing Circle, and perhaps ſomewhat readier. In which caſe, the Demonſtration would be the ſame as the foregoing.

PROBLEM XIX.

To inſcribe a Square in a given Pentagon.

Draw the Diagonal BE, of the Pentagon ABCDE; alſo the Perpendicular AF. Draw BG perpendicular to BE (Pr. 7. 1.); make it equal, alſo, and draw AG. From the Section H, where AG cuts the Side BC, of the Pentagon, draw HI parallel to CD. At the Extremes H and I, draw HL and IK, cuting AB and AE, and draw KL.

HIKL is a Square, inſcribed in the Pentagon, as required.

DEM. Produce the Perpendicular AF, and draw GM parallel to CD; alſo, draw AI. Then, becauſe HI is parallel to GM, and IK to BG, the Triangles AGM, AHF; alſo, ABG, AIK are ſimilar. - - - P. 2. 6. El.

Wh. BG: KI :: AG : AI; and AG: AI :: GM: HF;

conſ. BG: KI :: GM: HF; by Ratio of Equality. 8. 5. El.

But, BG is equal to the Diagonal BE, and GM to half BE; conſequently, FH is equal half KI; wherefore, HI = IK; and therefore HIKL is a Square, inſcribed; for it is equilateral, and the Angles are all right ones.

PROBLEM XX.

About a given Square, to circumſcribe a Pentagon.

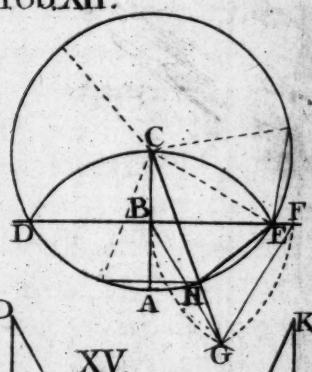
ABCD is the Square given, to be circumſcribed.

Biſect AB, and draw EF, perpendicular to AB. 6. 1. Produce CB, indefinite; and, with the Radius EB, deſcribe the Arch of a Quadrant EG, which divide into five equal Parts. With the ſame Radius, on C and D, deſcribe the Arks HI, HJ. Make HI, HJ, each equal a fifth part of the Quadrant EG; and, through C and D, I and J, draw KL and MN, indefinite; alſo, through B, and the 2nd Diviſion of the Ark EG, draw FL, cuting KL, at L. Make KL equal FL, and draw KM parallel to CD; and through A, draw FN; FLKMN is the Pentagon required.

DEM. In the Iſoſceles Triangle AFB, the Angles, at A and B, are each two fifths of a Right angle; by Con. And, in the Scalene, BLC, the Angle CBL + FBE are equal to a Right angle; for ABC is a right one. 1. 1. El.

But, CBL is 3 fifths of a Right angle, and BCL 1 fifth; conſequently, the Angle L is 6 fifths, equal F.

Prob. XII.



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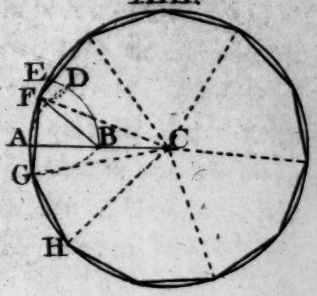

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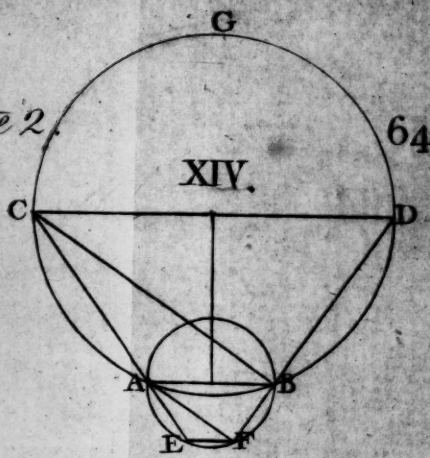

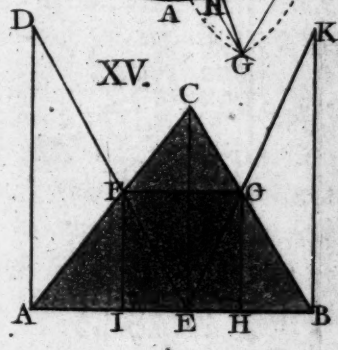

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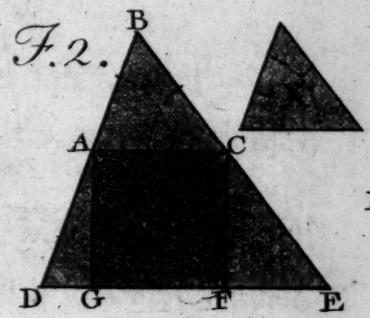

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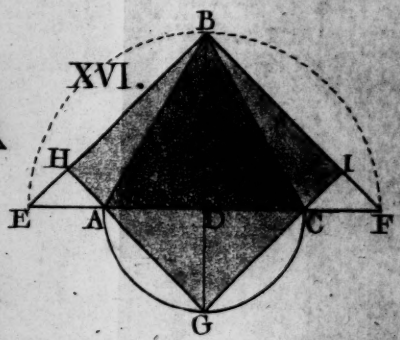

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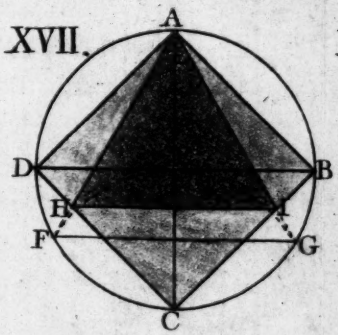

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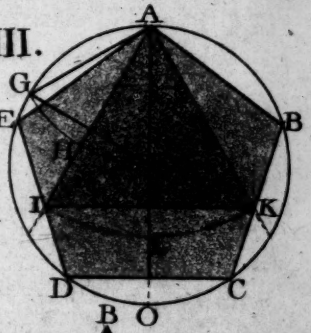

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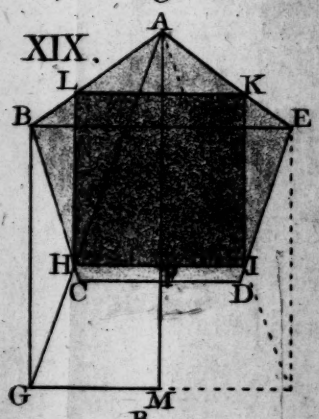

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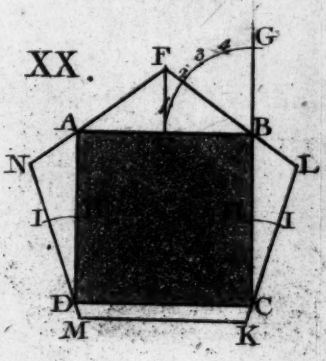
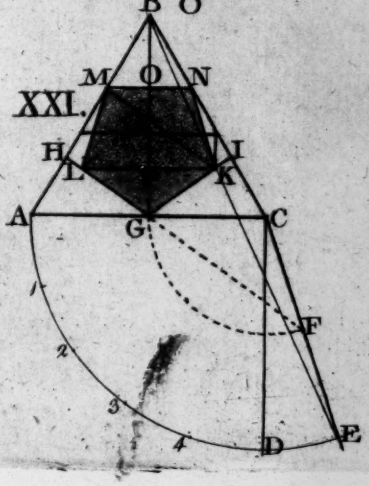


XXI.



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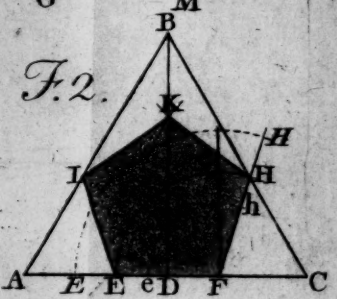
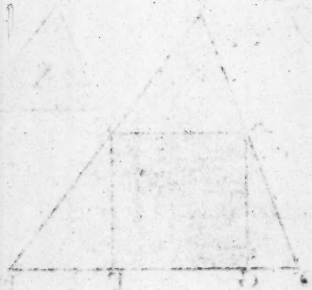
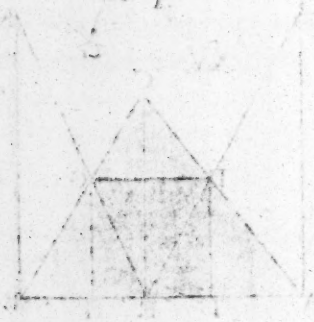
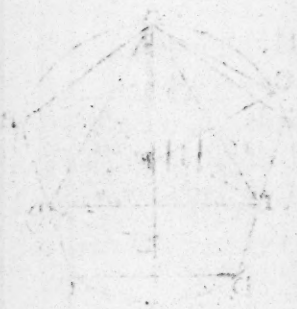
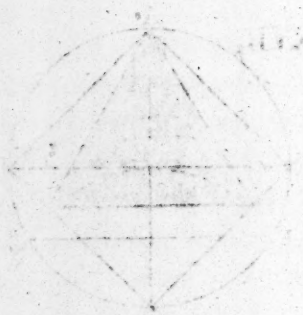










By the ſame reaſoning, the Angle N is equal L, equal F.

And, becauſe KM is parallel to CD, it is evident, that the Angles at K and M are each equal to 6 fifths of a Right angle; ADN, BCL being, each equal one fifth.

Therefore, the Pentagon, FLKMN, is equiangular; and, it is alſo equilateral, ſeeing it may be circumſcribed by a Circle; conſ. FN=FL, alſo, KM and MN.

PROBLEM XXI.

To inſcribe a Pentagon, in an equilateral Triangle.

Draw the Perpendicular CD (7. 1.); which, make equal to AC, by deſcribing the Ark ADE.

Divide the Quadrant AD into five equal parts, and make DE equal one of the Parts. Draw CE, and biſect it, in F; make CG equal CF; i. e. biſect AC. Through G, draw FH, and make CI equal AH; alſo, draw GI, and BE, interſecting at K.

Make GL equal GK; alſo, draw KM parallel to GH, and MN parallel to AC; and, laſtly, draw KN.

LMNKG is the Pentagon required, to be inſcribed.

The Angle ACE is 6 fifths of a Right angle; which is the Angle of a Pentagon. See 2nd Method, Pr. 1. S. 2.

And, CG was made equal to CF; wherefore, FG may be conſidered as a Diagonal, which is parallel to the oppoſite Side; and GL, a Side of the Pentagon, is in FG produced, which is parallel to the Diagonal KM; alſo MN was made parallel to AC, conſequently parallel to the Diagonal KL. GL was made equal to GK, and LM to GL; conſequently, MN and KN are each equal to GK.

Therefore, the Pentagon, inſcribed, is equilateral and equiangular.

I have not ſeen a Demonſtration of any of theſe Problems; and have deviſed thoſe here given; but I do not aſſert that they all are ſtrictly geometrical; this, at leaſt, is not.

This inſcription of a Pentagon is very curious and quite geometrical (granting the Arch of the Quadrant divided into five equal parts); but, 'tis evident, that 'tis not the largeſt poſſible. For, being inverted, having a Side of the Pentagon in a Side of the Triangle, it is manifeſt that it would be larger. That part of the Perpendicular, from O to P, being applied from G to Q, and a Line drawn through Q, parallel to the Side AC, of the Triangle, would be the Diagonal of the Pentagon inverted, which, 'tis obvious, may be larger than the other, as the Perpendicular from K evinces; conſequently the Triangle would contain a larger Pentagon, in that poſition. I have no where ſeen ſuch an Inſcription of it; therefore it may not be unacceptable to make an attempt at it; but I advertiſe the Reader, that 'tis not to be depended on.

AC being biſected, at D, by the Perpendicular BD, and AD divided in extreme and mean Proportion, at E; DF being made equal to DE, the leſſer Segment, EF is ſomewhat more than a Side of the Pentagon to be inſcribed, about four, hundredth Parts. Take EF ſo much leſs, and draw FG perpendicular. On F, with any Radius, as EF, deſcribe the Ark EGH, cuting the Perpendicular, and make GH equal to one fifth Part of the Quadrant EG, and draw FH, which is a Side, and EFH an Angle of the Pentagon; HI being drawn, parallel to AC, is a Diagonal, in that poſition. Then, with the Radius of a Side, on H and I, deſcribe two Arks, interſecting at K; draw HK and IK, and join EI; the Pentagon EIKHF is the largeſt poſſible in the Triangle ABC.

If either a larger or a leſs Radius were taken, as *E*F or eF, the Angle would be the ſame determined; for F*H*, or Fh produced, would cut the Side BC, of the Triangle, in the ſame Point, H.

PRACTICAL GEOMETRY.

SECTION V.

On PROPORTIONALS, *and dividing* Lines *proportionally.*

PROBLEM I. 13. VI. Euclid.

To find a mean Proportional between two given Lines; X and Z.

IT is required to determine a Line, to which, either of the Lines, X or Z, ſhall have the ſame Ratio or Proportion, as that Line has to the other.

Or, the Square of which, ſhall be equal to a Rectangle under the two given Lines.

Draw AC indefinite; make AB equal to one of the given Lines, as X, and make BC equal to the other, Z.
Biſect AC, in the point D; on which Center, and with the Radius AD, equal DC, deſcribe a Semicircle.
At the Point B, draw a Perpendicular, to AC, cuting the Arch at E; and BE is the Line ſought.

DEM. For, draw AE and EC; AEC is a Right angle. 12. 3.
And, the Perpendicular, BE, in a right-angled Triangle (AEC) is a mean Proportional, between the Segments of the Baſe, AB, BC, made by the Perpendicular. C. 1. 7. 6. El.
Conſequently, AB is to BE, as BE is to BC. - ſame.
and, AB x BC is equal to the ſquare of BE; Cor. to 9. 6.

SCHOL. *This is the very ſame, in the operation, as P. 4. S. 6. for the Side of a Square is a mean Proportional between the two Sides of a Rectangle having an equal Area. See Cor. to 9th. B. 6.*

OTHERWISE; thus.

X and Z are the two given Lines.

Make AB equal to X, and AC equal Z; biſect AB, and draw the Semicircle AEB, or the Ark BD, only.

Draw CD perpendicular, and make AE equal AD; AD, or AE, is a Mean, between AC and AB. Q. E. F.

For, the Triangles ADC, ADB are ſimilar. 7. 6. El. wherefore, as AC: AD (eq. AE):: AD: AB, therefore AE is a mean Proportional, as required.

PROBLEM II. 11. VI. Euclid.

To find a third Proportional to two given Lines.

X and Z are the given Lines.

It is required to find a third Line, to which the greater (Z) ſhall have the ſame proportion, as X has to Z; or, which ſhall have that proportion to the leaſt, X.

Make a right-angled Triangle, ABC, whoſe Catheti or Legs, are equal to the two given Lines, i. e. make AB equal to one (X) and BC equal to the other. - 17. S. 1.

Produce AB and CB, indefinite. Make ACD and CAE Right angles; i. e. draw CD and AE perpendicular to AC, cuting AB and CB, produced, in D and E.

Then will BD be a greater, and BE a leſs third Proportional, to the two given Lines, X and Z. Q. E. F.

DEM. For (as in the laſt) EB is to AB, as AB to BC; and AB: BC:: BC: BD. - - - P. 7. 6. El.

Wherefore, EB, AB, BC, and BD are in continued Proportion. - - - - - Def. 7. 5.

Conſequently EB is a leſs, and BD a greater third Proportional, to the two Lines AB and BC, equal X and Z.

PROBLEM III. 11. II. Euclid.

To divide a Line in extreme and mean Proportion.

AB is the given Line.

It is required to cut it ſo, that the leſſer Segment ſhall have that Proportion to the greater, as the greater Segment has to the whole Line.

Or, the Rectangle under the whole Line and the leaſt Segment ſhall be equal to the Square of the greater.

Biſect AB, in C; draw BD perpendicular to AB; make it equal to BC (half the given Line) and draw AD.

On D, with the Radius DB, deſcribe an Ark, BE; and, on A, the Ark EF; i. e. make DE equal DB, and AF equal AE; F is the Point ſought. Q. E. F.

N. B. AF, the greater Segment, is the difference between half the Line AB, and the Hypothenuſe, AD, of a Right-angled Triangle, ABD; conſtructed on the whole given Line, and half the Line, for the Baſe and Perpendicular. - by Pr. 17. S. 1.

This Problem is otherwiſe performed (and demonſtrated) in Prop. 11, of the ſecond Book of Elements; thus.

Conſtruct the right-angled Triangle ABC, as before, making BC half AB. Produce CB, indefinite; and make CD equal AC, and BE equal to BD.

So ſhall AB, be divided in the Point E, as required.

See the Demonſtration, as above.

PROBLEM IV.

Two Lines being given, to cut off, from one of them, a part which ſhall be a mean Proportional, between the remainder and the other given Line.

X and Z are the two Lines given.

Draw the Right line AB, indefinite; take AC equal to either of the given Lines (as X) and CB equal to the other. On AB deſcribe a Semicircle, and draw CD perpendicular to AB.

Biſect AC (or CB) and, with the Radius DE (or DF) deſcribe the Ark DF (or DG); ſo ſhall CG (or CF) be a Mean, between the remainder, AG (or FB) and the other given Line, CB.

CONSTRUCTION. Make CE and EF each equal to half CB. Make CG equal to CG (Fig. 1.) and EI to EG; make CK equal to CF, and CL to CG, and conſtruct the Rectangles AK, IL, and the Square CM.

DEM. EI=ON, ON=OD, and EG=ED (Con.); alſo, the Semicircle, deſcribed on E, with the Radius EI, paſſes through D; for, EI was made equal to ED.

Then, CD is a mean Proportional, between AC and CB; and alſo, between IC and CG (7. 6. El.); where fore, the Square CM is equal to the Rect. IL, and alſo to AK; - - - - - - - - Cor. 9. 6. El. conſequently, thoſe Rectangles are equal; - Ax. 3. 1. and AC:CL::CI:CK. - - - C. 1. to 8. 6.

But CL=CN, and CK=CF;

wh. AC:CN::CI:CF; moreover, AN:NC::IF:FC, equal CB; and, IF=NC; therefore, AN:NC::NC:CB

PROBLEM V.

A mean Proportional being given, and the Sum of the Extremes, to which it is the Mean; to determine the Extremes, ſeverally.

AB is the Sum of the two Extremes, and G the given Mean.

Biſect AB, and deſcribe a Semicircle. Make BC perpendicular to AB, and equal to G; draw CD parallel to AB, cuting the Arch at D; and, laſtly, draw DE parallel to BC; ſo ſhall AE be the greater, and EB the leſſer Extreme, as required.

For AE:ED::ED:EB (7. 6.); and ED is equal to BC, equal G (15. 1. El.); for, BCDE is a Rectangle. Therefore AE:G::G:EB.

PROBLEM VI.

The Difference between the Extremes, of three proportional Lines being given, and the Mean; to determine the Extremes.

G is the Mean given, and AB is the difference of the two Extremes.

On either Extreme (as B) of AB, make BD perpendicular and equal to G.

Produce AB, at both Extremes; and, having biſected AB, at C, draw CD; and with that Radius, on C, deſcribe a Semicircle, cuting AB, produced, at E and F.

AE and AF are the Extremes required.

For, AC=CB; conſequently AE=BF; and conſequently AB is the Difference between AE and AF.

And, EB:BD (equal G)::BD:BF.

PROBLEM VII. 12. VI. Euclid.

To find a fourth Proportional, to three, given, unequal Lines; X, Y, and Z.

It is required to find a Line, to which, the third (Z) ſhall have the ſame proportion, as the firſt has to the

ſecond. Or, the ſame proportion to the firſt Line, as the ſecond has to the third.

Draw two Lines, AD and AE, making any Angle at pleaſure. Make AB equal X, and AC equal Y; alſo make BD equal Z. Draw BC, and DE parallel to it; cuting AE in E; CE is a fourth Proportional, greater than Z.

i. e. As AB is to AC, ſo is BD to CE. Q. E. F.

Or, having made AF equal to Z, equal BD, draw FG parallel to BC; AG is the Proportional ſought, equal CE.

Dem. For, becauſe BC is parallel to DE, the Sides of the Triangle ADE are cut proportionally;
wherefore, AB is to AC, as BD is to CE. - - P. 2. 6.
But, AB is equal to X, AC is equal Y, and BD equal Z;
therefore, as X is to Y, ſo is Z to CE.

If a leſs Proportional be required; make AF equal to Z, AC equal Y, and FD equal to X.

Join FC, and draw DH parallel to it.

Then, will CH be the Proportional ſought.

Dem. For, as AF (equal Z) is to AC (equal Y) ſo is FD (equal X) to CH. - - - by the ſame.

After the ſame manner, a third Proportional may be found.

Draw AF and AG, making any Angle, as before; and, let AB & AC be made equal to the given Lines, reſpectively.

If the greater Proportional be required, make BF equal to AC, and draw FG, parallel to BC.

CG is the Proportional ſought. But if the leſſer be wanted; make CD equal AB, and draw DE parallel to BC.

BE is the Proportional required.

Dem. For, as AB is to AC, ſo is BF to CG;
and, as AC is to AB, ſo is CD to BE. - - 2. 6. El.
But, BF was made equal to AC, and CD to AB.
Conſ. as AB : AC :: AC : CG; and, AC : CD :: CD : BE.

N. B. A fourth Proportional, to three given Right lines, may be required, and found, in various orders of the given Lines.

Let X, Y, and Z be three given Lines; of which, X is the leaſt, and Z the greateſt of the three.

By the firſt, it is, as X is to Y, ſo is Z to a fourth;
and by the ſecond, as Z is to Y, ſo is X to a fourth.
But it may be, as Y is to either X or Z, ſo is the other to a fourth;
alſo, as X is to Z, or Z to X, ſo is Y to a fourth.

It may alſo be obſerved, that it is not neceſſary to draw the Lines, making an Angle, for the operation, longer than the greateſt given Line; except when a Proportional is required greater; for the meaſures may all be ſet off from the Vertex. e. g.

Let AB and AC make any Angle (BAC) at diſcretion.

Make AB equal to Y, and AC equal Z; alſo make AD equal X; join BC, and draw DE, parallel to BC, cuting AC in E.

Then, as AB, or Y, is to AC, or Z; ſo is AD, or X, to AE, a fourth Proportional.

But, if DC be joined, and BF drawn, parallel to DC; it is then, as AD, or X, is to AC, or Z, ſo is AB, or Y, to AF, a fourth.

Note. This method, of ſeting off all the meaſures from the Angle, is the moſt eligible, when we are clear in the manner of placing them; being leſs liable to error than when they are ſet forward on a longer Line; on account of the Parallels being nearer together. The Demonſtration is in the 4th of 6th El.

A third Proportional may alſo be found after the ſame manner. For, if AD be the firſt, and, if AB and AF be each made equal to the ſecond; DE being joined, BC drawn parallel to DE gives AC, a greater third Proportional; or, if AC be the firſt, AD is a leſſer third.

A third or a fourth Proportional may be very elegantly found after this manner.

Let X, Y and Z be three given Lines.

If you require a leſs Proportional, X, or Y, is the firſt Term taken; if a greater be required, Z or Y muſt be the the firſt, contrary to the order, after the former method.

Draw a Right line, AC, at pleaſure. Make AB and BC equal, reſpectively, to the firſt and ſecond Terms, X and Y; or Y and X. Through the Point B, draw, at pleaſure, DE, and make BD equal to the third Term (Z).

Deſcribe a Circle through the three Points A, C, and D, cuting DE, at E. - - - - - - Pr. 2. 3.

Then, BE is a fourth Proportional; in the order, as Z to Y, ſo is X to BE.

DEM. For, the Rectangles under the Segments of Chord Lines, cuting each other, are equal; i. e. the Rectangle, under AB and BC, is equal to that under BD and BE.

Conſ. as BD: AB, or BC, :: BC, or AB: BE. - 9. 6. El.

Therefore, BE is a fourth Proportional.

If a third Proportional be required, AB and BC muſt be each equal to that Term, of two given Lines, which you require to be the middle Term of the three.

For, if three Quantities are Proportionals, the middle Term is a Mean between the other two.

A greater Proportional will be obtained, by taking Y and Z, inſtead of Y and X, firſt, on AC.

IN analogous or equal Proportion of four Quantities, ſince the firſt has the ſame proportion to a ſecond, as a third has to the fourth; and conſequently, the firſt is to the third as the ſecond to the fourth; a Rectangle under the two extreme Terms (in equal Ratios of Right lines) is equal to a Rectangle under the two middle Terms (9. 6. El.); which, in Numbers, is eaſily proved.

Take any four proportional Numbers, either continual, e. g. as 3, 6, 12, and 24; or in diſcrete Ratio, as 3, 5; 9, and 15.

It is evident, that the firſt Number, 3, has the ſame proportion to the ſecond, 5, as 9, the third, has to the fourth, 15. For, if 3 be multiplied three times, it is equal 9; and 5 multiplied three times, is equal 15; conſequently, 3 has the ſame Proportion to 9, the third Number, as 5 has to 15; which will ever be, when any two Numbers are multiplied equally.

For, 3 is to 3, to twice or ten times 3, as 5 is to 5, twice or ten times. In either Caſe, the two extreme Terms, i. e. the firſt and the laſt, viz. 3 and 15, and the two middle Terms, 5 and 9, remain the ſame; only, the middle Terms have changed places; but, the ſecond multiplied by the third or the third by the ſecond is the ſame thing; and is always equal to the fourth multiplied by the firſt, or the firſt by the fourth, each being equal to 45.

Wherefore, if four Lines are Proportionals, as above; a Rectangle under the firſt and the fourth, the two Extremes, is equal to a Rectangle under the two mean or middle Terms; that is, the Rectangles have equal Areas, ſeeing, the Area of a Rectangle is produced by the multiplication of one Side by the other. Hence, a fourth Proportional may readily be found, as follows:

X, Y, and Z are the given Lines; X being the leaſt.

It is required to find a fourth Proportional, which ſhall have the ſame Proportion to Z, as X has to Y.

In this Caſe, X and Z will be the two Extremes; for ſince X is leſs than Y; conſequently, the Proportional required, will be leſs than Z, and is, properly, a Mean.

Make a Rectangle, ABCD, under the two given Lines X, and Z. - - - - - - - Pr. 17. 1.

Produce any two Sides, from the ſame Angle, as AB and AD; on either of which, make BE or DF, equal Y.

Draw EG or FH, through the Angle C, cuting the other Side, produced, in G or H; then is DG, or BH, the Proportional ſought.

Complete the Rectangle AK; produce BC and DC, to I and L.

DEM. The Rectangle CK is under BE (equal Y) and DG, the fourth Proportional required.

AC and CK are Complements of the Par. AK. Def. 38.

But, the Comps. in every Parallelogram are equal. 17. 1.

Th. the Rectangle CK is equal to the Rectangle ABCD.

But, if the Proportional were required to be to Z, as Y to X; then the Rectangle, or any Parallelogram, ABCD, muſt be under the two Lines Y and Z; which will, in this Caſe, be the two Means; and DG, the Proportional ſought, is now one of the Extremes; being the greateſt of the four; which, in the former Caſe, was one of the middle Terms.

Thus may a fourth Proportional be found, either greater or leſs than either of the given Lines, X and Z.

For, if a leſs Proportional were required, which ſhould be an Extreme of the four; the Rectangle, or other Parallelogram, muſt be made under X and Y.

SCHOL. *When three Lines are given; a fourth Proportional is generally underſtood, to be either greater than the greateſt, or leſs than the leaſt of the three; but if another mean Proportional be required,*

to three Lines given, as it muſt be between the two Extremes of the three; ſo it will be, either greater or leſs than the middle Line, as that is either greater or leſs than a true Mean, between the other two. And if it be already a true Mean, there can no other be found, on either Side; for the Square of a mean Proportional, being equal to the Rectangle under the two Extremes (C. 2. 6.) *conſequently, no other Rectangle but a Square, which ſhall have the true Mean for its Side, can be equal to a Rectangle under the two Extremes.*

PROBLEM VIII.

To find two mean Proportionals, to two given Lines.

Let X and Z be the given Lines.

It is required, to find two other Lines; which not only contain an equal Rectangle, but, are in continual Proportion.

Conſtruct the Rectangle ABCD, on the two given Lines; and on its Center, E, deſcribe a Circle, circumſcribing it. Produce the two Sides AB and AD, indefinite.

Apply a Ruler to the Angle C, and move it on that Point, until FG is equal to CH, and draw FH through C.

BF and DH are the two Proportionals ſought.

For, DC, DH, BF, and BC are in geometrical Progreſſion. And, a Rectangle under the two Means, DH and BF, is equal to the Rect. ABCD, under the two given Lines, X & Z.

Or, briefly thus, without the Circle.

Make a Right angle FAH; in which, from the Angle A, make AB & AD reſpectively equal to the given Lines, X & Z.

Join BD, which biſect in E; and, draw DC perpendicular to AH, or parallel to AF, and BC to AH; apply a ſtreight Ruler to the Point C, making EF equal to EH. Draw FH, and it is done. Q. E. F.

DEMONSTRATION.

FG is equal to CH (Con.) conſ. FC is equal to GH - Ax. 6. 1. wherefore, the Rectangles CFG, GHC, are equal - Ax. B. 2. But, AF × BF = CF × FG; alſo, AH × DH = GH × CH. C. 16. 3 conſequently, the Rectangles AFB, AHD, are equal - Ax. 3. 1. Wherefore, as AF : AH :: DH : BF, reciprocally. - C. 1. 8. 6. But, as AF : AH :: BF : BC (Th. 4) wh. BF : BC :: DH : BF. Ax. 13 Again. It has been ſhewn (changing Terms) DH : BF :: BF : BC alſo, that BF : BC :: AF : AH; i. e. as DC : DH. - P. 4. 6.

Therefore, the four Lines, DC, DH, BF, and BC are in continued Ratio; and conſequently, DH and BF are two Means, between DC and BC; equal X & Z, by Conſtruction.

OTHERWISE.

AB and AD are the two given Lines.

Conſtruct the Rectangle ABCD, as before, on AB and AD. Produce AB, CB, and CD; and draw the two Diagonals, interſecting at O; on which, deſcribe a Semicircle, in ſuch wiſe, that the Curve ſhall paſs through the Points F and G, in CB and CD, produced; where a Right line, joining thoſe Points, paſſes through A.

Draw the Lines EF, EC, EO, and OG.

DEM. The Diags. AC, BD, biſect each other in O; 15. 1. El. wherefore, AO, BO, CO, and DO, are equal among themſelves; alſo, EO = OG, being Radii; and, becauſe AE is parallel to CG, and BD cuts them both, the alternate Angles ABD, BDC, alſo, EBD, BDG, are each equal, between themſelves. - - 4. 1. El.

Wh. the Triangles EBO, OGD are congruous; 11. 1. conſequently, the Angle BOE = DOG; and therefore (being vertical) EO and OG are in a Right line. 2. 1. El.

Now, AD, and DG, are parallel, reſpectively, to BF and BA; and, the Angle ADG is equal to ABF; Con. wherefore, the Triangles AGD, FAB, are ſimilar; and the Angle DAG is equal AFB.

But, BE is equal DG (as above) and AD=BC, wherefore, the Triangles AGD, BEC, are congruous; and conſequently, BEC is ſimilar to ABF.

But, EFG is a Right angle; conſequently, the Angle EFB=FAB; and conſ. the Triangle FEB is ſimilar to FAB, and to BEC. Wherefore, as BC : BE :: BE : BF, and BF : BA; and therefore; BE and BF are two mean Proportionals between AB and BC. Q. E. D.

THIS Problem may be briefly performed thus.

On the two given Lines, AB and BC, conſtruct a right-angled Triangle, ABC (Pr. 17. 1.) and biſect the Hypothenuſe, AC at D. Draw AE perpendicular, and CF parallel to AB; and, on D, deſcribe the Ark EF, ſo, that the Chord EF paſſes through the Angle B.

Then ſhall AE and CF be the two Means required, between AB and BC, as is evident from inſpection of the Figure; DE being equal to DF, as EF to EH; Method 2.

N.B. Theſe methods, though very ingenious, are not perfectly geometrical; ſeeing, the Points required cannot be aſcertained, but by trial; yet, they are the beſt I have met with; for thoſe methods which are performed by an Inſtrument, are not practical otherwiſe, and conſequently cannot be called geometrical.

The method for finding a third Proportional (Pr. 2.) exhibits the reverſe of this; for AB and BC, the two given Lines, in that Problem, are two Means, between the two Extremes BE and BD. For, BD : BC :: BC : BA :: BA : BE, i.e. BD : BC : BA : BE ÷. P. 7. 6.

PROBLEM IX.

To cut two Lines ſo, that the four Segments ſhall be in continued Proportion.

AB and X are the two Lines given.

On either Extreme, as B, of AB, make a Right angle, ABC, and make BC equal to the given Line X.

Deſcribe a Semicircle, on AB; and draw AC, cuting the Arch at D. Draw DE perpendicular, and DF parallel to AB; ſo ſhall AB, and BC (eq. X.) be cut in two parts, in E and F, as was required.

For, the Triangles ADE, DCF are ſimilar; - 2. 6. El. wherefore, AE : ED :: DF : FC. But, DF is equal to EB; and, ED is a mean Proportional between AE and EB.

Therefore, AE : ED : DF : FC in continued Ratio.

PROBLEM X.

To continue a progreſſive Proportion, between two given Lines, infinitely; and to repreſent the ſum of them all. X and Z are the two given Lines.

Draw an indefinite Right line, AF; take AB equal to X, and BC equal Z.

Make BAG a Right angle; and draw BH parallel to AG; alſo, make AG equal AB, and BH equal BC.

Through the Points G and H, draw AF, cuting AC, produced, at F. AF is the whole Sum of the infinite Proportionals, required.

Draw CI perpendicular; make CD eq. CI, and draw DK. Make DE equal DK, and draw EL, perpendicular to AF. Then CI is a third, DK a 4th, and EL a 5th Proportional.

After the ſame manner, it may be continued, *ad infinitum.*

For, ſince AG is leſs than AF; ſo, BH is leſs than BF; and conſequently, CI, DK and EL, will ſtill have the ſame Ratio to CF, DF and EF; wherefore, the laſt may always be taken from the remainder, and therefore, AF, is equal to the whole Sum of the infinite Proportionals; and AB, BC, CD, &c. or, AG, BH, CI, &c. are in a progreſſive, geometrical Proportion.

DEM. Firſt, AF : BF :: AB : BC; i. e. as AG : BH. 4.6.El.
wherefore, AF : AB :: BF : BC. - - - 4. 5.
conſ. AF – AB (eq. BF) : AB :: BF – BC (eq. CF) : BC - 7.5.
wherefore, AB : BF :: BC : CF - - - 5. 5.
conſequently, AB + BF : BF :: BC + CF : CF. - 6. 5.
i. e. AF : BF :: BF : CF; and as CF : DF, &c.
But, AF : BF :: AG : BH; and, BF : CF :: BG : CI, &c. - 4.6.
and AF, BF, CF, &c. are in continued Ratio; as above;
th. AG : BH : CI : DK, i.e. AB : BC : CD : DE, :: AF : BF, &c. ÷

But, AF : AG :: BF : BH, &c. and AF is greater than AG.
wherefore, DF is greater than DK, and EF than EL;
conſequently, EL may ſtill be taken from EF, *ad infinitum.*
th. AF is the whole ſum of infinite Proportionals, to X and Z.

PROBLEM XI.

To cut off from a given Right line any Portion required.

AB is a Line given, from which it is required to cut off a fifth Part, from the extreme A, or two-fifths from B.

Draw AC, making, with AB, any Angle, at diſcretion. With any opening of the Compaſſes, ſet off, from A, five equal Diviſions from A to C. Join the extreme B and the laſt Diviſion, C, by a Right line; and, through the firſt and third Diviſions, on AC, draw 1D, and 3E parallel to BC.

Then is AD a fifth Part, and BE two-fifths, of AB. Q.E.F.

DEM. Becauſe 1D, 3E are parallel to BC, the Triangles A 1D, A 3E, are ſimilar to ABC;
wherefore, AD : A1 :: A3 : AE, and, as AB : AC. 2.6.El.

But A1 is one-fifth, and 3C two-fifths of AC; and conſequently, AD is one-fifth, and BE two-fifths of AB.

PROBLEM XII.

To divide a Right line into any number of equal Parts, or as another Line is divided, in any Ratio.

AB is the given Line, to divide into five equal Parts.

From either Extreme, as A, draw AD, making any Angle, BAD, at pleaſure; and, at the other extreme, B, make the Angle ABC equal BAD, or BC parallel to AD.

On AD and BC take, at pleaſure, four equal Diviſions, from A and B, at a, b, c, D; and, d, e, f, C.

Join, aC, bf, &c. as in the Figure, which will divide AB into five equal Parts, as required; at 1, 2, 3, and 4.

DEM. The Diviſions on AD and BC being equal, they are conſequently equal on AB; the Lines aC, bf, &c. being parallel. - - - - C. 15. 1. El.
for, AD is parallel and equal to BC. - - P. 4. 1.

N. B. If the Diviſions on AD and BC were either greater or leſs, AB would be divided the ſame; which is obvious by taking A*a* 3-fourths of Aa, and joining *af*; for, Bf is 3-fourths of BC.

I have never ſeen this method uſed for dividing a Right line in any given Ratio; which may be applied with ſucceſs.

Let AB be a Line given to be divided; and Z, a Meaſure known, i. e. a Right line divided in the Ratio, required.

Make DAB, ABC equal Angles, as be fore. Transfer the Meaſures Za, ab, and bc, to AD and BC, in the order required; viz. make AE equal Za, and EF equal ab; alſo, make BG equal bc, and GH equal ab, and join the Points EH, FG; which will divide AB, in the Ratio of Z, in the Points *a* and *b*.

Note. The two extreme Diviſions FD and CH are of no uſe in the operation; AC and DB, being joined, will be parallel to EH and FG. There is no regard had to the parallelism of the Lines EH and FG, but to join the Points, and they are neceſſarily parallel.

DEM. Firſt, AF : BF :: AB : BC; i. e. as AG : BH. 4.6.El.
wherefore, AF : AB :: BF : BC. - - - 4. 5.
conſ. AF – AB(eq. BF) : AB :: BF – BC(eq. CF) : BC - 7. 5.
wherefore, AB : BF :: BC : CF - - - 5. 5.
conſequently, AB + BF : BF :: BC + CF : CF. - 6. 5.
i. e. AF : BF :: BF : CF; and as CF : DF, &c.
But, AF : BF :: AG : BH; and, BF : CF :: BG : CI, &c. - 4.6.
and AF, BF, CF, &c. are in continued Ratio; as above;
th. AG : BH : CI : DK, i.e. AB : BC : CD : DE, :: AF : BF, &c. ÷

But, AF : AG :: BF : BH, &c. and AF is greater than AG. wherefore, DF is greater than DK, and EF than EL; conſequently, EL may ſtill be taken from EF, *ad infinitum.* th. AF is the whole ſum of infinite Proportionals, to X and Z.

PROBLEM XI.

To cut off from a given Right line any Portion required.

AB is a Line given, from which it is required to cut off a fifth Part, from the extreme A, or two-fifths from B.

Draw AC, making, with AB, any Angle, at diſcretion. With any opening of the Compaſſes, ſet off, from A, five equal Diviſions from A to C. Join the extreme B and the laſt Diviſion, C, by a Right line; and, through the firſt and third Diviſions, on AC, draw 1D, and 3E parallel to BC.

Then is AD a fifth Part, and BE two-fifths, of AB. Q.E.F.

DEM. Becauſe 1D, 3E are parallel to BC, the Triangles A 1D, A 3E, are ſimilar to ABC;
wherefore, AD : A1 :: A3 : AE, and, as AB : AC. 2.6.El.

But A1 is one-fifth, and 3C two-fifths of AC; and conſequently, AD is one-fifth, and BE two-fifths of AB.

PROBLEM XII.

To divide a Right line into any number of equal Parts, or as another Line is divided, in any Ratio.

AB is the given Line, to divide into five equal Parts.

From either Extreme, as A, draw AD, making any Angle, BAD, at pleaſure; and, at the other extreme, B, make the Angle ABC equal BAD, or BC parallel to AD.

On AD and BC take, at pleaſure, four equal Diviſions, from A and B, at a, b, c, D; and, d, e, f, C.

Join, aC, bf, &c. as in the Figure, which will divide AB into five equal Parts, as required; at 1, 2, 3, and 4.

DEM. The Diviſions on AD and BC being equal, they are conſequently equal on AB; the Lines aC, bf, &c. being parallel. - - - - C. 15. 1. El.
for, AD is parallel and equal to BC. - - P. 4. 1.

N. B. If the Diviſions on AD and BC were either greater or leſs, AB would be divided the ſame; which is obvious by taking A*a* 3-fourths of Aa, and joining *a*f; for, Bf is 3-fourths of BC.

I have never ſeen this method uſed for dividing a Right line in any given Ratio; which may be applied with ſucceſs.

Let AB be a Line given to be divided; and Z, a Meaſure known, i. e. a Right line divided in the Ratio, required.

Make DAB, ABC equal Angles, as be fore. Transfer the Meaſures Za, ab, and bc, to AD and BC, in the order required; viz. make AE equal Za, and EF equal ab; alſo, make BG equal bc, and GH equal ab, and join the Points EH, FG; which will divide AB, in the Ratio of Z, in the Points *a* and *b*.

Note. The two extreme Diviſions FD and CH are of no uſe in the operation; AC and DB, being joined, will be parallel to EH and FG. There is no regard had to the parallelism of the Lines EH and FG, but to join the Points, and they are neceſſarily parallel.

A Line may be readily divided, equally, or in any Ratio, thus.

AB is the given Line; and Z, is a Line divided in the Proportion required, at 1, and 2.

At any diſtance from AB, at diſcretion, draw CD parallel to AB; and make CE, EF, and FD equal, reſpectively, to the Diviſions on Z. Through the Extremes of the two Lines, AB and CD, draw CA and DB, and produce hem till they interſect, at G. Draw EG and FG, cuting AB in e and f. Then, is the given Line, AB, divided (in e and f) in the ſame Ratio as CD (in E and F) as the given Line, Z, is divided, in the Points 1, 2. - 2.6.El.

If the Meaſure given had been leſs than the Line given to be divided (as in this Example it is greater) AB would be divided the ſame.

Let c d, leſs than AB, be divided in the known Ratio *e* & *f*. Draw, as before, A c, and B d, through their Extremes, meeting at G; and, from G, through the Diviſions *e* and *f*, draw the Lines G*e*, G*f*, till they cut the given Line, AB, in the ſame Points e and f. Q. E. F.

The difference, 'tis evident, is but little in the operation, the effect is the ſame. In the firſt Caſe, when the Meaſure is greater than the Line to be divided; then, the Vertex G will fall on the oppoſite Side, from CD; but, when it is leſs, as c d, the Point G will fall on the ſame Side with c d; as is obvious in the Figure.

N. B. Any other Line being drawn, as g h, parallel to AB, or CD, will be divided in the ſame Ratio, by the Lines GC, GE, &c.

2. The ſame thing may be very readily and accurately done after this manner.

AB is the given Line, to be divided; and HI is a Line divided in the given or known Ratio, in 1 and 2.

At either extreme of the given Line, draw AE, in any Angle, at pleaſure. An acute Angle, not too ſmall, is beſt.

Make the diviſions at C, D, and E, equal, reſpectively, to the diviſions on HI (According as you require them on AB, begin at either end, H or I.)

Join the Point E, and the other extreme, B, of AB; draw CF and DG, parallel to EB, cuting AB in F and G.

So ſhall AB be divided in the ſame Ratio, as HI. - 2. 6.

If the Meaſure given had been leſs than AB, as Ae, divided in c and d, the Diviſions on AB would be the ſame.

N. B. If it were required to divide AB into any number of equal Parts; make ſo many equal Diviſions on CD, in the firſt, or on AE, in the laſt method, at pleaſure; and proceed as directed.

AB may be readily biſected by either method; by the laſt, making Af, fE, or fe, two equal diviſions, and drawing fg, as before, parallel to EB or eB.

The firſt method is demonſtrable, from Prop. 6. 6. and Corol. For the Triangles CGE and AGe, alſo EGF and eGf, &c. are ſimilar. Conſ. Ae:ef :: CE:EF, and as fB to FD.

The laſt method is eaſily deduced from P. 7. S. 1. and the Demonſtration of it, from Prop. 2. 6. El.

APPL. It is almoſt needleſs to give an Application of theſe Problems, which ſpeak their uſe ſufficiently. They are extremely uſeful in the practice of Perſpective, as well as in geometrical Drawings of all kinds; as Plans, Elevations, or Sections.

AB is conſidered as a finite Line, already drawn, in ſome Plan, &c. and the Ratio or Proportion given, is certain known meaſures, or diviſions, to be repreſented on AB, from either a greater or leſs Scale of Proportion.

3. There is another method, by which a Line may be divided from a greater Meaſure given; which, not ſo much for its utility as the ſingular elegance of it, I give, as thus.

Let AD be divided in the given Ratio, at B and C; and let X, or Z, be a Line given, to be divided, in equal Ratio.

Deſcribe three Circles, on the three Diameters AB, AC, and AD.

Take the Line X, or Z, in your Compaſſes, and, ſeting one Point in A, cut the greateſt Circumference, at E or F, with the other Point, and join AE or AF; which, will be divided in the ſame Ratio as AD, in b and c.

Draw Bb, Cc, and DE, or DF.

DEM. The Angles AbB, AcC, AED, or AFD, are Right; wh. △s AbB, AcC, &c. are ſimilar, the Lines Bb, Cc, &c. being parallel, and the Angle at A common. C. 3. 2. 6. Therefore, AE, or AF, is divided in the ſame Ratio as AD.

PROBLEM XIII.

A Right line (AB) being divided at pleaſure (in C); to deſcribe a Circle, to any Point in the Circumference of which, if Right lines are drawn, from each Extreme of the given Line, they ſhall have the ſame Ratio to each other, as the Segments of the given Line.

From the greater Segment, CB, take the leſs, AC; i. e. make CD equal AC. Make AE a third Proportional, to BD and CD (equal AC). With the Radius EC, on the Center E, deſcribe a Circle; and to any Point, F, in the Circumference, draw AF and BF.

Then, AF is to BF as AC is to CB. Draw EF.

DEM. Becauſe, EA : AC :: AC (eq. CD) : DB; - Con. conſ. EA : EA + AC (i.e. EC) :: AC : AC + DB (i. e. CB) that is, as EA : EC :: AC : CB. - Conv. to 6. 5.

But, EF = EC; wherefore, EA : EF :: AC : CB, i. e. as EC : EB, or EF : EB; and, as the Angle FEB is common, the Triangle AEF is ſimilar to EFB; therefore, AF : BF :: EA : EF. (eq. EC) i. e. as AC to CB.

Q. E. F. & D.

PRACTICAL GEOMETRY.

SECTION VI.

On Transformation *of* Figures.

PROBLEM I.

To make a Rectangle equal to any Parallelogram.

RECTANGLES are Parallelograms, which may have various Dimenſions, yet contain the ſame Area; which reſults from the Product of its Sides, multiplied one into the other; ſo that, being a Rectangle, it is immaterial what are the meaſures of its Sides, if it contain a certain Area. But if it were required to have it of a certain length or breadth, 'tis but to divide the Area by the length or width, the Quotient will be the other Side. Or, two mean Proportionals being found, by the 8th of Sect. 5, it will approach nearer to a Square: The Side of a Square, equal to any Rectangle, is a true Mean.

ABCD is an oblique angled Parallelogram; a Rhomboides, the Sides being of different lengths.

It is required to make a Rectangle equal to the Parallelogram; by which, its Area may be determined.

Draw a Perpendicular from B, cuting the oppoſite Side; which Side being produced, until it be cut by another Perpendicular, from A, at F; ABEF is the Rectangle required, equal to the Parallelogram ABCD; by Prop. 18. 1. El.

For, it is evident, and obvious, that the Triangles ADF, BCE are congruous; wherefore, if BCE be taken away, and its equal be added, the Area of the Rectangle is the ſame as of the Rhomboides.

Or; AD being taken, inſtead of AB, draw DB and AG, perpendicular to AD, cuting CB produced, at B and G. The Rectangle AGBD is alſo equal to the Parallelogram ABCD; conſequently, the two Rectangles are equal.

In the former, the Triangles ADF, BCE are congruous: in this, DBC is congruous with AGB.

PROBLEM II. 42. I. Euclid.

To make a Rectangle equal to a given Triangle.

ABC is the given Triangle.

Biſect any Side, as AC, in D (Pr. 8. 1.). Draw BF, parallel to AC (1. 5.); alſo, draw DG, from the point of ſection, perpendicular to the Baſe, AC. - - - 6. 1. Laſtly, draw CF parallel to DG.

The Rect. DGFC is equal to the Triangle ABC. Q.E.F.

Or, the Rectangle AHIC, conſtructed on the whole Baſe and half the perpendicular Altitude, BK, is equal to ABC.

Dem. For, either is half the Rectangle AEFC, which is double the Area of the Triangle ABC. - - 19. 1.

N. B. If an oblique angled Parallelogram were required, make the Angle CDG or CAB equal to the Angle given. The Parallelogram ABGD, between the ſame Parallels, AC and EF, and having the ſame Baſe DC or AD, half AC, is equal to the Rectangle AEGD, therefore to the Triangle ABC.

SCH. *From hence, and the 19th of the 1st Book of Elements, the whole Theory of Menſuration of Superficies is deduced; as all Figures whatever, except Parallelograms, are reſolved into Triangles, in Menſuration.*

From the 19th of the firſt Book we learn, that every Triangle is equal to half a Parallelogram of the ſame Baſe and Height; conſequently, the Rectangle DE, *or* DGFC *is equal to the Triangle* ABC, *which is on half its Baſe,* AC, *and the ſame height,* GD, *equal* BK; *or* AHIC, *on the whole Baſe,* AC, *and half its height.*

COR. Hence, the Rule for meaſuring a Triangle is to multiply the Perpendicular, BK (equal GD) by half the Baſe, AC, or the whole Baſe, AC, by half the Perpendicular BK (equal AH); the firſt gives the Rectangle DGFC, the other is the Rectangle AHIC.

For, the whole Baſe, AC, multiplied by the Perpendicular BK, gives the Area of the Rectangle AEFC, which is double of the Triangle (19. 1.); conſequently, half that Sum is the Area of the Triangle ABC, and alſo of the Rectangle DF, or AI.

PROBLEM III.

To make a Rectangle equal to a given Trapezium; ABCD.

Draw either Diagonal, as AC; to which, draw the Perpendiculars BE and DF, (Pr. 7.1.) Biſect the Perpendiculars, in G and H (Pr.8.). Through the Points G and H, draw IK and LM parallel to AC (5. 1.) and through A and C, draw IM and KL, perpendicular to AC. 10.

The Rectangle IKLM is equal to the Trapezium ABCD.

DEM. For, the Rect. AIKC is equal to the Triangle ABC, and the Rect. ACLM is equal to the Triangle ACD. 2. Conſ. the Rect. IKLM is equal to the Trap. ABCD. Ax. 3.

OR. Having drawn a Diagonal, as before, biſect AC, in the Point G (Pr. 8.). Draw BI, and MK (through D) parallel to AC; and, through G and C, draw HL and IK, perpendicular, cuting BI and MK.

Then, the Rect. HIKL is equal to the Trap. ABCD.

For, the Rect. GHIC is equal to the Triangle ABC. And the Rect. GLKC is equal to the Triangle ACD. 2.

Cor. Hence, to find the Area of a Trapezium is to multiply the Diagonal by half the Sum of the two Perpendiculars (Fig. 1.); or, the Sum of the two Perpendiculars (BE and FD) by half the Diagonal (GC, Fig. 2.) for, to multiply the whole Diagonal by the Sum of the two Perpendiculars, will give the Area of the Rectangle MNIK; half of which, is the Area of the Trapezium ABCD.

It may appear ſtrange, to thoſe who have not conſidered it, that the Area of any Figure ſhould be obtained without meaſuring its Sides; which are of no uſe in this operation; for, the whole buſineſs of Menſuration is to find a Rectangle equal to any Figure; as, the multiplication of any two Numbers, applied to meaſure, denotes a Rectangle of ſuch Dimenſions.

Now, ſince it ſeldom happens, that a Trapezium has either Right angles or parallel Sides, it is plain, that they cannot be of any uſe towards obtaining its Area; whereas, the Diagonals and Perpendiculars are at right angles with each other.

A Diagonal divides any Quadrilateral into two Triangles; and every Triangle is equal to half a Parallelogram having the ſame or an equal Baſe, and the ſame Altitude. (19. 1.)

Hence, it is eaſy to account for the Rules given for meaſuring a Trapezium, as two Triangles having a common Baſe; which is a Diagonal of the Trapezium. For, every Trapezium is equal to half a Parallelogram which circumſcribes it, having two Sides parallel to either Diagonal; and all Parallelograms having the ſame Baſe and Altitude are equal (18. 1.) conſequently, they are equal to a Rectangle of thoſe Dimenſions.

PROBLEM IV.

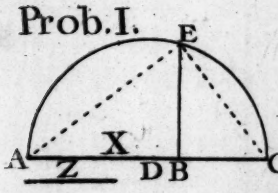
To find the Side of a Square, equal to a Rectangle.

ABCD is the given Rectangle.

Produce any Side, as AB, making BE equal to the adjoining Side, BC. Biſect AE in F; with the Radius AF

Section V.

Prob. I.



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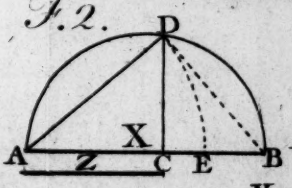

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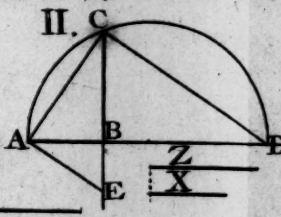

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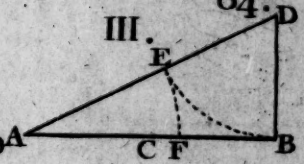

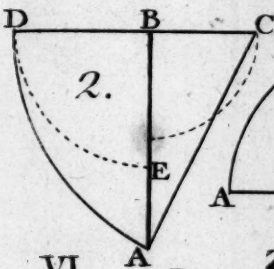

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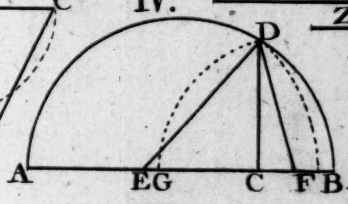

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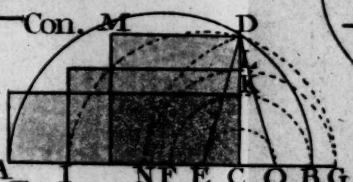

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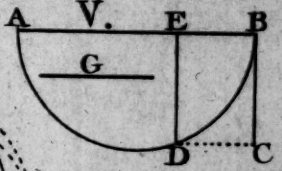

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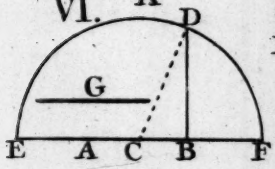

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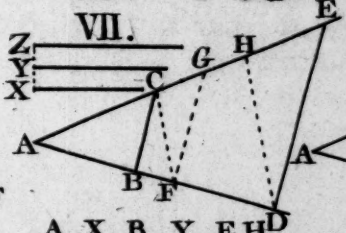

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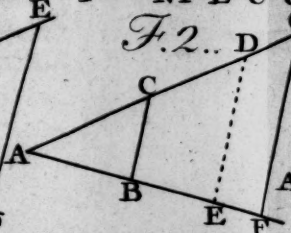

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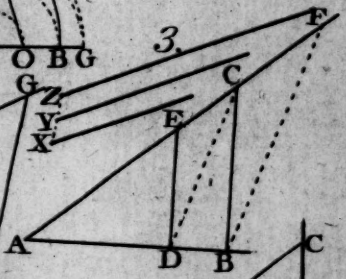

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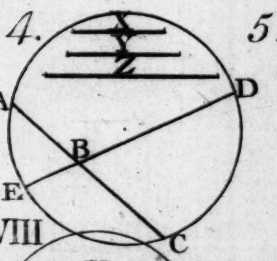

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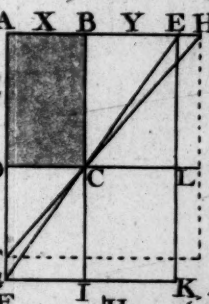

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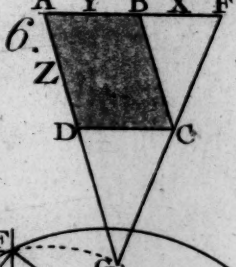

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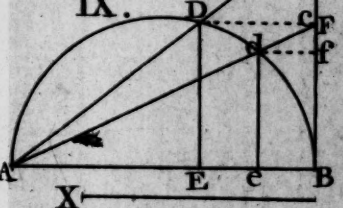

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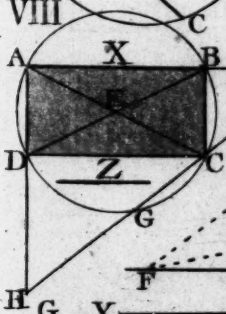

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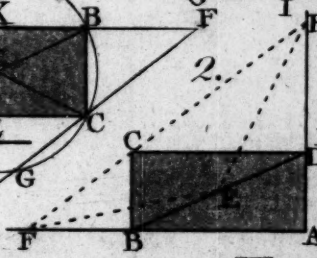

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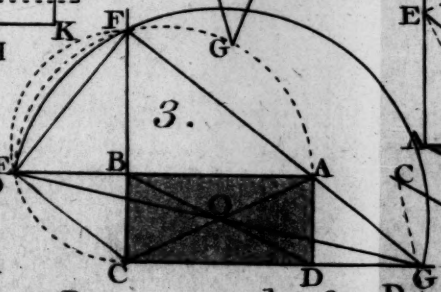

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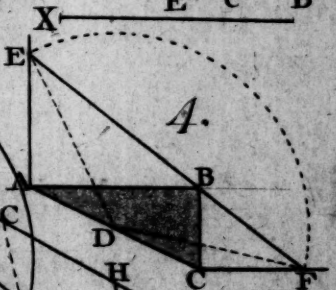

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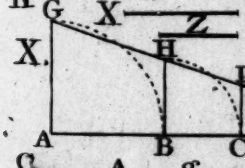

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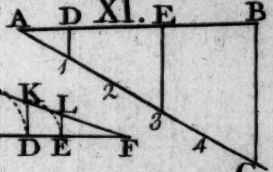

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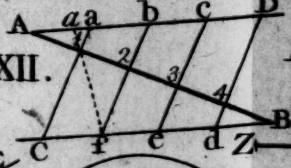

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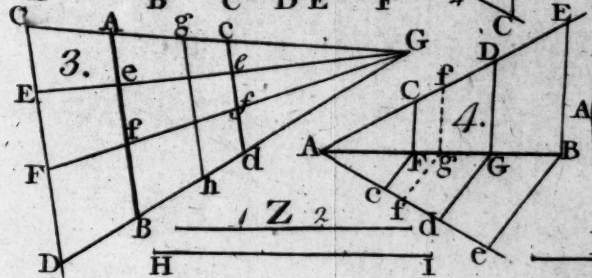
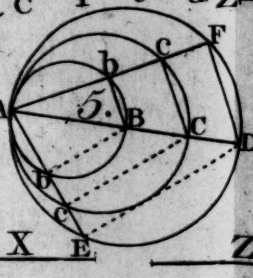
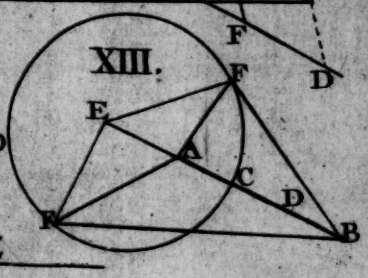




XIII.



deſcribe a Semicircle. Produce CB, till it cuts the Arch, at G; or, at the point B, draw BG perpendicular to AE.

Deſcribe the Square BGHE on the Line BG; which will be equal to the Rectangle ABCD. - - - 14. 3. El.

For, BG is a mean Proportional between AB and BE. Conſequently, the Rectangle under AB and BE (i. e. DB) is equal to the ſquare of BG (i. e. BH) by Cor. to 9. 6.

N. B. In the performance of this Problem, it is only neceſſary to draw a Right line (AE) in which, take AB and BE equal to the meaſures of the two Sides, and proceed as above.

PROBLEM V.

To lengthen, or ſhorten, either Side of a Parallelogram, at pleaſure, and to contain the ſame Area.

As the multiplication of Lines, geometrically, is to form a Rectangle of any two given Lines, P. 17. S. 1. (for it is, in reality, multiplying one into the other); ſo this Problem comprehends geometrical Diviſion. For, a Rectangle being given (as in this Problem) is a Quantity given, to be divided; and any Line of a different length, given, is the Diviſor; by which the Rectangle being divided, the Product reſulting from the operation, is the Quotient ſought, which conſtitutes four proportional Lines; for, the multiplication of the Quotient into the Diviſor, will produce a Rectangle equal to the given one; by P. 9. 6. El.

ABCD is the given Parallelogram, to be divided; and BE, or DF a given Line, or the Diviſor.

Produce any two contiguous Sides, as AB and AD, from the ſame Angle (A) indefinite; and having made BE equal to the Diviſor given, through C, draw EF, cuting AB, produced, at F, which gives DF, the Line ſought. Q. E. F.

DF is the greater, and BE a leſſer fourth Proportional with AB and AD; conſequently, they are the two Extremes; AB and AD the two Means. - See Pr. 8. S. 5.

Wherefore, BE × DF = AB × AD. (9. 6. El.). For,

DEM. EH being drawn parallel to AF, and FH to AE; BC produced to I, and DC to G, there is formed the large Parallelogram AFHE, of which ABCD and CGHI are Complements.

But, the Comps. of Parallelograms are equal. 17. 1. El. Therefore, the Parallelogram, conſtructed on BE and DF, is equal to the given one, ABCD; for, Rectangles conſtructed on thoſe Lines are equal. - 9. 6. El.

This may, to ſome, be made more ſatisfactory, thus.

Let the Rectangle ABCD be a Quantity given, to be divided, whoſe Area is 24; the Side AB being 6, and AD 4 equal Parts; it is required to be divided by 3.

Produce any Side, as AB, and make BE equal 3 (half AB); alſo produce AD indefinite. Through E and the Angle C, draw EC, and produce it, cuting AD produced, in F; DF will be equal 8 of the diviſions, twice AD, which is the Quotient ſought.

Compleat the Rectangle AEHF; produce BC to I, and DC to G; the Rectangle CH is equal AC.

For, CI is equal DF, i. e. equal 8; and, CG is equal BE equal 3; 8 multiplied by 3, is equal 24, the Area of CH; which is equal to the Dividend, or the Rectahgle ABCD.

If the Diviſor had been 8, equal DF, the Quotient would have been 3, equal BE.

If a fractional Number were given for the Diviſor, the Quotient would, moſt probably, be fractional likewiſe; for the Rectangle under the Diviſor and the Quotient will always be equal to the given Rectangle, which is the Dividend.

As in Diviſion of Numbers, the Quotient multiplied by the Diviſor is (when there is no remainder) equal to the Dividend; which proves the work to be true.

N. B. It is immaterial how the given Rectangle (or other Parallelogram) is ſituated, in this operation, or on which Side is taken the Diviſor; the Quotient will be the ſame.

PROBLEM VI.

On any Side of a ſcalene Triangle, to make an Iſoſceles equal to the given Scalene; ABC.

Biſect AC (or any other Side) at D, and draw DE perpendicular to BC; and, from the remote Angle, B, draw BE parallel to AC, cuting the Perpendicular, at E; draw AE and CE, and it is done. Q. E. F.

For, becauſe BE is parallel to AC, the Triangles, ABC, AEC, are equal (18. 1. El.); and AE is equal EC. C. 4. 9. 1.

PROBLEM VII.

To make a Triangle equal to another, by a Line drawn from any Point in a Side, to the Baſe.

ABC is the Triangle given, and D the given Point, in AB.

Produce AC, indefinite; draw DC, and BE parallel to DC, cuting AC, produced, at E, and join DE.

The Triangle ADE is equal to ABC.

For, becauſe DC is parallel to BE, the Triangles DBC, DEC, are equal; conſ. ADC + DBC = ADC + DEC. - Ax. 6. 1.

If the Point be taken in any Side produced, the Proceſs will be the very ſame, inverted.

Let B be taken in the Side AD, produced, of the Triangle ADE.

Draw BE, and DC parallel to BE, cuting AE at C, and draw BC. Then is ABC the Triangle required, equal to ADE; as it has been demonſtrated.

PROBLEM VIII.

To make a Triangle equal to a given one, by Lines drawn from any Point within or without the Triangle to the Baſe.

ABC is the Triangle given, and D a Point aſſumed within it, from which, two Lines, being drawn to the Baſe, AC, ſhall form a Triangle equal to ABC.

Draw DA and DC, and BE, BF, parallel to them, reſpectively, cuting AC, produced at both Extremes, at E and F; draw DE and DF.

DEF is the Triangle required. Join BD.

2nd. Let EDF be the given Triangle, and B a Point aſſumed, out of it. Draw EB and BF, and AD, DC, parallel to them, reſpectively; then, draw AB and BC.

ABC is the Triangle required.

DEM. The Tri. ADE=ABD, and, CDF=DBC. 18. 1. El. wh. ADC+ADE=ADC+ABD; alſo, EDC+CDF=ABDC+DBC; and conſequently, EDF is equal ABC.

PROBLEM IX.

To make a Triangle equal to a given one, on the the ſame Baſe, and having a given Angle.

ABC is the given Triangle; the Angle is at diſcretion.

Make the Angle ABD (at either Extreme) as required. Draw CD parallel to AB, cuting BD, and join AD.

ADB is the Triangle required; as is manifeſt. P. 18. 1. El.

If BD were given in length, more or leſs, as in Fig. 2 and 3. Draw CD, and BE parallel to CD, cuting AC, at E.

Draw AD, and EF parallel to AD, cuting AB and F; and join DF. BDF is the Triangle required.

DEM. Becauſe CD is parallel to BE, the Triangle EDB=ECB (18. 1.); and, as EGB is common, EDG=GCB; wh. FEGB+EDG=FEGB×GCB.

Again; becauſe FE is parallel to AD, the Triangle FAE=FDE; wh. FEDB+FAE=FEDB+FDE; therefore, BDF is equal to ABC. - Ax. 6. 1.

PROBLEM X.

To draw a Line in a given direction, cuting a Triangle, ſo, as, from any Angle, to make an equal Triangle.

ABC is the given Triangle.

It is required to draw a Line parallel to GH, cuting the Sides AB and AC, ſo, as to form another Triangle, equal to ABC.

Draw CD parallel to GH, cuting AB (produced if neceſſary; Fig. 2.) at D. Make AE a mean Proportional, between AB and AD (Pr. 1. 5.) and draw EF parallel to CD.

The Triangle AEF is equal to ABC. Q. E. F.

DEM. The Triangles ACD, ACB, having a common Vertex, and their Baſes in the ſame Right line, are to each other as AD to AB (1. 6. El.); and AEF (on the Mean, AE) is ſimilar to ACD (2. 6.). Therefore (by the following Lemma) the Triangle AEF is equal to ABC.

LEMMA.

If the Baſes of three Triangles are Proportionals; and, if the firſt be ſimilar to the ſecond, and have the ſame height as the third, the ſecond is equal to the third. But, if the ſecond be like the third, it is equal to the firſt.

In the Right line AD, take AB, BC, and CD, in continued Proportion; AB to BC, as BC to CD. Let ſimilar Triangles (X and Y) be conſtructed on AB and BC; and on CD, any other (as Z) having equal height with X; then will the Triangle Z be equal to Y. But, if Y and Z be ſimilar, then is X equal Y.

DEM. The Triangles X and Y are ſimilar; wherefore, the Ratio of X to Y is duplicate of AB to BC; 12. 6. i. e. as AB to CD. - - - - Def. 12. 5. El. But, X:Z :: AB:CD(1.6.); and, X:Y :: AB:CD, as above; and conſequently, Y is equal to Z; by Ax. 5. 5.

By the ſame reaſoning, it is evident, that if Y and Z are ſimilar (Fig. 2.) and X have equal height with Z; X is equal to Y.

PROBLEM XI.

From a given Point, without a Triangle, to draw a Right line, cuting two Sides, ſo as to make an equal Triangle.

ABC is the Triangle, and D the given Point.

Produce AC, and draw DEF perpendicular to AC. Make EF equal to DE, and draw FG parallel to AC, cuting AB at G; and, on AG, make the Triangle AGH equal ABC (7.). Draw DI, parallel AB; and divide AI ſo, at K, that AK ſhall be a mean Proportional, between the remainder KI and AH. Draw DL, through K, cuting AB produced; the Triangle AKL will be equal to ABC.

DEM. Becauſe AH, AK, and KI are Proportionals, of which AK is the Mean, and the Triangles DIK, KAL are ſimilar, KAL=AGH, by the preceeding Lemma. For AGH is of equal height with DIK, by Conſtruction. But, AGH=ABC (Con.); conſequently, AKL=ABC.

PROBLEM XII.

To make a Triangle equal to any quadrilateral Figure, on any Side of it, and retaining either Angle made by that Side; or having a given Angle.

ABCD is the Quadrilateral, given.

It is required to make a Triangle equal to it, on the Side AD, retaining the Angle A; or having a given Angle.

Produce AB; draw the Diagonal DB, and CE parallel thereto, cuting AB produced, at E, and draw DE.

AED is the Triangle required, equal to ABCD.

DEM. Becauſe CE is parallel to BD, the Triangle BED is equal to BCD. - - - 18. 1. El.

Wh. ABD + BED = ABD + BCD; conſ. AED = ABCD.

If any other Angle were required, inſtead of the Angle A, make the Angle BAF as required (6. 1.); draw DF parallel to AB, and join FE. AEF is equal to AED, 18. 1. which was made equal to the Quadrilateral ABCD.

PROBLEM XIII.

To reduce any Poligon, whatever, to a Triangle, which ſhall be equal to the Poligon.

ABCDE is the Poligon given, a Pentagon.

Draw the Diagonals AC and CE. Produce EA, indefinite, and draw BF parallel to AC (the adjacent Diagonal) cuting EA, produced, in F; and draw FC.

Again. Produce the Side DE, indefinite; and, parallel to CE, draw FG, cuting DE produced, in G; draw GC, and it is done. Q. E. F.

The Triangle GCD is equal to the Pentagon.

Dem. For, the Triangle FCE is equal to ABCE, - 12.
and CDE is common to both; wherefore, FCDE is equal to ABCDE. - - - - Ax. 7. 1.
But, the Triangle GCD is equal to FCDE; - 12.
for, the Triangle CGE is equal to CFE. - P. 18. 1.
conſ. CGE+ECD=CFE+ECD. - - Ax. 6. 1.
Th. the Triangle GCD is equal to the given Pentagon.

Or, FE being produced, and DH drawn parallel to CE, cuting it in H, draw HC; the Triangle FCH is equal to the given Pentagon, equal to the Triangle GCD.
For, the Triangle CHE is equal to CDE. - 18. 1. El.

Thus may a Triangle be readily conſtructed, whoſe Altitude ſhall be equal to a Perpendicular, from any Angle to any Side; as CD or CK. Or, any Angle, as D, and an adjoining Side, CD, of the Poligon, may be retained in the Triangle.

Another Example, in a more complex Figure.

Let ABCDEFGH be the given Figure.

Firſt, draw the Diagonals BH, BG, GC, GD, and DF. Produce the Side GH, and, draw AI, parallel to the Diagonal BH; cuting GH produced, at I, and draw BI.

Now, ſince AI is parallel to BH, the Triangle BIH,=BAH. 18. 1. wherefore, the Triangle BIG, is equal to the Trapezium BAHG.

Next, produce the Side BC, towards K, indefinite. Draw IK, parallel to the Diagonal BG, cuting CB produced, at K; draw GK.

The Tri. BKG=BIG, having the ſame Baſe BG. 18. 1. But, the Triangle BIG is equal to the Trapezium BAHG; wh. the Tri. BKG, is equal to the Trap. BAHG. - Ax. 3.

Conſequently, the Triangle BKO, which is added, is equal to the Trapezium AHGO, taken away; for, BOG is common to both.

Next, produce GF; draw EL, parallel to FD, and draw DL. The Triangle FLD is equal to FED. - 18. 1. El.

Then, produce CD; and, draw LM, parallel to GD; cuting CD produced, in M; draw GM.

Then is GMD equal GLD; conſ. GMC=GLDC.

Laſtly; parallel to GC, draw MN, cuting KC produced, in N, and draw GN. Then, GNC is equal to GMC, equal GLDC, equal GFEDC.

But, the Triangle BKG is equal to the Trap. BAHG; and, BKG+BGC+GNC is equal to KGN. - Ax. 2.

The Triangle KGN, is equal to the given Figure.

From hence, and Prob. 2, it is evident, that a Rectangle may be conſtructed equal to any given right-lined Figure; by which its Area is readily determined; for, having reduced the given Figure to a Triangle (KGN) it is equal to a Rectangle, on KP, half the Baſe, KN, and GR, the perpendicular Altitude of the Triangle.

By Prob. 4. it is eaſily reduced to a Square; or to a Triangle under any given Angle and Side, by the 9th.

SCHOL. *If any one will compare this, with the 14th Propoſition in the 2nd Book of Euclid, and go through the operation, both ways; I am confident he will give this the preference. To ſay nothing of the inaccuracy of the other, this may be done in a fourth part of the Time. Each Triangle (there are ſix, in this Figure) goes through two operations, viz. the 2nd and 5th of this; and are added, ſeparately into one Sum, or Rectangle. But, by this, the Figure is reduced one Side at every operation; from a Hexagon to a Pentagon, from a Pentagon to a Trapezium, from a Trapezium to a Triangle; each being equal to the original Figure: and, from a Triangle to a Rectangle, by the 2nd.*

PROBLEM XIV.

To make an equilateral Triangle, equal to any other.

On any Side, as AB, of the given Triangle, ABC, make an equilateral Triangle, ABD (14. 1.); and produce the Side DB, indefinite; draw CE parallel to AB, cuting it, at E, and draw AE. On DE deſcribe a Semicircle, and draw BF perpendicular. With the Radius BF deſcribe the Arch FGH, cuting BA produced, at G; on which, with the ſame Radius, deſcribe the Ark BH.

Draw GH and BH. GHB is the Triangle required.

DEM. The Triangle AEB=ABC(18. 1.); and DB, BF, BE are three proportional Lines; wherefore, the Triangle GHB, which is conſtructed on the Mean (BF) being ſimilar to ABD, on one Extreme, is equal to AEB, on the other, by the Lemma to Pr. 10; for they have the ſame height, AI. Therefore, the equilateral Triangle GHB=ABC, which is equal to AEB.

PROBLEM XV.

To make a Triangle ſimilar to any given one, and equal to any other.

ABC and X, are the given Triangles.

It is required to make a Triangle ſimilar to X and equal to ABC.

On any Side, as AC, make the Triangle ADC ſimilar to X (by Pr. 1. S. 8.). Draw BE, parallel to AC, and draw CE. AEC is equal ABC. - - - 18. 1. El.

Make AF a mean Proportional to AE and AD; - 1. 5. and draw FG parallel to CD.

AFG is the Triangle required.

DEM. The Triangle AEC=ABC (18. 1.); and ADC is similar to X. Wherefore, since AE:AF::AF:AD, the Triangle AFG, constructed on the Mean, being similar to ADC, is equal to AFC, by Lemma; for, ACE and ACD, on AE and AD, have equal height, the Vertex, C, being common to both.

Therefore, AFG, similar to X, is equal to ABC, which is equal to ADC.

PROBLEM XVI.

To make any regular Poligon, equal to any right-lined Figure, given.

The given Figure being red... Triangle, by 13. let ABC be the Triangle, '... to construct a regular Pentagon, equal to th... ABC.

On AC (any Side) constr... ...sceles Triangle, ACD, whose Angle at the Ver... ... of 72 Degrees; those at the Base (AC) being 54,

Produce DA, and draw BE paral... AC, cuting it at E.

Draw EC, and AI parallel to D... EAI, equal ADC, is the Angle of a regular Pentagon, ...btended by a Side, at the Center of a circumscribing Circle.

Divide AE into as many equal parts, as the Poligon required has Sides; i. e. take AF (for a Pentagon) a fifth part of AE, and draw FC.

Then, with the Radius AG, a Mean proportional between AD and AF, describe a Circle cuting AE and AI, and draw HI, which is the Side of a Pentagon in that Circle, equal to the Triangle ABC, and consequently to the original given Figure.

DEM. AF is a fifth part of AE; wherefore, the Triangle ACF is equal a fifth part of ACE; i. e. of ABC. 1. 6. El.

PROBLEM XIV.

To make an equilateral Triangle, equal to any other.

On any Side, as AB, of the given Triangle, ABC, make an equilateral Triangle, ABD (14. 1.); and produce the Side DB, indefinite; draw CE parallel to AB, cuting it, at E, and draw AE. On DE deſcribe a Semicircle, and draw BF perpendicular. With the Radius BF deſcribe the Arch FGH, cuting BA produced, at G; on which, with the ſame Radius, deſcribe the Ark BH.

Draw GH and BH. GHB is the Triangle required.

DEM. The Triangle AEB=ABC(18.1.); and DB, BF, BE are three proportional Lines; wherefore, the Triangle GHB, which is conſtructed on the Mean (BF) being ſimilar to ABD, on one Extreme, is equal to AEB, on the other, by the Lemma to Pr. 10; for they have the ſame height, AI. Therefore, the equilateral Triangle GHB=ABC, which is equal to AEB.

PROBLEM XV.

To make a Triangle ſimilar to any given one, and equal to any other.

ABC and X, are the given Triangles.

It is required to make a Triangle ſimilar to X and equal to ABC.

On any Side, as AC, make the Triangle ADC ſimilar to X (by Pr. 1. S. 8.). Draw BE, parallel to AC, and draw CE. AEC is equal ABC. - - - 18. 1. El.

Make AF a mean Proportional to AE and AD; - 1.5. and draw FG parallel to CD.

AFG is the Triangle required.

DEM. The Triangle AEC=ABC (18. 1.); and ADC is ſimilar to X. Wherefore, ſince AE:AF::AF:AD, the Triangle AFG, conſtructed on the Mean, being ſimilar to ADC, is equal to AFC, by Lemma; for, ACE and ACD, on AE and AD, have equal height, the Vertex, C, being common to both.

Therefore, AFG, ſimilar to X, is equal to ABC, which is equal to ADC.

PROBLEM XVI.

To make any regular Poligon, equal to any right-lined Figure, given.

The given Figure being reduced to a Triangle, by 13. let ABC be the Triangle, 'tis required to conſtruct a regular Pentagon, equal to the Triangle, ABC.

On AC (any Side) conſtruct an Iſoſceles Triangle, ACD, whoſe Angle at the Vertex, D, is of 72 Degrees; thoſe at the Baſe (AC) being 54, each.

Produce DA, and draw BE parallel to AC, cuting it at E.

Draw EC, and AI parallel to DC; EAI, equal ADC, is the Angle of a regular Pentagon, ſubtended by a Side, at the Center of a circumſcribing Circle.

Divide AE into as many equal parts, as the Poligon required has Sides; i. e. take AF (for a Pentagon) a fifth part of AE, and draw FC.

Then, with the Radius AG, a Mean proportional between AD and AF, deſcribe a Circle cuting AE and AI, and draw HI, which is the Side of a Pentagon in that Circle, equal to the Triangle ABC, and conſequently to the original given Figure.

DEM. AF is a fifth part of AE; wherefore, the Triangle ACF is equal a fifth part of ACE; i. e. of ABC. 1. 6. El.

But, AG is a mean Proportional between AD and AF; and, the Triangle AHI, on AH, equal AG, is ſimilar to ACD; wherefore, it is equal to ACF (by the Lemma); for, ACF and ACD have equal height, CL.

Therefore the Pentagon GHK, which contains five Triangles equal to ACF, is equal to AEC (eq. ABC) which is equal to five times ACF.

N. B. If a Hexagon had been the Poligon required, the Triangle ACD, and conſequently AHI would be equilateral whoſe Angles are each of 60 Degrees.

This Problem may be otherwiſe performed, thus.

Let ABC be the Triangle, equal to ſome given Figure, and it is required to make a regular Heptagon equal to it.

As the Heptagon does not divide the Circle equally, in Degrees, let one be conſtructed of any proportion, as at X; and draw ac and bc, making an Iſoſceles Triangle, acb.

On AC, a Side of the given Triangle, make a Triangle, ADC, ſimilar to acb, whoſe Vertex is D; and draw BE parallel to AC. EC being drawn, there is made the Triangle AEC equal ABC.

Divide AE in ſeven equal Parts, and ſet off one from A to F, on DA, produced. Find a mean Proportional (AG) between AE and AF; and, on A, deſcribe the Ark GH, cuting AD. Draw HI parallel to CD cuting AC; AI is the Side of a Heptagon, equal to the Triangle ABC; for, a Circle being deſcribed on H, with the Radius AG, will contain AI ſeven times, applied to the Circumference.

The Demonſtration of this is nearly the ſame as before; for, FA: AG:: AG: AD; and, the Triangle AHI is ſimilar to ADC; it is therefore equal to ACF, equal to a ſeventh part of ACE, which are of equal height.

PROBLEM XVII.

Two right-lined Figures being given; to conſtruct one ſimilar to either, and equal to the other.

X and Y are the given Figures; it is required to conſtruct one equal to X, and ſimilar to Y.

Let a Side of each Figure be in the ſame Right line AH.

Reduce each Figure to a Triangle, ABC, and CDE.

Make the Triangle AFG equal to ABC, and equal in height to DO (7.). Make EH equal to AG, and EI a mean Proportional between CE and EH.

Draw CI; and KL, MN, parallel to CI; and, on LN, conſtruct the Pentagon Z ſimilar to Y; by 1ſt of S. 8.

I ſay, that the Pentagon Z, is equal to the Trapezium X.

DEM. The Triangle AFG is equal to ABC (eq. X) and it has equal height with CDE, which is equal Y; and, EI is a mean Proportional between AG and CE; wherefore, the Poligon Z, conſtructed on EI, being ſimilar to Y, is equal to ABC, equal AFG, equal X.

This Problem may be otherwiſe performed, as follows.

It is immaterial what the original Figure is, to which an equal one is required; ſuppoſe it reduced to the Triangle ABC; X is the other given Figure.

A ſimilar one is required, equal to ABC.

Reduce the Figure X to a Triangle, AKL, and draw the Diagonal LM. Make, on AC, the Triangle ADC ſimilar to KLM (1. S. 8.). Draw BE parallel to AC, and join EC. Divide AE, in F, as KA is divided, at M. 12.5.

Make AG a mean Proportional between AF and AD; draw GH parallel to DC; and, on GH, conſtruct the Tarpezium GI, ſimilar to LN; ſo ſhall the Poligon AGI be ſimilar to X; and it is equal to the Triangle ABC.

DEM. For, the Triangle AGH is ſimilar to ADC, therefore to KLM; and AEC is equal to ABC. 18. 1. El.

But, AF:AG::AG:AD; wherefore, AGH=ACF, which has the ſame height as ACD. Alſo, ACE:ACF (eq. AGH)::KLA:KLM; for, AE:AF::AK:KM. But, KLA=KLN; and, KLN is ſimilar to AGI; conſequently, AGI is equal AEC; that is, to ABC.

OR; it is thus performed, according to Euclid.

X and Z are the two Figures given.

It is required to make a Pentagon ſimilar to X, and equal to the Trapezium Z.

On any Side of the Pentagon, as AB, conſtruct a Rectangle, AE, equal to the Pentagon; by 13, 2, and the 5th of this. Then, on the Side BE, deſcribe another Rectangle, BF, equal to the Trapezium, Z; by the ſame Problems. Biſect DF, at C; and, on that Center, deſcribe a Semicircle, with the Radius CD, cuting BE, at G.

On EG, if a Pentagon be conſtructed, ſimilar to X, making EG the correſponding Side to AB, it will be equal to the given Trapezium Z. Q. E. F.

DEM. EG is a mean Proportional between DE and EF; and, the Ratio of ſimilar Figures is duplicate of their correſponding Sides, i. e. as DE to EF (13. 6.); but the Ratio of DB to BF is as their Baſes, DE to EF; - 1. 6. conſ. the Ratio of the two Pentagons will be the ſame; for, they are reſpectively equal to the Rectangles, by Con. Therefore, a Pentagon conſtructed on EG, ſimilar to X, will be equal to the Trapezium Z.

By the 13th any Figure is readily reduced to a Triangle; by the 2nd a Rectangle is obtained equal to a Triangle; and, by the 5th another Rectangle may be determined equal to that, having one Side equal to AB, or any given Line, giving AD for the other Side of the Rectangle AE; which, is a fourth Proportional to AB and the two Sides of the Rectangle first found, equal to the Pentagon; as by the last method of Pr. 7. S. 5. Consequently, the Rectangle BD is equal to the Pentagon X, and BE to the Trapezium Z.

PROBLEM XVIII. 44. I. Euclid.

To reduce any given Figure to a Rectangle, or other Parallelogram, under a given Side and Angle.

Let the original Figure be reduced to the Triangle ABC; X is the given Angle, and Z the Side given.

Make the Parallelogram DEFC, equal to the Triangle ABC, whose Angles, at E and C, are equal to the given Angle, X (Pr. 2.). If a Rectangle were required, DE must be perpendicular to AC.

Produce AC and DE, indefinite; and make CH equal to the given Line, Z; through the Angle F, of the Parallelogram DF, draw HI, cuting DE produced, in I.

EI is the other Side of the Parallelogram, required; Pr. 5.

Draw IL parallel to AH, and HL to DI meeting at L; produce EF to G, and CF to K. KG is the Par. fought.

DEM. The Par. KLGF (having its Angles, at K and G, equal to the given Angle X, and a Side, FG, equal to a given Line, Z) is equal to the Par. DEFC; - P. 17. 1.
which, is equal to the Triangle ABC, - - Pr. 2.
and consequently to the original Figure.

APPL. This Problem, may be applicable in exchanging an irregular piece of Ground in one Place, for the same Quantity in another, suppose to build on; which, by reason of the contiguous Buildings, is confined to a certain Angle, which is the given Angle; the length in Front may be considered as the

given Side; the Queſtion is, what depth of Ground from the Front is required, to be equal in its Area to the other.

Having firſt reduced the original given Figure to a Triangle, it is then convertible into a Parallelogram under any Side, by this or the 5th; or, into a Square by the 4th.

PROBLEM XIX.

To make a Triangle equal to a Circle.

Draw a Diameter, AB (C is the Center); and, at either extreme, as B, draw the Tangent DB (3. 3.) and produce it towards E.

Take D at pleaſure, and make DE to AB as 22 to 7 *; or, more exactly, as 355 to 113. Draw CD and CE.

The Triangle DCE is equal to the given Circle.

As the Circle has never, yet, been ſquared, poſitive Demonſtration cannot be given; but it is demonſtrable, that a Triangle having one Side equal to the Perimeter of any regular Poligon, and its height, from that Side, equal to a Perpendicular from the Center to a Side of the Poligon (See Art. 7 of the Appendix, Part I.); and conſequently, a Triangle, having one Side equal to the Circumference, and its height to the Radius of the Circle, is equal to that Circle; which may be conſidered as a Poligon of a number of Sides, multiplied till they are equal to the Circumference.

If BD, or BE, were made equal to DE, the Triangle CBD or CBE would be equal to DCE, having the ſame height; by 18. 1. El.

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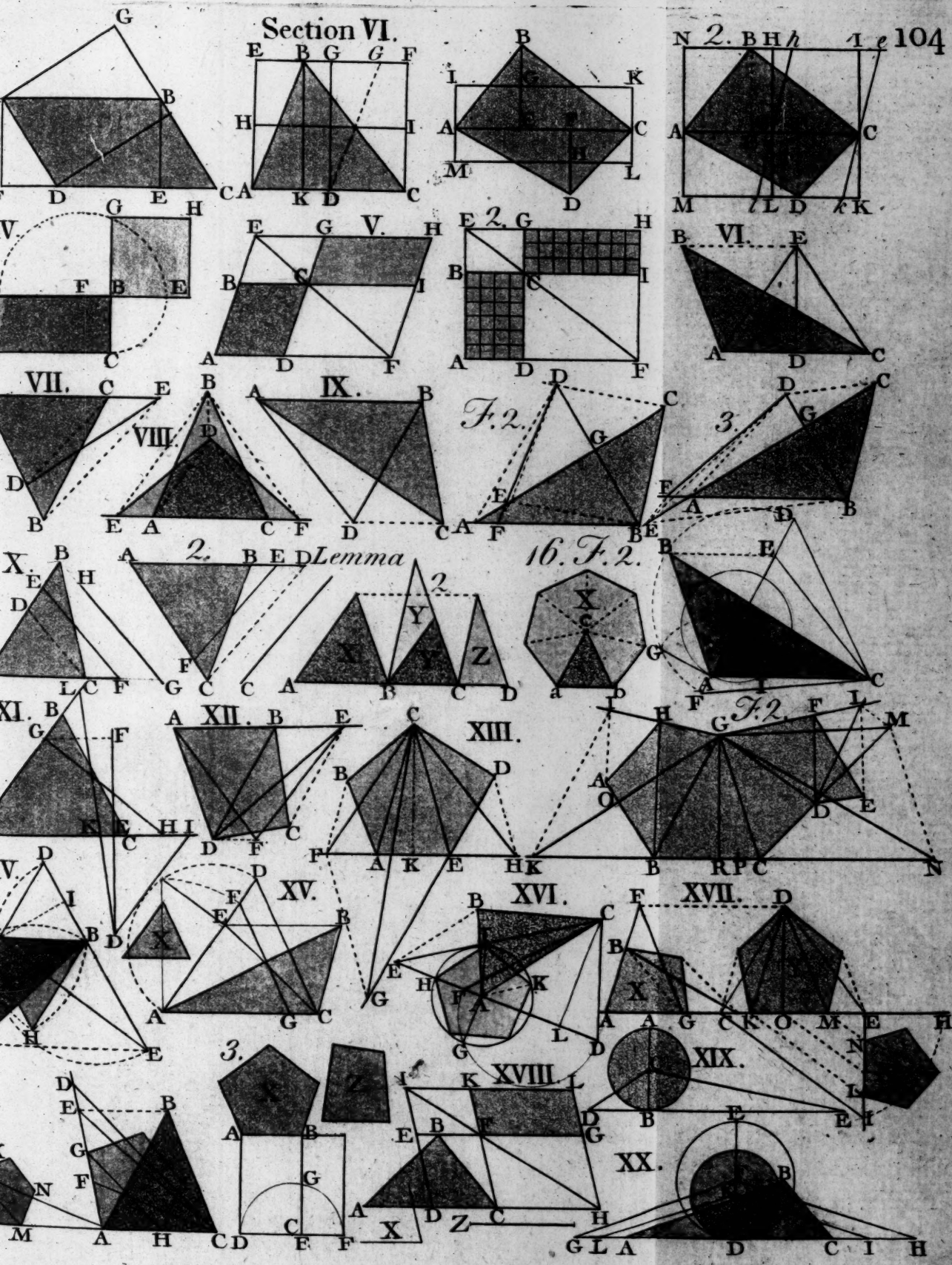
* The Diameter being divided into ſeven equal Parts, DE muſt be equal to twenty and two; i. e. to three Diameters, and $\frac{1}{7}$.

Section VI.


V.
VII.
VIII.
IX.
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2. Lemma
16. F. 2.
F. 2.
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XI.
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XV.
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XVII.
XVIII.
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XX.

PROBLEM XX.

To deſcribe a Circle equal to a given Triangle.

ABC is the Triangle given.

Biſect any Side, as AC; to which, draw DE perpendicular.

Draw BF parallel to AC, cuting DE, at F; on which, with the Radius DF, deſcribe a Circle, and reduce it to a Triangle, GFH. - - - - - - 19.

Take DI a mean Proportional, between DC and DH. 1. 5.

Draw IK parallel to FH; and, on K, with the Radius DK, deſcribe a Circle; I ſay, it will be equal to the Triangle ABC.

DEM. The Triangle DKI is ſimilar to DFH; IK being parallel to FH; wherefore, they are in the duplicate Ratio of their correſponding Sides; and of the Perpendiculars BK, BF (6. 6. El.); and, the Circles are to each other in the ſame Ratio. - - - Cor. 1. and 4. 13. 6.

But, the Circle whoſe Radius is DF is equal to the Triangle GFH; conſequently, that on DK is equal to DKI, doubled; to IKL. And therefore to the given Triangle, ABC, equal IKL.

This Section contains the Eſſence of practical Menſuration of Superficies; which conſiſts in reducing every Figure to a Rectangle, or Triangle; ſo that, the multiplication of one Side into the other, by any meaſure of length, ſhall produce the Area of the Figure, however irregular, with the greateſt eaſe, expedition, and certainty.

PRACTICAL GEOMETRY.

SECTION VII.

On Diviſion *of* right-lined Figures.

PROBLEM I.

To divide a Triangle in two or more equal Parts, by Right-lines, from any Angle required; as B, of the Triangle ABC.

BISECT AC (the Side oppoſite) and draw BD, which divides the Triangle ABC into two Triangles, ABD, DBC, which are equal.

If it were required to be cut into three or more equal Parts; divide AC accordingly, and draw B1, B2; ſo may it be divided into any number of Parts, at pleaſure, equal or unequal, in any Ratio required. For; however the Baſe, AC, is divided, the Triangle is divided in the ſame Proportion, by Lines drawn from B. - - P. 1. 6. El.

PROBLEM II.

To divide a Quadrilateral in two or more equal Parts, by a Line drawn from any Angle.

ABCD is the given Figure to be divided, from the Angle B.

Draw the Diagonal BD, and CE parallel to BD, cuting AD produced, at E, and draw BE.

The Triangle ABE is equal to the Trapezium AC. Biſect AE, and draw BF; which divides the Figure as required, in ABF, and FC.

If AE be divided into three or more equal Parts; Lines drawn from B to the ſeveral Diviſions will divide AC the ſame; if they fall within the Baſe, AD.

This needs no further Demonſtration; being evident, from the foregoing.

OR thus, without reducing the given Figure to a Triangle.

Draw the Diagonal AC, and triſect it, in E and F; draw BE and BF, DE and DF; alſo, draw BD.

Draw FG and EH parallel to BD, and join BG and BH; which will divide the Trapezium, as required.

DEM. The Triangles ABE, EBF, and FBC are equal; alſo ADE, EDF, and FDC, - - - - 1. 6. El. conſequently, ABE+ADE=EBF+EDF, &c.

FG and EH are parallel to BD; wh. EBH=EDH, and EIH is common; conſequently, EBI, added, is equal to IDH taken away; and therefore, the Triangle ABH is a third part of ABCD.

After the ſame manner, BCG may be proved to be a third part; conſequently, BGH is a third part; and therefore the Trapezium is divided into three equal Parts, by the Lines BG and BH. Q. E. F. & D.

The Figure ſhews, that it is applicable if four or more Parts are required, equal or otherwiſe, at pleaſure.

PROBLEM III.

To divide any right-lined Figure into three or more Parts, in any Ratio required, equal or otherwiſe; by Lines drawn from any Angle.

It is required to divide the Pentagon ABCDE into three Parts, in the Ratio as X is divided; from the Angle B.

Reduce the Pentagon, given, to a Triangle, FBG. 13. 6.

Divide FG, the Baſe of the Triangle FBG, as the Line X is (by 12. 5.); draw BH and BI, which divides the Pentagon as required. Q. E. F.

DEM. The Triangle FBG = to the Pentagon. See Pr. 13. 6. And, the Side FG was divided in the ſame Ratio as the Line X; conſequently, the Triangles, FBH, HBI, IBG are to each other as their Baſes FH, HI, IG; P. 1. 6. El. and, the Pentagon is divided in the ſame Ratio as FBG.

For, the Triangle BAE is equal BFE; conſequently, BAH = BFH, for EBH is common to both; alſo, BCD = BGE, and BDI is common; wh. BCI = BGI; and therefore, the Pentagon is divided, by the Lines BH and BI, in the ſame Ratio as the Triangle FBG.

If it ſo happened that BH or BI fell without the Baſe, ED, of the Pentagon; or of AD in Fig. 2, as BE, it does not divide the given Figure, as required; although the Triangle ABF is, which is equal to it, in ABD, DBE, and EBF; for, the Triangle GBC, cut off the Trapezium, by the Line BE, is greater than the Triangle EBF; and DBG, of the Trapezium, is leſs than DBE, by the Triangle DGE.

Wherefore, EI being drawn, parallel to BD, and BI joined, the Triangle GBI is equal DGE; for BID is equal to BED, and BGD is common; BCI is therefore equal to EBF, and therefore, BI and BD divide the Trapezium (or other Figure, equal ABF) as required.

PROBLEM IV.

To divide a Triangle in two or more equal Parts, by Right lines drawn from Points given in any Side.

Let D be the given Point, in the Side AC, of ABC.

Biſect the Side AC, in E, and draw BE. Draw DB, and EF parallel to it, cuting BC in F, and draw DF.

The Trapezium ABFD, is equal to the Triangle DFC.

DEM. Becauſe AE=EC, the Tri. ABE=EBC. - 18. 1.
But, EF is par. to DB; wh. the Tri. BFD=BED; ſame. conſ. BFD+ABD is equal to BED+ABD. - Ax. 6.
And, the Triangle ABE is equal to EBC; - - above. wh. the Trap. ABFD, is equal to the Tri. ABE, eq. EBC, and conſequently, to the Triangle DFC.

Therefore, the Right line DF divides the Triangle ABC into two equal Parts. Q. E. F. & D.

If it were required to be divided into three equal Parts, as in Fig. 2, from the Points D and E; triſect AC, at F and G, and draw BF, BG. Then, the Triangle is triſected, by the Lines BF and BG. - - - - Pr. 1.

Draw BD, and FH parallel to it; join DH; alſo, draw BE, and GI parallel; join EI. The Right lines DH and EI triſect the Triangle ABC, as required.

DEM. ABF is a third Part of ABC, and BHD=BFD; conſequently, ABHD is equal ABF.

By the ſame reaſon, FBC is biſected, by BG, and BIE=BGE; wherefore, FBIE=EIC. - Ax. 6 and 7. 1.
But, DFL=BHL; wherefore, DHIE=FBIE; and therefore, DHIE=EIC; conſequently, the Triangle ABC is triſected, by the Lines DH and EI. Q. E. F.

After the ſame manner, a Triangle may be divided in any number of equal Parts, at diſcretion; by Right lines drawn from given or determined Points in any Side of the Triangle.

And, 'tis evident that by the ſame means (as in the foregoing Problem) it may be divided into any number of Parts in any given Ratio, as well as in equal Parts.

PROBLEM V.

To divide a Trapezium into two equal Parts, by a Right line, drawn from a Point given in a Side; without reducing it to a Triangle.

ABCD is the Trapezium given, and E the given Point.

Through the Angle B, draw BF, parallel to AD. - 5. 1.
Biſect AD and BF, in G and H; and draw GH and HC.
Then the Pentagon ABCHG is equal to the Trap. GHCD.

Dem. For (having drawn BG and GF) the Triangle ABG is equal to GFD; BGH=HGF, and BCH=HCF. 18. 1.
Therefore, AH+BCH=HD+HCF. - - Ax. 6. 1.

2ndly. In Fig. 2, let ABCHG be equal GHCD, as before.
Draw GC, and HI parallel to GC, cuting BC in I, and draw GI. GI biſects the given Trapezium.

For, becauſe HI is parallel to GC, GIC=GHC. 18. 1.
Wherefore, GIC+GCD=GHC+GCD. - - Ax. 6.

Again. Draw EI, and GK parallel to EI, cuting BC in K; and, from E, the given Point, draw EK.

I ſay, that EK divides the Trapezium ABCD equally.

For, the Trapezium GHCD was proved half ABCD; and, GICD was proved equal to GHCD.
The Triangle EKI is equal to EGI. - - 18. 1.
Therefore, the Trapezium ABKE is equal to EKCD.

If E had been given near the Angle A, at the leſſer end of the Trapezium, the proceſs would be more operoſe.

Let Fig. 3, be ſuppoſed the ſame Figure divided by GI, into two equal Parts, from the middle Point G (as in Fig. 2.) which is neceſſary to be the firſt done, in all caſes.

Join the given Point, E, and I, as before; draw GF, parallel to EI, cuting CD in F, and draw EF and IF. Then, the Pent. ABIFE is equal to the Tri. EFD+FIC.

But, ABCD, is not equally divided by one Right line. Therefore, draw IH, parallel to CD, cuting EF; draw CH. The Pentagon ABCHE=ABIFE=the Trap. EHCD.

Laſtly; join EC, and draw HK parallel to EC, cuting CF in K, and draw EK; which Line divides the Trapezium ABCD into two equal Parts, as required.

For, ABCKE is equal to ABCHE. But, ABCHE was equal to ABIFE, equal to ABIG; which, was proved equal to half the Trapezium ABCD.

PROBLEM VI.

To divide any right-lined Figure in two equal Parts; by a Right line drawn from any Point, in a Side.

Firſt, let the given Figure, ABCD, be a Trapezium, as in the laſt. P is the given Point.

Reduce ABCD to a Triangle, ABE, equal to it, by 13.6. and divide AE in two equal Parts, at F; draw BF, which divides the Triangle in two equal Parts; and conſequently, the Trapezium ABCD.

Join BP, and draw FG parallel, cuting BC at G; PG, being drawn, will biſect the given Trapezium, as required.

This being evident, from the preceding Problem, it is obvious, that, to biſect the Quadrilateral, from the given Point, P, nothing more is requiſite than to determine AE, and biſect it, in F; FG being drawn, parallel to BP, cuting BC; PG divides the Trapezium in two equal Parts. Fig. 2.

After the ſame manner, the Poligon ABC may be divided into two equal Parts from a given Point P, in the Side AC. Being reduced to a Triangle, DBE, equal

to the Poligon, and the Baſe, DE biſected, at F; BF divides the Poligon in two equal Parts.

Then (as in the firſt) FG, parallel to BP, gives the Point G; to which, a Line drawn from P cuts the given Figure in two equal Parts, as required.

Thus may any right-lined Figure (having no indented Angle) be biſected; from a Point, at diſcretion, in any Side.

When there is an indented Angle (as at C) in the Figure. It is required to divide ABDN into two equal Parts, by a Line drawn from a given Point, P, in the Side AN.

Reduce the given Figure to a Triangle, EBF, equal to it.

Biſect EF, at G, and draw BG; the Triangle EBG is equal to half the given Figure, equal ABC.

Draw PB, and GH parallel to it; alſo, draw PH and BH; the Triangle BHP is equal to BGP; conſequently, ABH is equal to half the Figure, equal EBG.

But, BHC is without the Figure, therefore no part of it. Join BD, or draw a Line cuting CD, at diſcretion, and CK parallel to it, cuting BH, produced, at K; the Triangle CDK is equal CBK; conſequently, HDK is equal to CBH.

Draw KL parallel to CD, and join LD; the Triangle HLD is equal HKD.

Laſtly, draw LM parallel to PD, cuting CD at M, and draw PM; which divides the given Figure, as required.

Thus may *any* right-lined Figure, whatever, be divided into two equal Parts, or otherwiſe, by a Right-line, from a Point given, in a Side; which needs no further Demonſtration, as each ſtep is evident.

PROBLEM VII.

To divide a Poligon, of any denomination,* into three, or more, equal Parts, or any other Ratio; by Right lines, from given Points in any Side.

* The Figure may have any number of Sides, be either ordinate or inordinate; but let it be underſtood, not to have any indented Angles, unleſs it be ſo expreſſed.

It is required to divide the Hexagon ABC into three equal Parts, from the Points E and F, in the Side AF.

Reduce the given Figure to a Triangle (DBG) on that Side (Pr. 13.6.). Triſect DG, and draw B1, B2. Join BE; draw 1H parallel to BE, and draw EH. Alſo, join B and F; draw 2I parallel to BF, and laſtly, draw FI.

EH and FI triſect, or divide ABC into three equal Parts.

DEM. The Triangle DBG is equal to the given Figure; and it is triſected by the Lines B1 and B2. - P. 1. 6. El.

The Triangle BKH = EK1; and BLI = FL2; for, BHE = B1E and BIF = B2F (18. 1. El.) and BKE, BLF are com.

Therefore, the Pentagon AH = △DB1, the Trapezium EI = △1B2, and, the Pentagon ICF = △2BG; i. e. they are each equal to a third part of the given Figure ABC; and conſequently, it is triſected by the Lines EH and FI.

After the ſame manner, it may be divided into more Parts, equal amongſt themſelves; or in any Ratio required.

PROBLEM VIII.

To divide any right-lined Figure in two Parts, from a given Point, in any Side; or, as a given Line is divided.

ABC is the Figure to be divided, and X the given Line, divided in the Ratio required.

Reduce ABC to a Triangle, DBE, equal to it.* 13. 6.

* The 13th Prob. Sect. 6. has been frequently referred to, and it muſt be obvious, that this Section is almoſt wholly built on it; it is therefore unneceſſary to be ſo particular, now, as in the foregoing Problems.

Divide the Baſe, DE, at F, as the Line X is divided; 12.5. and draw BF; which divides the Triangle, and alſo the Poligon, in the ſame, given Ratio; viz. as the Baſe, DE, of the Triangle.

Draw PB, and FG parallel; PG, being drawn, divides the Poligon ABC, as required; as is evident from inſpection of the Figure.

For, BF divided it in the Ratio required; as DF to FE. Conſequently, if the Triangle BEP be taken away, and its equal, BGP be added, AGP : GCP :: DBF : FBE; i. e. as DF to FE; that is, as the Parts of the given Line, X.

After the ſame manner, it may be divided into three or more Parts, in any Ratio, whatever.

PROBLEM IX.

To divide any right-lined Figure in two or more Parts, which ſhall be, amongſt themſelves, in any given Proportion; or, as any Side is divided.

X is the Figure given, to be divided (a Pentagon) into three Parts; as its Baſe, AC, is divided, at E and F.

Through B, draw HI, parallel to AC; and, from the two outer Angles of the Figure, draw Lines, parallel to AB and BC reſpectively, cuting it, at H and I.

Through A and C, draw HK and IK, meeting at K; and, through E and F, draw KD and KG, cuting HI, at D and G. Then, draw DL and GM parallel, reſpectively, to BE and BF, cuting the Sides of the Figure, at L and M. Laſtly, draw EL and FM; the Poligon X is divided, by thoſe Lines, as its Baſe at E and F.

Dem. The Trapezium AHIC is equal to the Figure X.

For, the Triangle cut off, by the Diagonal AB, is equal

AHB, and BIC equal to that cut off by BC ; - 18. 1. El.

But, HI is divided, at D and G, as AC, at E and F; 12. 5. wherefore, the Trapezia AD, DF, and FI are, between themſelves, as AE, EF, and FC, their Baſes;* and, the the Pentagon ELMBF is equal to the Trapezium DEFG (for, the Triangle ELB=EDB, alſo, BGF=BMF); conſequently, the Trapezium AL=HE, and MC=FI. Therefore, the given Figure is divided, by the Lines EL and FM, as AC is divided at E and F, Q. E. F. & D.

PROBLEM X.

To divide a Triangle into three equal Parts, by Lines drawn from the ſame Point in any Side.

Firſt. It is required to divide the Triangle ABC, into three equal Parts, from the middle Point, D, in the Side AC.

AC being biſected, draw BD. Alſo, triſect AC, at E and F, and draw BE and BF ; which divide ABC into three equal Parts (Pr. 1). Draw EG and FH parallel to BD; and laſtly, draw DG and DH, which divides the Triangle ABC as required; in AGD, DGBH, and DHC.

This needs no Demonſtration, as (by the Proceſs) it is evident, from the preceding Problems.

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* There is no Propoſition in the whole Elements directly to the Point, the 1ſt of the 6th being applicable to Parallelograms, only; but it may be made manifeſt by the ſimilar Triangles HKD, AKE, &c. for, HKD : DKG, or GK1,:: AKE : EKF, or FKC; viz. as HD to DG, or G1, i. e. as AE to EF, or FC.

Wherefore, taking away the Triangles AKE, &c. from HKD, &c. there remains the Trapezia HE, EG, and GC, in the ſame Ratio as the Triangles, viz. as AE, EF, and FC.

For, the Triangle AGD is equal to ABE, and DHC to FBC (Ax. 6 and 7. El.); conſequently, the Trapezium DGBH is equal to EBF; that is, each to a third part of the Triangle ABC; therefore, the Triangle is triſected by DG and DH.

2ndly. When the Point given is not in the middle of a Side, as at E, Fig. 2. In which caſe, it muſt be firſt divided from the middle, as in the preceding caſe; then;

Draw EG, and DF parallel to EG, cuting AB at F, and join EF. Alſo, draw DI parallel to EH, and join EI; EF and EI cut the Triangle ABC, as required, in AFE, EFI, and EIC; which is alſo evident, from the foregoing; in which caſe, the Triangles AGD, DHC, being equal, are equal in height. - - - 18. 1. El.

PROBLEM XI.

To divide any Figure into three equal Parts, by Lines drawn from any Point in any Side of it.

X is the given Figure (a Pentagon) and P the given Point.

Reduce the given Figure to a Triangle, DBE, equal to it. Divide the Baſe, AC, into three equal Parts, at A and C; and draw AB and BC, dividing the Triangle DBE into three. Draw PB, and AF, CG, parallel to it. PF and PG, being drawn, divide the Pentagon into three equal Parts.

By the ſame means it may be divided in any Ratio; by dividing DE in the Proportion required; at A and C.

Dem. The Triangle DBE is equal to the given Figure; and the three Triangles, ABD, ABC, and CBE, are equal, between themſelves, or otherwiſe, as required.

But, the Triangle BFP = BAP, & BGP = BCP; 1.6.El. wherefore, the Trapezium, PFG = △ ABC; and conſequently, the Trap. AF = △ ABD, and CG equal CBE. Therefore, the Pentagon is divided, by PF and PG, as required.

PROBLEM XII.

To divide a Triangle into three equal Parts, by Lines drawn from a Point, taken without it.

ABC is the Triangle to be divided, by two Lines drawn from P.

Divide AC into three equal Parts, at D and E. Pr. 11.1. Draw BD and BE, which triſect the given Triangle. 1.6.El.

Draw PF and PG, ſo, as to make the Triangle AFD equal ABD, and EGC equal to EBC (11.6.); conſequently, the Pentagon DFGE is equal to the Triangle DBE; and the Triangle ABC is triſected, by the two Lines PF and PG, drawn from P. Q. E. F.

This Problem is ſo evident as to require no Demonſtration.

PROBLEM XIII.

To cut other right-lined Figures into three equal Parts, from a given Point without it.

ACE is the Figure given, to be triſected from the Point P.

Reduce the given Hexagon to a Triangle, DBG, equal to it. Triſect DG, at E and F, and draw BE and BF.

Produce BA and BC, till they cut DG, produced both ways, at H and I.

Draw PK, ſo, as to make the Triangle IKL equal to BIF (Pr. 11. 6.); alſo (if P be ſo ſituated, that a Line drawn from P, cuting BE, will alſo cut the Side AB) make a Triangle equal to BEH, by the ſame.*

The two Lines ſo drawn, from P, will triſect the given Hexagon, ACE. Q.E.F.

For, the Triangle BIF was made equal to KIL; wherefor, BOK=LOF; conſ. the Trapezium EK=△EBF; and therefore, the Pentagon KF, is one-third part of ACE.

And, if BE tends to P, the thing is manifeſt; otherwiſe, PB being drawn (as above) the Pentagon KAL will be biſected; as is demonſtrable, from the foregoing.

PROBLEM XIV.

To divide any Quadrilateral in two equal Parts, by a Line drawn parallel to any Side.

If it be a Rectangle, or other Parallelogram, nothing more is requiſite than to biſect one of the contrary Sides, and, from the point of biſection, to draw a Line parallel to the Side required.

It is required to cut the Quadrilateral ABCD (a Trapezium) having two Sides parallel (AD and BC) into two equal Parts, by a Line parallel to either of the other Sides.

Biſect AD, and BC, at E and F, and draw EF, which, biſect at G. Through G, draw HI, parallel to AB, or KL parallel to CD, and it is done. Q.E.F. This is manifeſt.

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* If P happens to be ſo ſituated, that BE tends to it, no more is to be done; but if a Line, from P, cuting BE, cuts the Side BC, it muſt be continued until it cuts AB, produced. In which caſe, making a Triangle equal to BEH, it will be ſomewhat near, but not ſtrictly true; and, although it is eaſy, by another operation, to draw a Line biſecting KAL, it is not ſo to make it tend to P, as required.

When it has no Sides parallel, as ABCD; Fig. 2. and 3.

Reduce the given Figure to a Triangle, ABE, equal to it. Biſect the Baſe, at F, and draw BF.

Produce AD and BC, meeting at G; and draw *B*F, Fig. 3. parallel to CD. Make GH a mean Proportional, between FG and AG (1. 5.); and draw HI, parallel to AB (Fig. 2.) or to CD, Fig. 3; ſo ſhall HI cut the Trapezium ABCD in two equal Parts, as required.

DEM. The Triangle ABE = ABCD, and ABF = half ABE. But, FG : GH :: GH : AG; and ABG, FBG; alſo, A*B*G and F*B*G, are equal in height; wh. GHI = FBG, Fig. 2; or to A*B*G, Fig. 3. Conſequently, AI = ABF, half AC.

PROBLEM XV.

To divide a Pentagon, or other Poligon, in two equal Parts, by a Line parallel to any Side.

It is required to cut the given Figure, ABD, in two qual Parts, by a Line drawn parallel to the Side CD.

Produce the two contiguous Sides, AD and BC, meeting at E; and, on either of them, as a Baſe, reduce the Pentagon to a Triangle, EBG. Biſect the Baſe (FG) at H, and draw BH; alſo, HI, parallel to CD.

Make a EK a mean Proportional, between EB and EI. KL, drawn parallel to CD, will biſect the Pentagon ABD.

DEM. The Tri. BFG = Pent. ABD; and, FBH = $\frac{1}{2}$BFG, equal ABD. HI was drawn parallel to CD; and EI : EK :: EK : EB. But, KL is parallel to HI; conſequently, the Triangle EKL is ſimilar to EIH; and, the Triangles BHI, BHE, have equal Altitude, wherefore, the Triangle EKL = EBH; conſ. BL = BFH; that is, the Pentagon BL is equal half ABD.

Therefore, KL, parallel to CD, biſects the given Figure.

After the ſame manner, may any other Poligon be biſected.

PROBLEM XVI.

To divide a Triangle into any number of equal Parts, by Lines drawn parallel to any Side.

ABC is the given Triangle; let it be divided into three equal Parts, by Lines drawn parallel to AB.

Triſect either Side, AC or BC, at D and E; draw BD and BE. On the ſame Side (AC) deſcribe a Semicircle, and draw EF and DG perpendicular to AC, cuting the Arch at F and G.

Deſcribe the Arks GH and FI; draw IK and HL parallel to AB; which Lines triſect the Triangle ABC.

DEM. CI is a Mean, between CE and AC; the Triangles CBE and ABC have equal height, and CIK is ſimilar to ABC; therefore it is equal to CBE, 1-third of ABC.

Alſo, CH is a Mean, between CD and AC; ABC and DBC are of equal height, and HLC is ſimilar to ABC; therefore, it is equal to DBC, i. e. to two thirds of ABC; and therefore, AL is equal one-third.

Conſequently, IK and HL triſect the Triangle ABC.

After the ſame manner, a Triangle may be divided into any number of equal Parts, or in any Ratio, by dividing AC in the Ratio required.

PROBLEM XVII.

From a Point given within a Figure, to divide it into any number of equal Parts; or in any Ratio.

P is the Point given, in the Hexagon ABC; from which it is required to divide it in three equal Parts.

Reduce the given Figure to a Triangle, JBH; then, from the Point given, and on the ſame Baſe, produced, make the Triangle EPF equal to JBH. - - Pr. 8. 6.

Triſect EF, at A and G, and draw AP, PG; alſo PD.

Make DI equal AG; produce the Side DC, draw IK parallel to DP, alſo KL to PI, and draw PL. DL is a third part of ABC.

Make DJ equal to DI, and draw JM, parallel to AP, cuting the Side AM, and draw PM; DM is another third part of ABC; and conſequently, MBL is a third part.

DEM. The Triangle APG=one-third of ABC, and DI was made equal AG; wherefore, DPI=APG. P. 1. 6. El.

The Triangle PKD=PID, and PLC=PKC; wherefore, ſubſtituting one for the other, the Trapezium DL is equal to DPI.

After the ſame manner, MD may be proved equal to DPJ, equal APG; i. e. to a third part of ABC. Q.E.D.

N. B. If the Point L falls beyond the Angle, another operation will be requiſite, to bring it back, into the Line BL.

By the ſame proceſs, any Figure, may be divided into any number of equal Parts, or in any given Ratio; by dividing EF into the number of Parts, as required.

PROBLEM XVIII.

To divide any Poligon into any number of Parts, in any given Ratio, by ſimilar concentric Figures; that is, about the ſame Point.

ABD is an irregular Pentagon; it is required to divide it into three Parts, which ſhall be, between themſelves, in the Ratio of two, three, and four; as the Line X is divided. Aſſume any Point, as P, within the Figure.

Draw AP, CP, &c. On any of the Radii, as AP, deſcribe a Semicircle, and divide AP in the Ratio as required, at E and F; draw FG, and EH, perpendicular to AP, and join PG and PH.

Make PI equal PH, and PK equal PG; draw KL and IM, parallel to AD, and continue them around the Figure, parallel, reſpectively, to every Side; ſo ſhall the Pentagons KNL, IOM, be ſimilar to ABD; about the ſame Point, C, and be in the Ratio between themſelves, as 2, 5, and 9.

The Poligon ABD is divided as two, three, and four; the inner Poligon is two, the middle Border three, and the outer one four-ninths of ABD.

The Demonſtration of this Problem is the ſame as the foregoing; the Triangle APD being divided by Lines parallel to AD, in the Ratio of the Line X; viz. as two, three and four.

Each Triangle, CPD, &c. is divided in the ſame proportion; conſequently, as any one is divided, taken ſeparately, ſo is the whole, taken collectively; - 2. 5. El. therefore, the given Poligon is divided in the Ratio required.

By the ſame proceſs, it is evident, that any Poligon may be divided in any Ratio, and into any number of Parts, equal or otherwiſe.

If the Figure given be a regular Poligon, or ſuch as may be circumſcribed by a Circle, touching every Angle, they may then be truly concentric.

PRACTICAL GEOMETRY.

SECTION VIII.

On reducing *and* enlarging Figures, *in general. Alſo, from certain* Data, *to determine Things appertaining to the* Figures *they relate to.*

PROBLEM I. 18. VI. Euclid.

On a given Line, to conſtruct a Figure, ſimilar to any given, right-lined, Figure.

Firſt; let the given Figure be a Triangle; ABC.

ON DE to conſtruct a Triangle; ſimilar, and alike ſituated, to ABC.

At the extreme D, of the given Line, make an Angle EDa, equal to CAB, of the given Triangle (4. 1.) and, at the other extreme, E, make the Angle DEb, equal to ABC.
Produce Da and Eb, meeting in F. - - - Poſt. 3.
The Triangle DEF will be ſimilar to ABC. Q. E. F.

DEM. For, the three Angles are reſpectively equal. C. 5. 10. 1.
Therefore, the Sides are proportional. - - P. 4. 6.
Conſequently, the Triangles are ſimilar. - Def. 1. 6.

Secondly; let the given Figure be a Trapezium; AC.

Draw either Diagonal, as BD; and, on the given Line, EG, make a Triangle, EFG, ſimilar to ABD, by the

firſt; then, on FG, make a Triangle, FGH, ſimilar to BCD; and it is done.

The Triangle EFG is ſimilar to ABD, and FGH to BCD; and they are alſo ſimilarly ſituated, each to each, reſpectively; therefore, the Trapezium EFHG is ſimilar to ABCD. Q. E. F.

Thirdly; ABCDE is the Figure given (a Pentagon) and ae the given Line.

Draw the Diagonals AC, AD, BE, and CE.

On the given Line, ae, make the Triangle abe, ſimilar to the Triangle ABE, in the original Figure, by the firſt. Alſo, make ace ſimilar to ACE, and ade to ADE. Join bc and cd, which compleats the Figure.

abcde is ſimilar to ABCDE. Q. E. F.

This method is deduced from the firſt, of this Problem; for, all right-lined Figures may be reduced into right-lined Triangles. It is demonſtrable from 4. and 13. 6. Elements.

It may be thus performed, when required larger.

On AB, or AE, the given Line, deſcribe a Pentagon, Abcde, congruous to the given one; by reducing the given Figure into Triangles, as in the Figure, and making Abc, Acd, and Ade, reſpectively equal to them.

Produce Ab, or Ae, and the Diagonals, Ac, Ad, indefinite. Draw BC, CD, and DE, parallel to the Sides bc, cd, and de, reſpectively; cuting the Diagonals, and Side AE, in the Points C, D, and E. ABCDE is ſimilar to Abcde.

After the ſame manner, any Poligon may be encreaſed or diminiſhed in any Proportion; demonſtrable by 2. 6. El.

APPL. This Problem is of great uſe in various Profeſſions. By it we take Altitudes and Diſtances, though otherways inacceſſible; the Surveyor takes his Bearings, and lays down a

Plan of the Ground he ſurveys; by it, the Mariner plans the Courſe in which the Ship ploughs the Ocean; and the Mechanic plans the Ground, on which he intends to build, &c.; in ſhort, it is almoſt of univerſal Uſe, which, to ennumerate, is not neceſſary in this place.

PROBLEM II.

To make a Triangle, equal to any number of Figures given, and of any kind; being right-lined.

X, Y, and Z are three Figures given; it is required to make a Triangle which ſhall be equal to them all.

Reduce all the given Figures to Triangles (13 6.); and, if they are not already ſo, make them equal in height; by 6 and 7. Sect. 6.

Draw AD, indefinite, and make AD equal to the three Baſes of the Triangles X, Y, and Z; as AB, BC, and CD.

Draw AE, at pleaſure; in which, take the Point E ſo, that its Diſtance from AD is equal to the Altitude of the Triangles, deſcribed, and draw ED.

ABD is equal to X, Y, and Z.

DEM. Draw BE and CE. The Triangle AEB is equal to X; having equal Baſes and equal height; BEC is equal to Y, and CED to Z, for the ſame reaſon; 1. 6. El. th. the Triangle ABD is equal to X, Y, and Z. Ax. 2.

PROBLEM III.

To cut off a Part from a Triangle, equal to any given Figure; or any portion required.

ABC is the given Triangle It is required to cut off, from ABC, a portion equal to the Pentagon X.

Reduce the Pentagon to a Triangle, equal, in height, to the given one, ABC; make AD, or CD, equal to the Baſe, EF, of the Triangle X; and, drawing BD, you have done what was required.

For, whether AD or CD were made equal to EF, the Triangle ABD, or DBC; is, accordingly, equal to X, the given Figure.

After the ſame manner, any portion, as half, a third, &c. of the Triangle ABC, or any part, whatever, may be cut off; by taking AD, or CD, in the proportion required to AC, or other Side of the Triangle, and drawing a Line, from that Point, to the oppoſite Angle; A, B, or C; as is manifeſt, and needs no further Demonſtration.

Squares, and other Figures, may be reduced, in any proportion, very expeditiouſly, and elegantly, in the manner following.

Let AB be the Side of a Square, or other Figure, given, to be reduced a third, or two-third parts; i. e. to determine the Side of a Square in that Ratio, to the given one.

Triſect AB, at 1 and 2; and, from either of the Diviſions, draw a Perpendicular, indefinite. Deſcribe a Semicircle on AB, cuting the Perpendicular at C, and draw AC and BC.

On AC, if a Square, or other Figure, be conſtructed, ſimilar to a given one, it will contain a third part of the Area; and the ſquare of BC is two-thirds of AB ſquare; and therefore, it is the ſame of any other Figure, the correſponding Sides of which (being ſimilar)are AB, AC, and BC.

If AB were given, for the Diameter of a Circle, to be reduced a fourth part or three-fourths, in Area; divide AB into four equal Parts, as at 1, 2, and 3. At the firſt,

or third Diviſion, draw a Perpendicular, indefinite, and on AB deſcribe a Semicircle, cuting it at C; AC and BC, being drawn, the firſt is the Diameter of a Circle, equal to one-fourth part of the given one, and, BC is the Diameter of one, equal to three-fourths, as propoſed.

The Properties of a right-angled Triangle (ACB) are ſo well known, and recur ſo often, as to render a Demonſtration unneceſſary. AC is a mean Proportional, between A1 and AB; wherefore, a Figure on AC is a third part, in the firſt, and a fourth, in the ſecond Example (of the Circle) of any ſimilar Figure whatever, on AB; by 13. 6. El.

PROBLEM IV.

To cut off, from any given Figure, and from any Angle of it, a portion by which it exceeds any other.

The Trapezium Z, and Hexagon BCE are the given Figures; of which Z is the leaſt.

It is required to cut off a part, from the greater (from the Angle B) by which it exceeds the leſſer; the remainder being equal to it.

Reduce the Trapezium to a Triangle, equal to it. 13. 6.

Produce AE (a Side adjacent to the Angle B) and make the Triangle ABF equal to Z, under a given Side and Angle (9. 6.). Join BE, and draw FG parallel thereto, cuting ED, produced (if neceſſary) and draw BG.

The Triangle BGE is equal BFE; and conſequently, the Trapezium ABGE is equal to the Triangle ABF.

Join BD, and draw GH parallel, cuting CD, and draw BH, which cuts the Hexagon as required; the part BHE

being equal to the Triangle ABF (equal ABGE) which was equal Z ; and conſequently, the part BC, cut off, is equal to the Difference.

This Problem is deducible from the 3rd. S. 7. which is evident; for, having reduced the Hexagon to a Triangle (as there) the Triangle ABF (in this) equal to Z, is a portion to be cut off, in a given Ratio ; equal to one, two, or more, of the Parts required, in that Problem.

PROBLEM V.

To enlarge or diminiſh a Square, or other Figure, double, triple, or quadruple; alſo, ſix or eight fold, &c. the Area of the other, given one.

Firſt. ABCD is a Square; it is required to enlarge it; that is, to find or determine the Side of a Square, containing twice, four, or eight times its Area.

Produce AB and AD, or any two contiguous Sides; alſo, draw the Diagonal AC, and produce it, indefinite.

With the Radius AC, deſcribe the Ark CE; i. e. make AE equal to the Diagonal, AC; the ſquare of AE is double of AD, a Side of the given Square, ABCD. 20. 1. El.

Again. Draw EF perpendicular to AE, cuting AC produced at E, and make AG equal AF; which is the Side of a Square double of AE ſquare, conſequently, quadruple of the given one.

If GH be drawn perpendicular, cuting AF produced, at H; it will give AH for a Side of a Square, containing twice AG ſquare, equal to four times AE, and, conſequently, eight times AD, ſquare.

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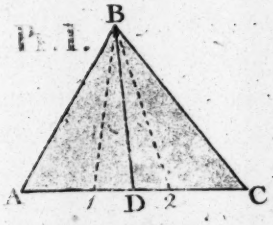

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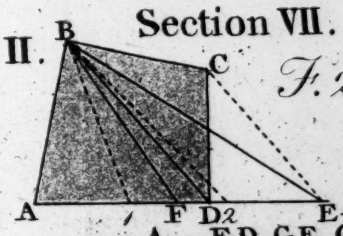

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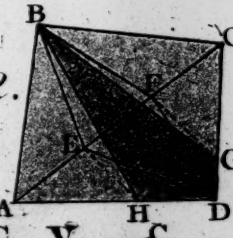

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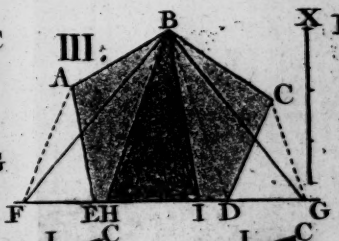

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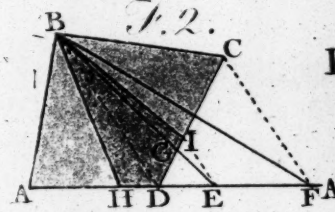

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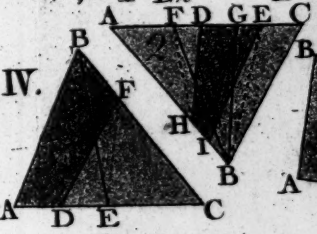

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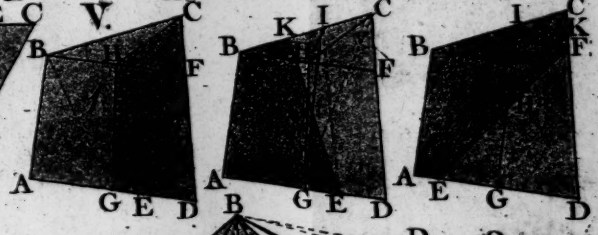

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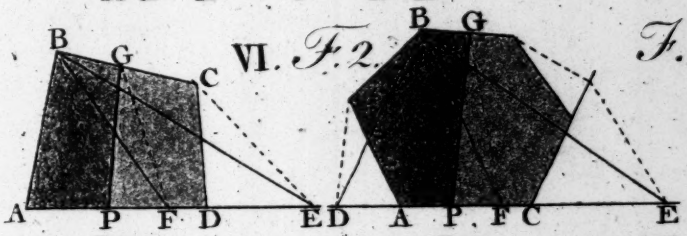

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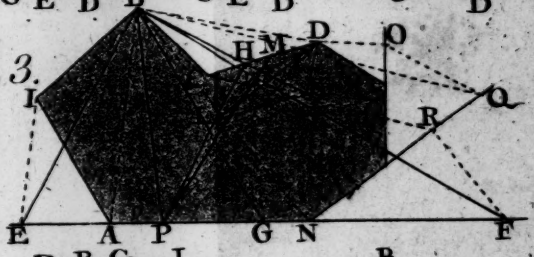

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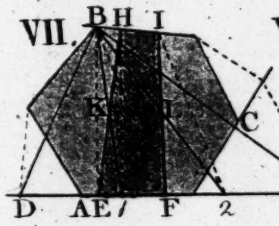

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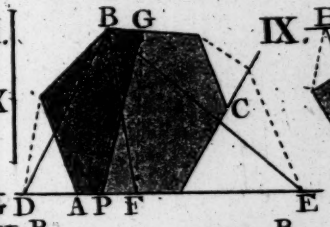

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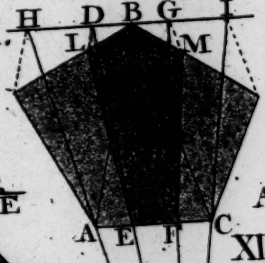

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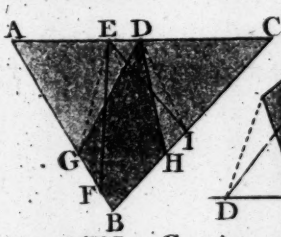

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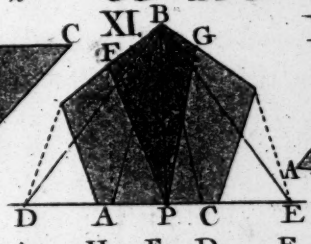

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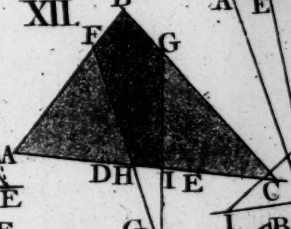

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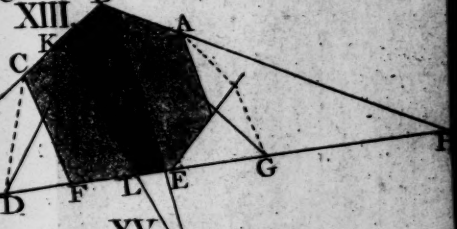

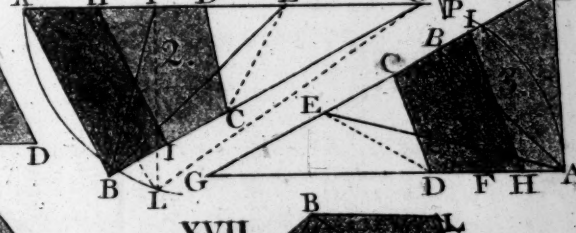

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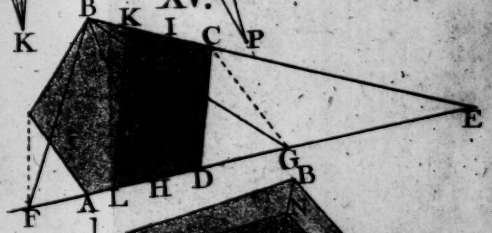

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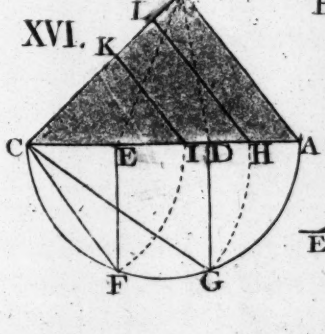

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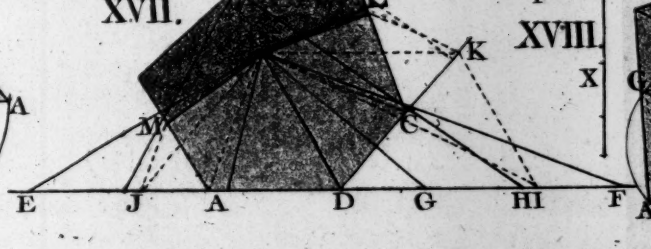

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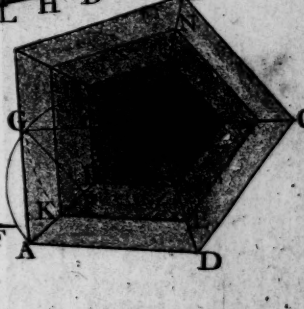


By the ſame means, it may be enlarged, by doubling it each time, to 16, 32, and 64 times, &c. infinitely; which is evident from, and demonſtrated in the 20th of the firſt Book of Elements.

Secondly. It is required to diminiſh the Square AIHG to Squares of a half or a fourth part of its Area.

Draw the Diagonal AH; and, with the Radius AG, deſcribe the Ark GF, cuting the Diagonal. FE being drawn, perpendicular to AG, or parallel to GH, is the Side of a Square, containing half the Area of the given Square, GI.

After the ſame manner (reverſe of the former) it may be decreaſed infinitely.

Thirdly. It is required to find the Side of a Square, triple of the given Square, KB.

Proceed as before, and find AD, the Side of a Square, double of KB, whoſe Side is AB. DE being drawn, perpendicular, produce KC, cuting it at F; draw AF, and make AG equal to AF; which is the Side of a Square, triple of the given one.

For, KF=AD, and DF=BC; wherefore, AF, equal AG, is the Side of a Square, equal to the two, on AD and DF (20. 1.). But, the ſquare of AD is double the ſquare of AB; conſequently, the ſquare of AG is triple of the given Square, KB.

Fourthly. Draw GH perpendicular; make AI equal to AH, the Side of a Square ſextuple of KB; for it is double the ſquare of AG, which is triple of the given one.

Fifthly. By the ſame Proceſs, ſimilar Figures may be conſtructed, double, triple, or quadruple, &c. of any given right-lined Figure, whatever; for Example:

ABD is a given Pentagon, which is required to be doubled.

Produce any Side, as AD; draw DE perpendicular thereto, and equal to AD; draw AE, and make AF equal to it; which is the Side of a Pentagon, ſimilar to ABD, and correſponding with AD; which is double in Area; for, ſimilar Poligons are to each other, as the ſquares of their correſponding Sides. - - - Cor. 2. 13. 6. El.

After the ſame manner, as for Squares, it may be tripled or quadrupled, &c. at diſcretion.

Sixthly. The Diameters of Circles, double, triple, and quadruple of a given one, are alſo determinable, by the ſame.

C is the Center of a Circle, whoſe Diameter is AB.

Draw CD perpendicular, and DE parallel to AB; alſo, draw BE parallel to CD; then, BCDE is a Square.

Draw CE, a Diagonal; and, with that Radius, deſcribe a Circle, which will be double the Area of the given one.

Make CF equal to the Diagonal, CE.

FG being drawn, parallel to BE, cuting DE, produced; CG is the Radius of a Circle triple of the given one; and thus it may be enlarged, at pleaſure.

For, the Areas of Circles are, to each other, as the ſquares of their Diameters, or Radii; by Cor. 1. 14. 6. El.

PROBLEM VI.

To find the Side of any right-lined Figure (ſimilar to a given one) which ſhall be to the given Figure in any proportion required.

Let AB be the given Line, or Side of the given Figure.

It is required to find the length of a Line, for a correſponding Side; on which, if a ſimilar Figure be conſtructed, its Area ſhall be to the given one in the Ratio of $3\frac{1}{2}$ to 1.

Produce the given Line, AB, indefinite. Make BD equal to $3\frac{1}{2}$ times AB. Biſect AD; and, on C, deſcribe a Semicircle, or the Ark AE only. At the Point B, draw BE perpendicular to AB, cuting the Ark at E; BE is the Line ſought. Q. E. F.

DEM. BE is a mean Proportional between AB and BD. 7.6.
Wherefore, the ſquare of AB, to the ſquare of BE, is duplicate of AB to BE. - - - 12. 6. El.
i. e. their Ratio, or Proportion to each other, is as AB to BD; and all ſimilar Figures are in the ſame Ratio, as the Squares of their correſponding Sides. - C. 2. 13. 6.

By this Problem, any right-lined Figure, whatever, may be increaſed or diminiſhed in any Proportion. e. g.

If you would decreaſe it a fifth, a fourth, a third, or a half, &c. make BF to AB in that Ratio; biſect AF, and deſcribe the Semicircle AGF, cuting the Perpendicular BE (at the Point B) in G; and BG is the Line ſought.

Hence, the Ratio between any two Figures may be known.

For, having reduced the given Figures to Squares; i. e. having found the Side of a Square, equal to each Figure, reſpectively, make AB and BE reſpectively equal to them, and forming a Right angle, ABE.

Produce AB or EB, indefinite; join AE, and make AED, or EAH, a Right angle; i. e. draw ED, or AH, perpendicular to AE, cuting AB, or EB, produced, in D or H.

Then, as AB is to BD, or, as HB to BE, ſo is one Figure to the other;—by Cor. 1. 13. 6. El.

For, the Squares of AB and BE are, reſpectively, equal to them.

N. B. If the Figures, whoſe Ratio is required, are ſimilar, make AB and BE, reſpectively, equal to, or in the Ratio of any two correſponding Sides, and proceed as above.

PROBLEM VII.

Two Figures being given; to enlarge either of them ſo, as to be equal to both; that is, to find a correſponding Side of a Figure, ſimilar to either, and equal to the Sum of both.

ABC and X are the given Figures. It is required to determine a correſponding Side with AB, of a Pentagon ſimilar to ABC, which ſhall be equal to them both.

Reduce both Figures to Triangles, ABD and RST, equal to them (13. 6.), and (by 9. 6.) make the Triangle BED, on the Side DB, and Angle DBE, equal to RST. Make AF a mean Proportional, between AB and AE. 1.5. On AF, if a Pentagon be conſtructed, as AFH, ſimilar to ABC, it will be equal to AED; which is equal to the two given Figures, by Conſtruction. Q. E. F.

DEM. ABD is equal to ABC, and BED to RST. by Con. wherefore, AED=ABC+RST, equal X; and, as ABD, BED, have a common Angle (D) inſiſting on the ſame Right line, AE, they have that proportion to each other, as AB to BE (18. 1.); and conſequently, ABD:AED::AB:AE; i. e. as ABC to the Sum of both.

But, AF is a Mean, between AB and AE; and, the Pentagon AFH is ſimilar to ABC; conſequently, equal to AED. For, the Areas of all ſimilar Figures, are in the duplicate Ratio of their correſponding Sides, AB, AF; and, AB:AF::AF:AE, wh. ABC:AFH::AB:AE. P.13.6.El. But, the Triangles ABD, AED are as AB to AE. above. Th. the Pentagon AFH=△AED, equal ABC+X. Ax.5.5.

PROBLEM VIII.

Two Figures being given; to reduce the greater to a ſimilar one, by ſo much as it exceeds the other.

ABC and Z are the two given Figures, as in the foregoing. It is required, to reduce the Pentagon ABC, by how much it exceeds the Hexagon, Z.

Having reduced both Figures to Triangles, as before, make DBE equal to RST, equal Z. - - - - Pr. 9. 6.

Make AF a mean Proportional, between AE and AD; 1. 5. and draw FG parallel to BD, cuting AB at G. Deſcribe, on AG, the Pentagon AGH, ſimilar to ABC. - Pr. 1.

AGH will be equal to ABE, the Exceſs of ABC to Z.

DEM. The Triangle ABD = Pent. ABC; and DBE to Z; and, AF is a Mean, between AE and AD; by Con. wherefore, the Triangle AGF, being ſimilar to ABD, is equal to ABE, by Lemma to 10. S. 6.

But, ABD = ABC, and DBE to RST, equal Z. Con. therefore, the Pentagon AGH = △AGF, equal ABE, the Exceſs of ABD to DBE; i. e. of ABC to Z.

For, AG : AB :: AF : AD; and, the Triangle AGF : ABD :: AE : AD (12. 6. El.); and therefore, AG is the correſponding Side with AB, on which, a Figure ſimilar to ABC, being conſtructed, is in the Ratio required. Q. E. D.

PROBLEM IX.

To find the Side of a Squre, equal to any Number of Squares, given.

Let X, Y and Z be the Sides of three given Squares.

It is required to find the length of a Line, on which if a Square be conſtructed, it ſhall be equal in Area to all the three.

Make a Right angle ABD (Pr. 7. 1.). From the Angle B, take BA and BC, equal to any two of the given Lines, reſpectively, as X and Y, and join AC.

The Square of AC is equal to the two Squares of AB and BC, or X and Y. - - - - - - 20. 1. El.

Again. Make BD equal AC, and BE equal Z; draw ED, which is the Line required.

DEM. For, ED ſquare = the two Squares of EB and BD.
But, BD = AC; and AC ſquare = AB + BC ſquare.
Wherefore, ED ſquare = AB□ + BC□ + EB ſquare.
i. e. ED ſquare = the three Squares, of X, Y and Z. - 20. 1.

APPL. By this uſeful Problem, Quantities may be increaſed in any Proportion at pleaſure.

By means of this Problem, Carpenters form a Right angle, in framing Timber, &c. For, having made AB equal 4 feet, and BC equal three; then, if AC meaſures 5 feet, ABC is a Right angle.

N. B. The Numbers 3, 4 and 5 being multiplied, ſeparately, by any one Number, at pleaſure, produce the ſame effect. e. g.

If AB be made 6 or 9 feet, and BC equal 8 or 12 feet; then will AC be equal to 10 or 15 feet, if the Angle ABC be a right one.

For, the Square of 9 is 81; and the Square of 12 is 144, added to 81 is 225; equal to the Square of 15. - 20. 1. El.

When the Timbers are long, it is neceſſary, to apply greater meaſures; which will give the Angle with greater accuracy.

COR. Hence, a Perpendicular may be drawn, very readily, at the extremity of a Right line; by a Scale of equal Parts. As BD, perpendicular to AB. e. g.

Take, by any Scale, 3, 6, 9, or 12 equal Parts, which, ſet off, from B to A; at which Point, B, a Perpendicular to AB is required.

Then take, for Radius, 4, 8, 12, or 16 Diviſions, off the ſame Scale; and, ſeting one Point of the Compaſſes in B, make a ſmall Ark, at D.

By the ſame Scale, take 5, 10, 15, or 20 equal Parts, and with one Point of the Compaſſes, at A, croſs the former Ark, at D, and draw BD; which will be perpendicular to AB.

For, the Square of AD is equal to the two Squares of AB and BD, added together; by 20. 1. El.

By the ſame, the Side of any right-lined Figure, whatever, may be determined, which ſhall be equal to any number of ſimilar Figures, given, equal or otherwiſe.

Or it may be as elegantly and readily performed, thus.

Z is the given Figure; and V, X, and Y, the Sides of three other Figures, ſimilar to, and correſponding with the Baſe of Z.

Draw BC perpendicular to AB, the Baſe; make BC equal to V, and draw AC. Make CD perpendicular to AC, and equal to the Line X; draw AD, and DE perpendicular, equal to Y, and draw AE. On AE, if a Pentagon be conſtructed, ſimilar to Z, making it the correſponding Side with AB, it will be equal to four ſuch, on AB, V, X, and Y.

DEM. Becauſe ABC is a Right angle, $AC\square = AB + BC\square$; for the ſame reaſon, $AD\square = AC + CD\square$; and $AE\square = AD + DE\square$; conſ. $AD\square = AB + BC + CD + DE\square$s; i. e. equal to the Squares of AB, V, X, and Y. 20. 1. El.

But, all ſimilar Figures are, to each other, as the Squares of their correſponding Sides. - Cor. 2. 13. 6. El.

Therefore, the Pentagon conſtructed on AE, being ſimilar to Z, is equal to Z and three other, on V, X, and Y.

By the ſame means, a Circle may be deſcribed, i. e. the Diameter of a Circle may be determined, which ſhall be equal to any number of given Circles.

V, X, Y, and Z, being the Diameters of four Circles.

By the ſame conſtruction, as before, AE is the Diameter ſought, of a Circle equal to the four given Circles; as is manifeſt.

For, the Areas of Circles are, alſo, as the Squares of their Diameters; the ſame as the Diagonals of ſimilar right-lined Figures, or as their correſponding Sides. Cor. 1. 14. 6. El.

Therefore, the Circle, whoſe Diameter is AE, is equal to four Circles, whoſe Diameters are equal to V, X, Y, and Z.

PROBLEM X.

To divide a Square into any number of Squares, in any Ratio required.

Divide AB, a Side of the given Square, in the Ratio required, at C and D; viz. as two, three, and four.

Deſcribe a Semicircle on AB, and draw CE and DF perpendicular to AB, cuting the Circumference at E and F.

Make AG equal to CD, and draw GH perpendicular; draw the Chords AE, AH, and BF, which are the Sides of Squares in the Proportion required.

Draw BE, BH, and AF.

DEM. The Angles AEB, AHB, and AFB, are all Right.

Wherefore, the Chord AE is a mean Proportional, between AC and AB; conſequently, a Square on AE is to the given one, as AC to AB, i. e. as two to nine.

And, the ſquare of AH is to the given one, as AG to AB.

Alſo, BF is a Mean, between AB and BD; wherefore the ſquare of BF is to the given one, as BD to AB, 4 to 9; that on AH as three to nine, and the firſt as two to nine.

But, two, three, and four, are to nine, as nine is to nine; therefore, the three Squares, on AE, AH, and BF are, together, equal to the ſquare of AB. - Ax. 1. 5.

Thus may any right-lined Figure, whatever, as well as Squares, be divided into any number of ſimilar Figures, in any proportion to the given one, and to each other; dividing any Side in the Ratio required.

Note. If there are four or more Diviſions; all the intermediate ones muſt be transfered to one Extreme or the other, as (in this Example) CD is transfered to AG.

PROBLEM XI.

To find the Side of a Square, equal to the Difference between two given Squares.

X and Z are the Sides of the given Squares.

Draw AC indefinite; in which, take AB equal X, and BC equal Z. Make ACD a Right angle, drawing CD, indefinite. On B, with the Radius AB, deſcribe the Ark AED, cuting CD at D; CD is the Side of a Square, equal to the difference between the ſquares of AB and BC, or X and Z. Draw BD.

DEM. BD (equal AB, equal X) is the Hypothenuſe of the right-angled Triangle DCB; the ſquare, of which, is equal to the ſquare of BC + ſquare of CD. by 20. 1. El. Conſequently, BD (equal X) ſquare, exceeds BC (equal Z) ſquare, by the ſquare of CD.

PROBLEM XII.

The Sum of the Side and Diagonal of a Square being given, to determine them ſeverally.

AB is the meaſure given, which is ſuppoſed to be equal to a Side and the Diagonal of a Square, in one Sum; it is required to divide AB, determining each, ſeverally.

Deſcribe a Semicircle on AB, the Sum given; and, from the Center, draw CD, perpendicular, cuting the Arch at D. Biſect AC; and, with the Radius DE, deſcribe the Ark DF, i. e. make EF equal DE, and draw DF, which is equal to the Diagonal; the Difference between DF and AB is, conſequently, equal to a Side of the Square; for, AB is the Sum of them both.

Or, it may be thus performed, and demonſtrated.

Make a Right angle at either Extreme of AB, as ABC; make BC equal to AB, and draw AC.

On C, with the Radius CB, deſcribe an Ark, through B, and cuting AC at D. Draw DE perpendicular to AC, cuting AB as required; for AE is the Diagonal, and EB a Side of the Square.

AD, the Difference between the Diagonal, AC, and AB, as the Side of a Square, is the Side of the Square ſought; of which, AB, is the Sum of both.

DEM. For, ADE is a Right angle, and DAE half right, equal AED; conſequently, the Triangle ADE is Iſoſceles, and AD is equal DE.

But, EB=ED; for they are Tangents to the Circle of which CD is Radius (C. 2. 16. 3. El.); wherefore, $AE\square = 2EB\square$; for it is equal to $AD\square + DE\square$; and therefore, AE is the Diagonal, and EB is the Side of a Square, AB being the Sum of both.

To determine them arithmetically.

Extract the Square root of twice the ſquare of the Sum, and ſubtract the Sum from that Root; the remainder is the Side required ($AC - AB = AD$); which being ſubtracted from the Sum given, the remainder is the Diagonal; $AB - AD = AE$. For, $AE\square = AD\square + DE\square$; i. e. to twice EB ſquare.

PROBLEM XIII.

The Difference between the Side and Diagonal of a Square being given; to determine both.

AB is the Difference given; how to determine them.

Make ABC a Right angle, and BC equal to AB. 7. 1. Draw AC, and produce it, indefinite; make CD equal to CB, equal AB. Produce AB, and draw DE, making ADE a Right angle. AD is a Side, and AE the Diagonal of a Square, of which, AB is the Difference, given; as is obvious from the Figure; which is the ſame as the foregoing, differently poſited, and from which the Demonſtration is evident.

DEM. The Triangle ABC being Iſoſceles, right-angled at B, and CD was made equal to CB; and, the Angle ADE a right one; conſequently, the Triangle ADE is ſimilar to ABC, and DE is equal to AD. - P. 9. 1. El.

Alſo; becauſe CD=CB, and the Quadrilateral BCDE is right angled at B and D, BE=DE; for, CE being drawn, the Triangles BEC, CED are congruous.

Therefore, AB is the Difference between DE (equal AB) a Side, and AE the Diagonal of a Square, which were required to be determined.

They may be thus determined, arithmetically.

To the Difference given, add the Square root of double the ſquare of the Difference, for the Side (AB + AC=AD); ſquare the Side found, double it, and extract the Square root, which gives AE, the Diagonal. - - 20. 1. El.

PROBLEM XIV.

The Hypothenuſe and the Area of a right-angled Triangle being given, to find the Sides containing it.

Z is the Area given, and AB the Hypothenuſe.

It is immaterial in what Form or Figure the Area is given; as Z, let it be reduced to a Rectangle, RS. 3. 6.

Double RS; as RT.

Find another Rectangle equal to RT, the meaſure of one Side being AB (5. 6.). Deſcribe a Semicircle on AB, and draw a Perpendicular from C, the Center.

Make CD equal TU (the other Side, found, of a Rectangle equal to twice Z). Draw DE parallel to AB, cuting the Arch at E; draw AE and EB.

AEB is the Triangle required, whoſe Area is equal to Z.

Dem. AEB is a Right-angled Triangle (12. 3.) whoſe Area is AC × CD, half AB × CD. But, AB × CD is equal to the Rectangle RT, which is double Z; Con. wherefore, AC × CD = Z, the given Area; and it is the Area of the Triangle AEB.

Draw CE, and EF perpendicular to AB.

Then, CE□ − EF□ = CF□; and AC + CF = AF. But, AF□ + EF□ = AE□; and AB□ − AE□ = BE□. Conſequently, the Square roots, AE and EB, are the Sides required.

An Iſoſceles, right-angled, Triangle (AGB) whoſe Hypothenuſe, given, is AB, has the largeſt Area poſſible.

The Area and the Hypothenuſe being given, in Numbers; how to find the Sides containing the Right angle, arithmetically.

Double the Area given, and divide it by the Hypothenuſe; it gives CD. For AB × CD is double the Area of AEB (19. 1. El.). Square half the Hypothenuſe (AC, equal CE); from which, ſubtract the ſquare of CD (eq. EF); the Square root of the remainder gives CF; which added to half the Hypothenuſe (AC) gives AF. Square AF and EF, and add them together, the Square root of which is AE, one Side. AF ſubtracted from AB, gives FB; the ſquare of which, add to EF ſquare; extract the Square root, we have BE, the other Side.

PROBLEM XV.

Either Segment of the Diagonal of a Rectangle, made by a Perpendicular from an Angle, being given, together with the continuation of the Perpendicular to the oppoſite Side; to determine the Sides.

AB is the whole Diagonal, which is cut by a Perpendicular, at C; and Z is the continuation of the Perpendicular.

Firſt; AC is the Segment given, to determine the Sides.

Draw CD perpendicular to AB, and produce it indefinite, on the other Side; i. e. make ACD a Right angle, make DC equal Z, and produce DC. Draw AD, making a right-angled Triangle. - - - - - 17. 1.

Make DAE a Right angle; AE, cuting DC produced, is one Side; make AEB a Right angle, cuting AC produced at B; EB is the other Side of the Rectangle, required.

This is obvious from the Conſtruction, the Angles at A and E being Right; conſequently, as AB is a Diagonal, AE and EB are Sides of a Rectangle, of which it is the Diagonal.

In this Caſe, it is but to find two third Proportionals; as by the firſt method of Pr. 2. S. 5; firſt, to CD and AC, then, to AC and CE; that is, to continue a progreſſive Proportion, to CD and AC, to four Terms; the two (CE and CB) ſo determined, give the Sides of the Rectangle.

For, CD : AC : CE : CB ÷÷; wherefore, if the other Segment, BC, were given, it is but to find two mean Proportionals (AC and CE) to BC and CD.

AC and CD being given in Numbers, the Sides are determinable, by a geometrical Progreſſion, in Numbers. For, AC□ + CD□ = AD□; conſequently, the Square root of AC and CD ſquare, in one Sum, gives AD; alſo the Squares of AC and CE being added together, in one Sum, the Square root gives AE; and, the Square root of CE and CB ſquare, in one Sum, gives EB.

It muſt be obvious that, from either Segment of a Diagonal given, the whole is determinable, as well as the Sides; being the Sum of the 1ſt and 3rd Terms, AC and CB, in geometrical Progreſſion; CE being the Mean.

PRACTICAL

GEOMETRY.

SECTION IX.

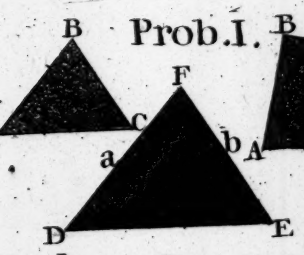
On geometrical Projection, *applied to the* Five regular Solids, *or* platonic Bodies.

AS the three laſt Books (13th, 14th and 15th) of Euclid treat, wholly, of the five regular Solids, for the ſole inveſtigation of which, it is ſaid, he compiled his Elements; I conceived that a clear Deſcription, and geometrical Conſtruction of them would be acceptable to the Public; either as a regular Section, or an Addenda to this practical Treatiſe. For, although I cannot perceive quite ſo much Divinity in, as the Antients atributed to them, there is, nevertheleſs, ſomething extraordinary, and worthy to be known, in their Conſtruction and Projection; and ſtill more in their Properties, and the Proportions of one to another, which runs through the whole; inſomuch that, I could not deviſe any Subject more fit, or ſo proper to conclude this Work with. I have received great pleaſure, and ſome advantage, from my own ſtudy and contemplations on them; therefore, have no doubt of their being well received by the Public, as I have no knowledge of any ſmall Tract which treats on them, at all; but, to do it familiarly, and ſatisfactorily, is not to be done in a curſory way.

In ſome Treatiſes on Perſpective, they have been handled rather elaborately, eſpecially by Highmore; as if no fitter Subject could be found, for diſplaying Perſpective on: But, although Brook Taylor himſelf has ſet the Precedent, I can by no means think ſo; being well aſſured, that (excepting in the Principles) there are other Objects far preferable. Indeed, none but Hamilton has conſidered their Properties, and Affinity to one another, or conſtructed them geometrically (nothing more being neceſſary towards delineating them perſpectively, than the Figure and the inclination of one Face to another) ſo that, but little knowledge of their Properties can be obtained from thoſe Works. And, excepting Dr. Barrow, I believe no modern Geometrician has favoured us with them, from Euclid; and very few, I am of opinion, give much attention to them, as treated by him.

The delineation and inveſtigation of the regular Solids are not only and merely amuſing, but I am well convinced, that they are of great utility to the practical Geometrician; and eſpecially to the Meaſurer, of ſolid Bodies. I adviſe the young Student, therefore, to make himſelf familiar with them, by delineating them in various Poſitions, and Situations to the Plane of Projection; for, although it be only by way of amuſement, at firſt, he will find ſomewhat more than entertainment to the Mind, more advantage to reſult from an intimate acquaintance and familiarity with them, than he imagined. I ſhall therefore ſay no more by way of Preface, or Preamble; for, without a propenſity in the Student, all that can be ſaid here, to recommend the ſtudy of them, will avail little or nothing: Being treated familiarly, thereby rendered facile, is the beſt recommendation, of which, each Perſon will judge for himſelf. I have done the beſt I can, to make them underſtood; and doubt not, by giving the attention requiſite, that a competent knowledge of all that is eſſential in them may ſoon be acquired.

Prob. I.



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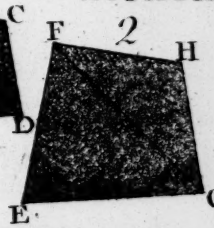

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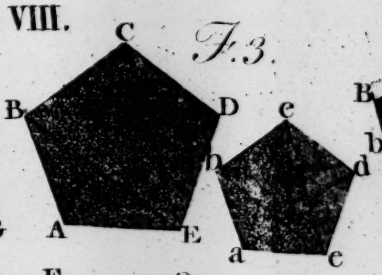

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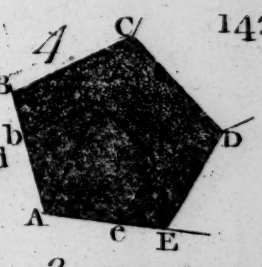

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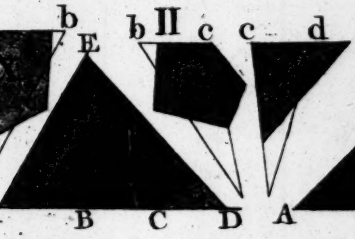

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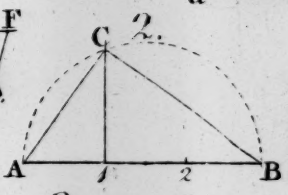

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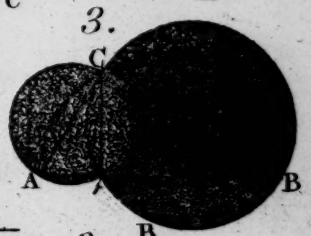

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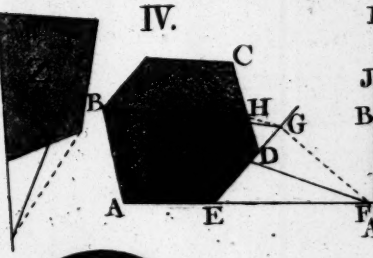

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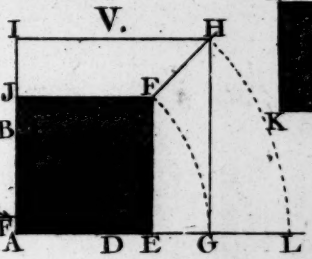

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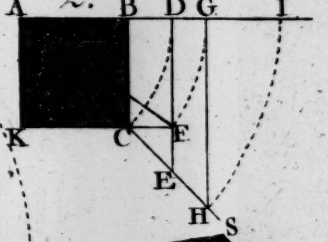

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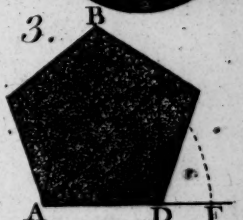

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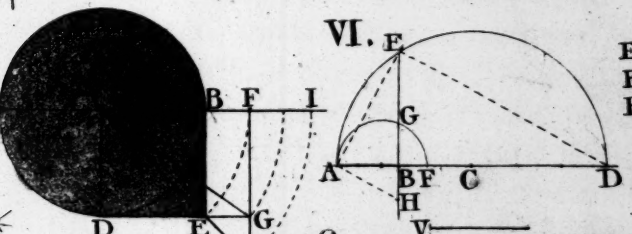

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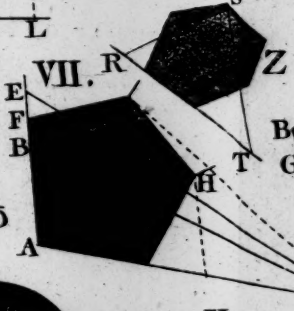

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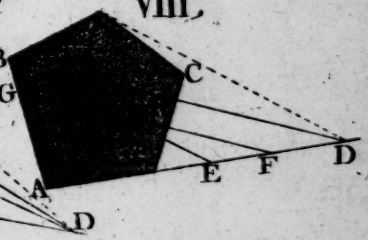

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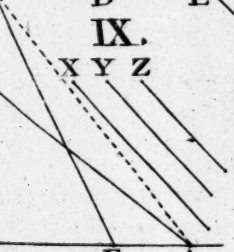

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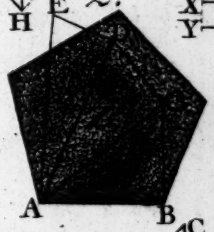

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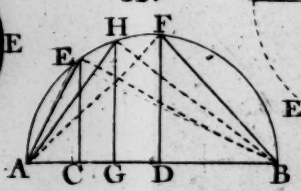

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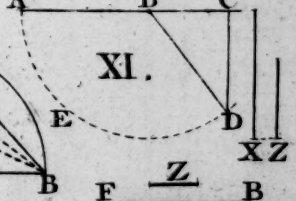

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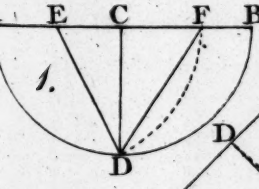

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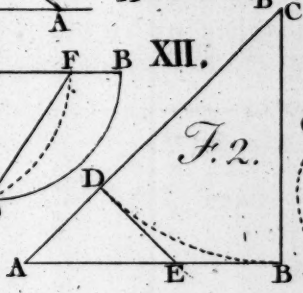

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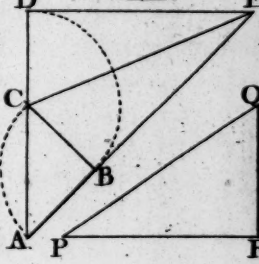

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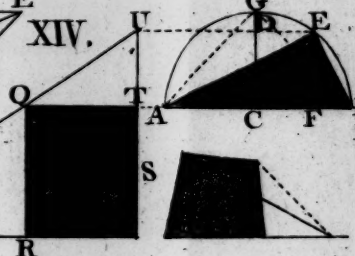

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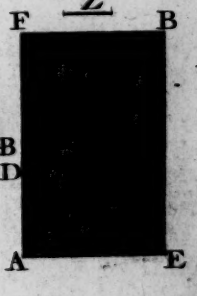

A REGULAR SOLID is one whoſe Faces are all equal and ſimilar Figures, and equally inclined to one another, whoſe Angles would be all in the Surface of a circumſcribing Sphere; or, a Sphere being ſuppoſed to be inſcribed, it would touch every Face in its Center.

No more than three ordinate Figures can form a regular Solid; viz. Triangles, Squares, and Pentagons; of the five, three are equilateral Triangles, one of Squares, and one of Pentagons. For, as by the 12th of Book 7, all the plane Angles, forming a ſolid Angle, are neceſſarily leſs than four Right angles; conſequently, no other ordinate Figure can form a ſolid Angle. Becauſe, the Angle of a Hexagon being equal to two Angles of an equilateral Triangle; viz. 120 Degrees, and three times 120 is 360 equal four times 90, i. e. four Right angles; therefore cannot form a ſolid Angle. And it is evident, that not leſs than three plane Angles can form a ſolid Angle (Def. 6. 7.); wherefore, a ſolid Angle may be formed of three, four and of five equilateral Triangles (being each of 60 Degrees) one leſs than three hexagonal Angles. Three Right angles form a ſolid Right angle; and the pentagonal Angle, being 108 Degrees, three of them make 324, which is 36 leſs than four Right angles (the largeſt of all); ſo that, it is manifeſt, no more than five regular Solids can be formed.

DEFINITIONS OF THE SOLIDS.

DEF. I. The firſt is called a TETRAEDRON*. It is a Pyramid formed of four equilateral Triangles; having as many Angles, and ſix Sides. Each Side being com-

* This Term, I think, is improper; for, as all the reſt are called by the number of their Faces (this only is not ſo, but by the Angles of a Face) I think it ſhould have been termed a *Quatra*, or *Quadraedron*.

mon to two Faces (their mutual Interſection) conſequently, as each Face is bounded by three Sides, and four times 3 is 12, half of which is the number of Sides of the Solid; and thus, the number of Sides in each Solid may be aſcertained.

II. The next is called a HEXAEDRON. It is a Cube, or Parallelopiped, compoſed of ſix Squares, all at right angles with each other, ſave the oppoſite ones, which are parallel. See Def. 9 and 10. 7. El.

The Hexaedron, having ſix Faces (each bounded by four Lines) has twelve Sides and eight angles; for each Side has two Extremes, and three terminate in each Angle; conſequently, the Angles are to the Sides as two to three, eight being two-thirds of twelve.

III. The third is called an OCTAEDRON. It is two quadrangular Pyramids, whoſe Baſes are Squares, which do not make a Face of the Solid; for they are equilateral Triangles, eight in number, and conſequently it has twelve Sides (the ſame as a Cube) though but ſix Angles, half the number of Sides; becauſe four Extremes are in each Angle.

IV. The fourth is called a DODECAEDRON; having twelve Faces (Pentagons) each being bounded by five Lines, it has therefore 30 Sides, and 20 Angles; three Sides terminating in each. This does not come within any Definition of a geometrical Solid.

V. The fifth, and laſt, is called an ICOSAEDRON. It has twenty Faces (equilateral Triangles) and conſequently, thirty Sides, the ſame number as the Dodecaedron, though but twelve Angles, becauſe, five Sides terminate in each Angle; therefore, they are, in number, to the Sides, as two to five; 12 and 30.

This Solid forms as many pentagonal Pyramids, as it has Angles, but not ſo many diſtinct ones; for, each has two Faces common to each contiguous Pyramid, or it muſt have ſixty Faces; three times the number it has.

VI. THE PLANE OF PROJECTION.

IN this laſt Section, I have ſhewn how to project, that is, to delineate or draw, geometrically, the five Solids, in plano, or on a Plane, in various poſitions to the Plane; which is called the *Plane of Projection*.

VII. The Plane of Projection may be either horizontal or vertical; being horizontal, the Projection thereon is called the *Ichnographic Projection*, the *Plan*, or *Seat* of the Object projected thereon, by Right lines, perpendicular thereto, from every Angle of the Object; as a b c d e f. No. 1.

The Lines Aa, Bb, Cc, &c. taken collectively, are called the SYSTEM of *projecting Rays*.

VIII. When the Plane of Projection is upright; as No. 2; the projecting Rays, Aa, Bb, &c. being perpendicular to it, are then horizontal; in this caſe (which differs in nothing from the former, but the poſition of the Plane to the Horizon; for, by turning them, either becomes the other) the Projection is called *Orthographic*; which exhibits the Elevation of an Object above the Ground Plane; yet, it is the Seat of the Object on that Plane.

N. B. The Seat of a Point on any Plane (if it be not in the Plane) is the Point in which a Line from the Point, perpendicular to the Plane, cuts it; as a, of A. And the Seat of a Line, not in the Plane, is a Line joining the Seats of its Extremes; as af, of AF. abf is the Seat of the Face ABF, on the Plane X; and alſo on the Plane V. The Lines AB, BC, &c. which bound the Faces of a Solid, are the *Sides* of it.

When the Projection is made by a Plane cuting the Object, perpendicular to the Horizon, or the Plane on which it reſts, it is called a SECTION; which may be

made variouſly; but moſt generally, it is perpendicular to the Faces which are cut; or through the Sides (common to two Faces) or both, frequently.

Theſe Preliminaries being ſettled, which are neceſſary to a clear underſtanding the different kinds of Projection, we now proceed to ſhew how to project Solids on a Plane; exhibiting the three Dimenſions, of length, breadth, and thickneſs.

PROBLEM I.

A Side being given, to deſcribe the Ichnography of a Tetraedron; its Seat on a horizontal Plane.

This Object being bounded by four equilateral Triangles, it muſt neceſſarily reſt on one of them; therefore, nothing more is requiſite, than to take the given Line AB in the opening of the Compaſſes, and, with that Radius, make an equilateral Triangle, ABC (Pr. 14. S. 1.) which is the Seat of a Tetraedron.

Find D the Center of a circumſcribing, or an inſcribed Circle (Pr. 3 or 4. 4.); from which, draw Lines to each Angle, DA, &c. D, being conſidered, as the Vertex of a Pyramid, elevated out of the Plane, of its Baſe; and A, B, C the Seats of its Angles on a Plane, or the Angles themſelves.

AD, BD, and CD may be conſidered as the three Sides, from the Baſe to its Vertex, and ADB, &c. the three elevated Faces projected on the Plane of its Baſe; which is the Figure it exhibits to an Eye perpendicular over the Vertex.

PROBLEM II.

To find the Altitude of a Tetraedron, and to determine the Inclination of one Face to another.

Having drawn the Plan, ABC, draw a Perpendicular from any Angle to the oppoſite Side, as AE; in which, find D the Center, as before; and draw DF perpendicular to AE (Pr. 8. S. 1.). Then, with the Radius AE, on E, deſcribe the Ark AF, cuting DF at F, and draw EF.

Or, without drawing DF, take AB for Radius, and, on A deſcribe the Ark BF, cuting the former at F.

DF is the Altitude, and AEF the Inclination of the Faces; which is readily conceived, by imagining the Triangle AFE turned up, on AE perpendicular; in which Poſition, it exhibits a Section made through two Faces, in which AE and FE are Perpendiculars, and through one Side, AF.

This Section exhibits, alſo, an Elevation of this Object; in which, only one Face is ſeen, obliquely.

Figure 2 is another Elevation, exhibiting a Face, having one Side direct; in which, AB, the Baſe, is a Side of the Solid; two are thus determined.

Biſect AB, and draw CD perpendicular, make CD equal to its height (Fig. 1.) and draw AD and CD, forming an Iſoſceles Triangle ADB.

If the Perpendicular CD be conſidered as a Side of the Solid, another Elevation is exhibited, ſeeing two Faces, equally; which, three, are all the regular Elevations this Object can exhibit; oblique ones may be various, which are not at all neceſſary to be conſidered.

PROBLEM III.

A Side (AB) being given, to project a Hexaedron.

The Hexaedron being a Cube whoſe Faces are Squares at right angles with each other; there is nothing more to be done, than to deſcribe a Square ABCD; - Pr. 16. 1. which is either its Plan or Elevation, being conſidered as horizontal or vertical.

The Dimenſions of this Solid are the ſame; in length, breadth, and height.

Another Elevation, exhibiting two Faces, equally, is the ſame as a Section, through any two oppoſite Sides, and the Diagonals of two oppoſite Faces; as the Rectangle ABCD. (2.). Being conſidered as one Plane, it is a Section; but if EF be drawn parallel to AB, biſecting BC and AD, it is an Elevation of two Faces.

PROBLEM IV.

To project the Hexaedron, on an Angle exhibiting three Faces, equally.

On the Side given, having conſtructed a Square, or a right angled Triangle only, ABC; on BC make an equilateral Triangle, BDC.

Draw a circumſcribing Circle (Pr. 4. 4.); and biſect each Ark (10. 1.) in E, F, and H. Draw the Chords, BE, DE, &c.; alſo, to the Center G, draw BG, CG, and DG, which compleats the Projection required.

This Projection is the ſame, whether it be conſidered as ichnographic or orthographic; the Figure is a regular Hexagon, and each Face a Rhombus.

PROBLEM V.

On a Side given, to project the Ichnography of an Octaedron.

On AB, the given Side, make an equilateral Triangle, ABC; and draw a circumſcribing Circle, as in the foregoing Problem.

Biſect each Ark, in D, E, and F; and draw the Chords AD, DB, &c.; alſo, draw DE, DF, and EF, which compleats the Plan.

If ABC be conſidered as the Face on which it reſts, DEF is its oppoſite Face, parallel to it, equal and ſimilar; and AD, DB, &c. are Sides of the Object, joining the upper and the under Faces; ADE, &c. being the declining Faces, from the upper Face, ADB, &c. are the reclining Faces from the Bottom, or Seat; as if the Solid were tranſparent, and thoſe Faces, &c. ſeen through it.

PROBLEM VI.

To determine the Altitude, and Inclination of the Faces to each other, in an Octaedron.

Take AB equal to the Perpendicular of an equilateral Triangle, which is a Face of the Solid; on A, with that Radius, deſcribe an Ark; with the Radius of a Side, on B, interſect the Ark, at C, and draw AC and BC.

On the common Baſe BC, make a ſimilar Iſoſceles Triangle BDC; and, draw DE, perpendicular to AB, produced.

DE is the Altitude required; and DBE is the Angle of Inclination of one Face to another.

The Plane of Projection may be conſidered as cuting four Faces of the Solid, perpendicularly, through the Perpendicular of each Face; of which, AB, AC, CD, and BD are Sections.

This conſtruction is the true orthographic Projection of it; in which, no more than two Faces are ſeen.

Another is thus conſtructed, exhibiting three Faces.

Make a Rectangle, ABDC, on its Side and Altitude. 17. 1. Biſect AB and CD (10 1.) in E and F, and draw AF and BF, CE and DE.

If AFB be conſidered as the hither Face, CED is its oppoſite, ſeen through; or CED being the hither one, AFD is the oppoſite; for either may be the other.

AB and CD are Projections of the upper and under Faces.

PROBLEM VII.

To project the Octaedron, reſting on an Angle.

On the given Side conſtruct a Square ABCD; draw the Diagonals, AC and BD, and it is done.

This Projection is either ichnographic or orthographic, by ſeting it on an Angle.

Fig. 2 is another Elevation, the ſame as in the laſt Problem, on an Angle.

Theſe are all the regular Projections of the Octaedron.

The Inclination of its Faces, are readily determined thus.

Take AB equal to the Diagonal of the ſquare of a Side; and on each Extreme, with the Radius of the Perpendicular of a Face, deſcribe Arks, interſecting at C, forming an Iſoſceles Triangle ABC.

Produce either Side, as BC; ACD is the Angle of their Inclination.

For AB is equal to AC, or BD, Fig. 1. and AC, BC, &c. are each equal to the Perpendicular of a Face. EC is a Side.

N. B. The Inclination of the Faces of the Tetraedron and Octaedron is the ſame (72 Degrees). The interior Angle of one is the exterior of the other; and, the Side of the Octaedron is equal to a Perpendicular in a Face of the Tetraedron.

The Hexaedron and Octaedron are equal in height, ſo that, the ſame Circle will circumſcribe a Face of either; which, in the firſt, is a Square, in the other, it is an equilateral Triangle. And it is remarkable, that the Hexaedron, having but ſix Faces, has eight Angles, the other but ſix, although eight Faces; yet they have, each, twelve Sides; but, four Sides terminate in each Angle of one, three only in the other.

Hitherto, the Projections of the firſt three Solids are very ſimple; we now proceed to the Dodecaedron, which affords a little more variety and ſomewhat more of difficulty. The two laſt (which follow) depend, greatly, on the diviſion of a Line in extreme and mean Proportion, with which, I preſume, the Student is already well verſed; if not, I adviſe him to reconſider that Diviſion of a Line, as he will-find it, in the 11th of B. 2. El. or practically, in the fifth Section, Prob. 3; and applied to the Pentagon in the 34th and 35th Prop. of the 6th of Elements.

PROBLEM VIII.

On a Side given, to project the Ichnography of a Dodecaedron.

On AB, the given Side, deſcribe a Pentagon. Pr. 1. S. 2.

Find the Center (9. 4.); and draw a Line through any Angle, as E, from the Center.

Make EF to EQ as the leſſer Segment to the greater, of a Line divided in extreme and mean Proportion. With the Radius QF deſcribe a Circle; and, radiating from the Center (Q) draw AG, BH, &c. cuting the Circumference, in G, H, I, and K.

Biſect the Arks FG, &c. in L, M, &c. and draw the Chords FL, GL, &c. which compleats the Plan, of either the upper or the under Faces; ſuppoſe it the upper half.

Then, deſcribe a Circle through the Angles of the Pentagon ADB; and applying a Ruler to the Center, Q, draw LR, MS, &c. and join RS, ST, &c. which compleats the under Faces, as ſeen through, being tranſparent.

PROBLEM IX.

To delineate the Elevation; or, to deſcribe the orthographic Projection, of a Dodecaedron.

X is a Side given.

Having deſcribed a Pentagon on the given Side, (1. S. 2.) take AB equal to a Perpendicular of the Pentagon; which, divide in extreme and mean Proportion, at C.

Produce AB, making BD equal AB; and, with that Radius, on B, deſcribe the Ark DE; alſo, on D, with the Radius of a Diagonal, cut it at E, and draw BE.

Produce DE to F, making EF equal X, the Side given. Draw AG, parallel to EF, and make it equal; then, draw FH and GH parallel, reſpectively, to AB and BE, cuting at H.

Make BI equal to BC, alſo HK and HL; and draw HC, HI, BK and BL, cuting at M and N; and, laſtly, join MN.

The Figure AHE is the true orthographic Projection of a Dodecaedron, whoſe Side given is X; on a vertical Plane; to which, the Sides AG, EF, and MN are parallel; and, four Faces, of which, AB, BE, FH, and GH, are Sections, are perpendicular to the Plane.

A Plane paſſing through AG and EF, being parallel to, may be conſidered as the Plane of Projection, cuting four Faces, perpendicularly; and exhibiting the Inclination of one Face to another, as AB and BE; either of which, as AB, being produced, DBE is the Angle of their Inclination.

OR, it may be thus projected. Make an Iſoſceles Triangle, ADG (on BA produced) whoſe Baſe, AG, is a Side, and AD, DG, each equal to BC, the greater Segment of AB divided as above; which, is the greater Segment of a Perpendicular, cut by a Diagonal parallel to that Side on which the Perpendicular falls.

Then, producing DG, make GH equal to AB; and draw BE parallel thereto, alſo FH to AB, and equal amongſt themſelves; EF will be equal and parallel to AG. The remainder is done as deſcribed in the foregoing.

OTHERWISE, thus, on AB; a perpendicular of a Face.

AB being divided in extreme and mean Proportion, make, on CB (the greater Segment) an Iſoſceles Triangle, CDB, whoſe equal Sides are CB, BD, and the Baſe, CD, the Diagonal of a Pentagon on the given Side; by which is obtained the Inclination of two Faces.

On CD conſtruct a Square, CDIK (16.1.); and, having produced BD, draw FH and GH through the other two Angles of the Square (I and K) parallel, reſpectively, to AB and BE, cuting at H.

Make BE equal AB; and draw AG and EF, parallel to CK and DI.

Then, draw Lines, from B and H, reſpectively, to the oppoſite Angles of the Square (as in the Figure) and draw MN, between their interſecting Points, which compleats this Projection, the ſame as before.

In this laſt Projection, there is not one Line, exterior, or out of the Figure; nor a Line drawn that is redundant, but the Sides of the Square; which are Diagonals of the four Faces exhibited to view.

A Line (OP) paſſing through the Angle E, cuting AB and HF (produced) perpendicularly, meaſures the diſtance between two oppoſite Faces; and it is cut in extreme and mean Proportion, in the Point E. OE:EP::EP:OP.

Or, it may be thus conſtructed, reſting on a Side, as elegantly, and as facile, as by any of the preceding Methods.

Having conſtructed a Pentagon, on any Line given, for a Side, take AB equal to a Diagonal. Or, without making a Pentagon, divide the given Side in extreme and mean Proportion (3. 5.) and, to it add the greater Segment; which is equal to a Diagonal.*

Deſcribe a Square, ABCD, of the Diagonal found, 16. 1. and draw the two Diagonals; which, produce at both Extremes, indefinite. Make AF to AE as the leſſer Segment to the greater, of a Line divided in extreme and mean Proportion. - - - - - - C. to 35. 6. El.

Draw FG and FI parallel, reſpectively, to AB and AD, and complete the Square FGHI, parallel to, and concentric with ABCD. Draw KL, through the Center (E) parallel to FI; and, from K and L draw Lines through the Angles of the interior Squares (KC, KD, &c.) as in the Figure; and, laſtly, draw KA, KB, alſo LC, LD, and MN, joining their Interſections, as before, which compleats it.

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* See Prop. 34, and Cor. to 35. B. 6. of Elements.

PROBLEM X.

To project another Elevation of the Dodecaedron, parallel to the Diagonals, exhibiting five of its Faces.

A Side (AB) being given, find a Diagonal, as in the laſt.

Having biſected AB, make CD and CE each equal to half the Diagonal; alſo, make DF and EG each equal to half AB, and, draw FI and GH perpendicular to FG; make either of them equal to the diſtance between two oppoſite Faces (from the laſt) and draw HI parallel to FG.

Divide FI in extreme and mean Proportion, and from both Extremes, make IK, and FL each equal to the leſſer Segment; through which, draw Lines parallel to FG.

Draw DO and EP perpendicular; or parallel to GH; alſo CQ. Then, drawing Lines joining the Points K, R, S, T, and N (as in the Figure) it will be compleated.

DE, and OP, are Sections of the Face it reſts on, and the oppoſite, parallel thereto; which are perpendicular to the Plane of Projection.

AKD, ASB, and BNE, are three Faces, reclining from the Face it reſts on; and ORQ, QTP, are upper Faces, inclined to the uppermoſt, to OP.

No more than one Side, AB (and alſo RT, a Diagonal) is, in this Projection, parallel to the Plane it is projected on; KL and MN are Sides, in a Plane perpendicular to it; and DE, OP, are meaſures of the Diagonals of the under and the upper Faces, parallel to AB.

The five Faces on the other Side, formed by Lines joining correſponding Points, may be ſuppoſed as if ſeen through, being tranſparent; exhibiting the ſame Figure inverted; ſhewing three Faces above, and two below.

PROBLEM XI.

To project the Ichnography of an Icoſaedron.

A Side (AB) being given, make an equilateral Triangle, ABC; which may be either the upper Face, or that it reſts on.

Find D, the Center of an inſcribed or circumſcribing Circle; and, through any Angle, as A, draw DE, making AE to AD, as the leſſer Segment to the greater, of a Line, divided in extreme and mean Proportion. 35.6. El.

With the Radius DE, deſcribe a Circle; and draw Lines through the other two Angles, B and C, from D, cuting the Circumference, at F and G. Biſect the Arks EF, EG, and FG, at H, I, and K, and draw the Chords EH, EI, &c. Then, drawing AH, AI, BH, BK, and CI, CK, the Plan is completed.

The Ichnography of the Icoſaedron is a regular Hexagon, that of the Dodecaedron a Decagon; each being bounded by half the number of Sides the other has Faces.

ABC being conſidered as the upper Face of this, as a Solid, make another equilateral Triangle, LMN, concentric with ABC, but in a ſubcontrary poſition (circumſcribing the former by a Circle, and biſecting each Ark on a Side of it, at L, M and N); then, joining KL, LF, LG, &c. as in the Figure, we have all the under Faces; ten, in number, the ſame as in the other, in an inverted Poſition to them; which may be conſidered, as being ſeen through them.

PROBLEM XII.

To project the Elevation, or Orthography of the Icoſaedron; on a Plane parallel to a Perpendicular of the Face the Solid reſts on.

Draw AC indefinite, repreſenting a Section of the Plane; in which take AB equal to the Perpendicular of a Face, and make BC equal to AB.

Make an Iſoſceles Triangle, CBD, whoſe Baſe (CD) is equal to the difference between a Side of the Figure, and the Diagonal of a Pentagon, conſtructed on a Side (equal EF, in the Ichnography); which gives the Angle of the Inclination of one Face to another, equal CBD.

Produce CD, making DE equal to the given Side; draw EF parallel to AB and make it equal alſo; then, draw FG and AG, reſpectively parallel to BD & DE, interſecting at G.

Draw FA and FD, alſo, BE and BG, interſecting at H and I; join H and I, which compleats the Orthography required; in which, three Sides, AG, DE, and HI are parallel to the Plane of Projection, which may paſs through the Sides AG and DE.

THIS Elevation may be otherwiſe projected, as follows.

On AB, the Perpendicular of a Face, make an Iſoſceles Triangle ABD, whoſe Baſe (AD) is the Diagonal of a Pentagon, conſtructed on a Side (equal EF, in the Ichnography); which alſo gives the true Inclination of two contiguous Faces, in this Solid, by producing AB.

Through the Angle B, draw KL parallel to AD; and, BF perpendicular thereto; alſo, through A and D, draw KN and LM parallel to BF, and complete the Square KLMN. Draw EF and FG parallel, reſpectively, to AB

and BD. Draw a Line through the Center (Q) parallel to AD; and, laſtly, draw AF and FD, BE and BG, cutting it at H and I, which compleats the Elevation.

There is another Method of projecting it, and the Dodecaedron alſo, more mechanically, from the Plan or Ichnography; drawing Lines perpendicular to AP, from Points therein, correſponding with D, E, F, G, H, and I; the affinity of which is perceivable in and practicable from the moveable Scheme; by which means, various Elevations may be conſtructed. Having found the Diſtance between two oppoſite Faces (equal OP) and divided it in extreme and mean Proportion, at D, making PR equal PD, the leſſer Segment; which is the ſame in both Solids, in equal circumſcribing Spheres; conſequently, the Baſe of the ſame Segment, i. e. the ſame Circle circumſcribes both, the pentagonal Face of the Dodecaedron, and the triangular Face of an Icoſaedron; as was obſerved of the Hexaedron and Octaedron, a Square and equilateral Triangle.

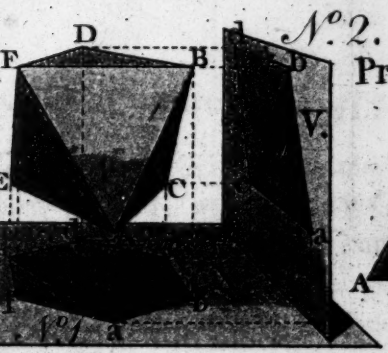
PROBLEM XIII.

To project the Orthography of an Icoſaedron on a Plane parallel to a Side of the Face it reſts on.

A Line (DE) being drawn, at diſcretion, take AB equal to a Side, and biſect it, at C.

Make CD and DE each equal to half the Diagonal of a Pentagon, whoſe Side is AB; and draw EF, perpendicular to DE.

Make EF equal to the Diſtance between two oppoſite Faces, from the former (OP; equal AK, BI, or CH, in the Plan); and complete the Rectangle DEFG.



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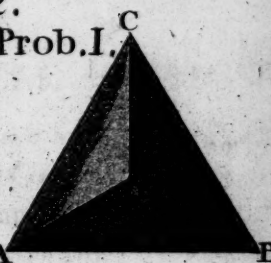

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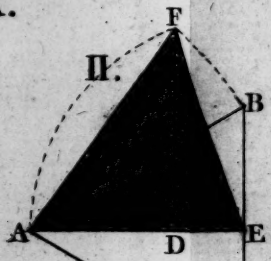

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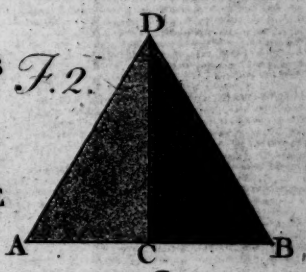

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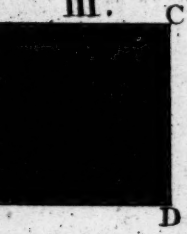

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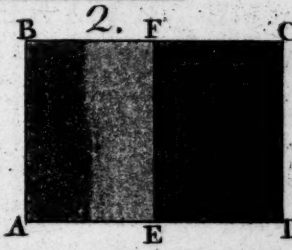

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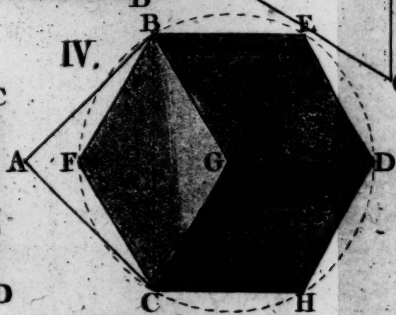

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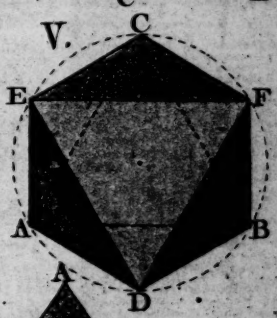

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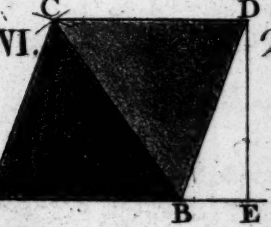

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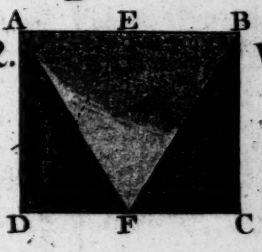

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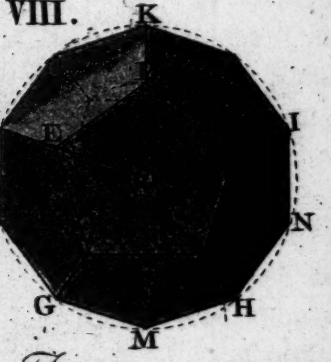

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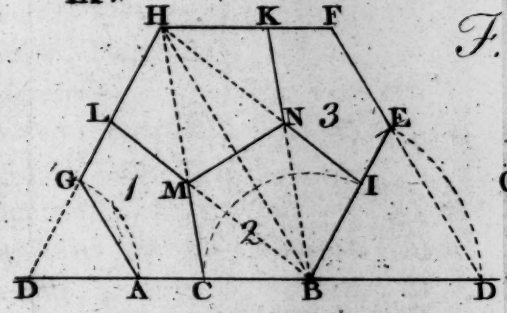

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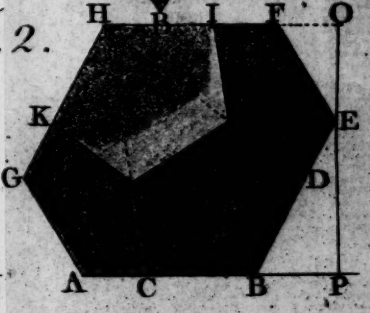

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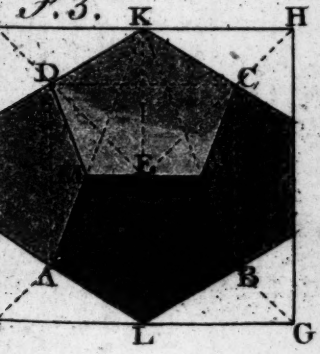

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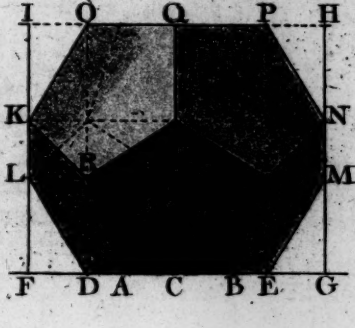

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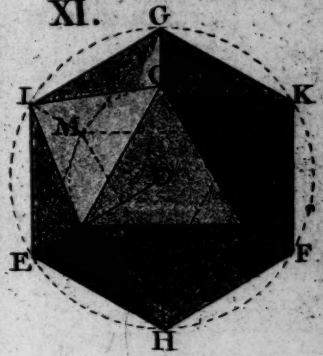

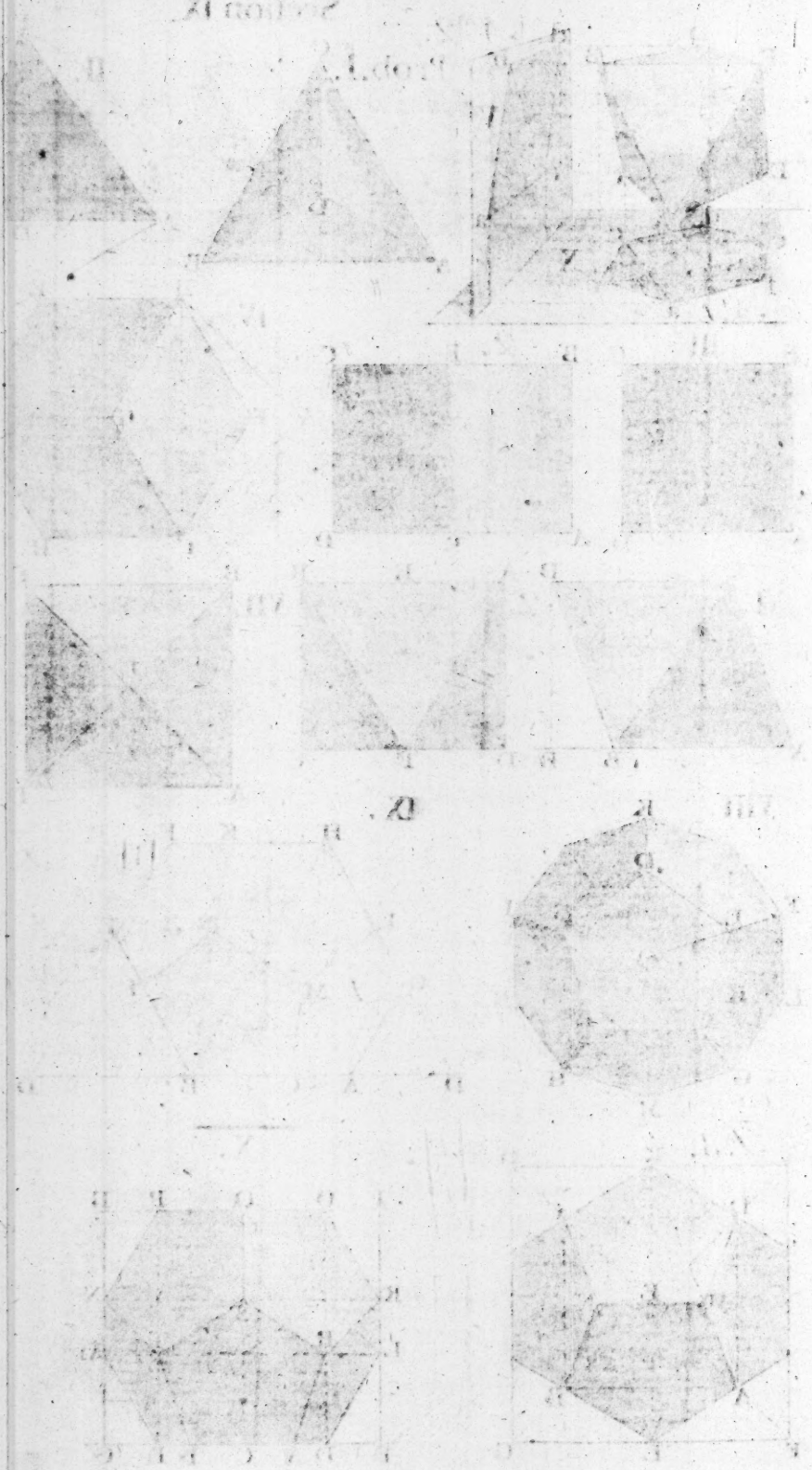

Divide DG, or EF, in extreme and mean Proportion, at H or I, K or L; make GH, or DI, equal to the leſſer Segment, and draw IK and HL parallel to DE.

Draw AM, BN, and CO, parallel to DG and EF; and, joining all the Points, thus determined, by Right-lines, viz. AH, AI, and AP, HP, HM, and HO, and the correſponding Lines on the other Side, this Elevation is compleated.

PROBLEM XIV.

To deſcribe an Elevation of the Icoſaedron, ſtanding on an Angle perpendicularly, on a Plane perpendicular to the Plane of Projection.

Draw a Right line at diſcretion, in which, take the Point A, for the Seat of an Angle in that Line, and draw AB perpendicular. A Side is given.

Make AB equal to the Diameter of a circumſcribing Sphere (equal AE in the Elevation, Pr. 12.); in which, take AC and BD each equal to the Side of a Decagon inſcribed in the ſame Circle which circumſcribes a Pentagon, conſtructed on a Side; AD, and BC will be divided in extreme and mean Proportion, in C, and D.

Make AE and AF each equal to half the Diagonal of a Pentagon conſtructed on a Side; through B, draw a Line parallel to EF, and complete the Rectangle EFGH.

Draw Lines (ab and cd) through C and D, parallel to EF; make CI and CK, each equal to half a Side of the Solid, and draw cI, ID, DK, and Kd; alſo, Aa, AI, AK, and Ab, cB and Bd, which compleats the Elevation required.

If DL and DM be each made equal to CI, or CK; Lines drawn between the Points, aM, MC, &c. as in

the Figure, will exhibit all the Faces in the other ſide, as ſeen through, ten in each; the whole being inverted, in reſpect of the former, on the hither ſide.

a b D is the Ichnography of the Icoſaedron, on an Angle, as in the Elevation; which conſiſts of two regular Pentagons conſtructed on a Side of the Solid, concentric, but ſubcontrary in Poſition; forming a Decagon; whoſe Sides a c, c I, I D, &c. are the Seats of correſponding Sides in the Elevation, in this Poſition.

The four laſt Problems, for the projection of the Icoſaedron, ſhew all the regular Projections that Solid exhibits, as the preceding three, of the Dodecaedron; the foregoing contain the whole of the other three; in which, the Angles of the inclination of their Faces are alſo determined, ſave the Hexaedron, which is a Cube; whoſe Faces being Squares, the Angles are all Right, and conſequently, the Faces are perpendicular to each other; for, three Right angles form a ſolid Right angle. Various other Projections, Elevations and Sections, may be made, which the genius of the Student will ſugeſt to him; but thoſe, here given, are all I conceive to be uſeful.

It remains only, to ſhew how to obtain a Face, or the Side of a Face of each Solid, contained in the ſame Sphere (the largeſt poſſible) which determines their Proportions to one another; ſhewing alſo the Properties and Affinity, which runs through all the five; for which, obſerve the following Problem.

PROBLEM XV.

The Diameter of a Sphere being given, to determine a Side of each of the five regular Solids, inſcribed; the largeſt that can be contained in that Sphere.

On AB, the Diameter given, deſcribe a Semicircle, AFB. Triſect AB, at D (11. 1.) and draw DE perpendicular to AB, cuting the Arch at E; AE is the Side of a Tetraedron, in a Sphere whoſe Diameter is AB.

The Squares of the three Sides of a Face is equal twice the ſquare of the Diameter; for, AE is a mean Proportional between AB and AD; conſequently, $2AB\square = 3AE\square$ (for $2AB = 3AD$); by 10. and 12. 6. El.

BE is the Side of a Hexaedron, inſcribed in the ſame Sphere. For BE is a mean Proportional between AB and BD; wherefore, $AB\square = 3BE\square$ (for $AB = 3BD$); alſo, $AE\square = 2BE\square$ (for, $AD = 2BD$, and DE is a Mean between them); conſequently, a Side of the Tetraedron is equal to the Diagonal of the Face of a Hexaedron, which is a Cube, the Face being a Square, whoſe Side is BE; that of the Tetraedron AE. Its height is AD.

If a Perpendicular be drawn from the Center (C) cuting the Arch at F, AF is the Side of an Octaedron; which was evident in its Conſtruction on an Angle (Pr. 7.) its Plan and Elevation being equal Squares; whoſe Diagonals (perpendicular to each other) paſs through the Centers of equal circumſcribing Circles; and conſequently, through the Center of a circumſcribing Sphere; the ſquare of its Diameter (AB) being equal to twice the ſquare of a Side (AF, equal BF).

If BE be divided in extreme and mean Proportion, at G; then is AG (the greater Segment) the Side of a Dodecaedron, inſcribed in the ſame Sphere, and BE is the Diagonal of a Face; wherefore, the Side of a Hexaedron is equal to the Diagonal of a Face of the Dodecaedron, which is a Pentagon; whoſe Diagonal is to a Side, as a Line divided in extreme and mean Proportion. P. 34. 6. El.

Laſtly, for the Icoſaedron, draw AH perpendicular, and equal to AB, the Diameter, and draw CH, cuting

the Arch at I, and draw AI; which is the Side of an Icoſaedron, inſcribed.

A Perpendicular, IK, being drawn, is the Radius of a Circle which circumſcribes a Pentagon, whoſe Side is AI; the Baſe of the Pyramid ſubtending a ſolid Angle of an Icoſaedron. And, AK:KI::KI:AK+KI.

AK is the Side of a Decagon, inſcribed in the ſame Circle; and, 2AK+IK=AB, the Diameter of the circumſcribing Sphere; in which, a Side of each Solid is determined, as propoſed.

PROBLEM XVI.

To determine the Diſtance between two oppoſite Faces, in the Dodecaedron and Icoſaedron, and alſo of the Octaedron, from the Diameter of a circumſcribing Sphere, given.

Deſcribe a Semicircle, on AB, the given Diameter; then, having found the Side of either Solid (15.) in that Sphere, find the Diameter of a Circle circumſcribing a Face of either (for, the ſame Circle circumſcribes both); on A, with the Radius AC, equal to that Diameter, deſcribe the Ark CD; i. e. make AD equal to the Diameter.

Biſect AD, and draw EF (from the Center) through the Point of biſection, cuting the Arch at F, and draw FG perpendicular to AB. Make BH equal to AG; GH is the Diſtance required; for, EG=EI (the Diſtance of a Face from the Center) and EH is equal EG.

The Dodecaedron and Icoſaedron, inſcribed in the ſame Sphere, are equal in height; and conſequently, the ſame Circle circumſcribes a Face of either. The Diſtance between two oppoſite Faces being determined, and divided

in extreme and mean Proportion, facilitates the Projection of theſe two Solids greatly, both being dependant thereon.

The Hexaedron and Octaedron are alſo equal in height, and the ſame Circle circumſcribes a Face of either; their diſtance aſſunder is determined as the other. And, as in them ſo in theſe, it is remarkable, that each has the ſame number of Angles as the other has Faces (12 and 20) yet an equal number of Sides, 30, each Solid. But, only three Sides terminate in each Angle of the Dodecaedron, and having 20 Angles, $20 \times 3 = 60$, half of which is the number of Sides. In the Icoſaedron, five Sides terminate in each Angle; $12 \times 5 = 60$, twice the number of Sides; for, each Side is common to two Faces, and terminates in two Angles; conſequently, their number of Sides is equal.

This holds in all the five Solids; ſo, that, multiplying the number of Angles into the number of Sides, terminating in each, gives double the number of its Sides.

AS the Side of the Hexaedron is equal to the Diagonal of a Face of the Dodecaedron, it is evident, that every Side of a Hexaedron, inſcribed in a Dodecaedron (12 in number) will be a Diagonal in each Face of the Dodecaedron; ſo that, it is manifeſt, all the Angles which are, in this caſe, common to them both, are in the Surface of the ſame circumſcribing Sphere. And, cuting off ſix, equal and ſimilar Solids, from the Dodecaedron, there will be formed a Hexaedron, eight Angles of one, remaining in the other; twelve being deducted, or cut off, which were all in the Surface of the ſame Sphere.

To obtain the ſuperficial Meaſure of any one of the Solids, is to find the Area of one Face, and to multiply that by the number of Faces.

The ſolid Contents of a Tetraedron is the Area of a Face multiplied by a third part of its height; P. 4. 8. El.

The Hexaedron being a Cube, it is but to multiply a Side twice into itſelf, for the ſolid Contents.

The Octaedron is two Pyramids on a common Baſe, a Square, of the Side; its Area multiplied by a third part of a Diagonal gives its ſolid Contents.

The Dodecaedron and Icoſaedron may be conſidered as ſo many Pyramids as they have Faces; whoſe common Vertex is in the Center of the Solid. Wherefore, the Area of the whole Superficies of either, multiplied by a ſixth part of its height, gives the ſolid Contents

The Proportions of the five Solids to a circumſcribing Sphere are as follow. The Diameter of the Sphere being 2, its Circumference is 6,28318; the Area is 3,14159.

	Sphere.	Tetraedr.	Hexaedr.	Octaedr.	Dodecaedr.	Icoſaedr.
Sides	2, Diam.	1,6329	1,1547	1,4142	,7136	1,0514
Surfaces	12,56637	4,6185	8,	6,9278	10,5146	9,5745
Solidity	4,18879	,5132	1,5396	1,3333	2,7851	2,5361

In the Scheme annexed is given the two laſt Solids, in ſuch wiſe, that the Affinity between the Ichnography and Orthography is obvious on inſpection; the letters of Reference, being the ſame to correſponding Sides and Angles, render a deſcription of the Proceſs unneceſſary.

The Lines (PQ and QR) on which the Elevations are projected, are conſidered as the Interſections of the Plane of the Elevation with the Plane of the Ichnography; the two, to each Solid, are perpendicular to each other; one parallel to a Side, the other to a Perpendicular in the Face it reſts on. The Planes being erected on thoſe Lines, into a vertical Poſition, and the Object, to be projected, ſtanding on its Seat; in which ſituation of the Planes, the projecting Rays, from each Angle of the Solid, would fall on the Plane of the Elevation perpendicularly, giving the Seat of each Angle thereon. The Line

drawn between the Seats of the two Extremes of a Side is the Seat of that Side, projected on the Plane, in the Elevation; and all thoſe Faces of the Solid, which are perpendicular to the Plane of projection, as well as that on which it reſts, are projected into Right lines; the Planes of the Faces being ſuppoſed to be produced, until they cut the Plane of projection. Two Faces ſo ſituated, being parallel they have parallel Projections.

To aſſiſt the Imagination, if the Student ſhould ſtill be unable to form a clear and adequate Idea of their Figures, here is added a geometrical projection of all the Faces of each Solid, in plano; ſo diſpoſed, that, being drawn on ſtiff Paper, or fine Paſteboad, and carefully cut out, they will fold into the true Figure of each. Thoſe given are too ſmall; but it is eaſy (by them) to draw others, to the proportion of the Projections, in the Plate; or to any Scale he pleaſes. Being placed on the Plan, properly, then, turning up the Plane of the Elevation, the Affinity between them may be clearly ſeen; nor can more be done to render it clearer.

THE following Proportions, between the Sphere, the Cylinder, and the Cone, may be intereſting to ſome.

The Superficies of a Sphere is double that of a ſquare Cylinder inſcribed.

A Sphere has the ſame Proportion, both in Surface and Solidity, to a conical Rhombus circumſcribing it, as the Side of a Square to the Diagonal.

The Superficies of a Cone circumſcribing a Sphere is quadruple of an inſcribed one; and, in its ſolid Contents, it is eight fold.

A Hemiſphere is double of a Cone, having the ſame Baſe and Altitude.

A Sphere has the ſame proportion, both in Surface and Solidity, to an equilateral Cone circumſcribing it, as four to nine.

The ſame ſeſquialteral proportion is continued between a Sphere, a circumſcribing Cylinder, and an equilateral Cone, as four, ſix, and nine.

FINIS.

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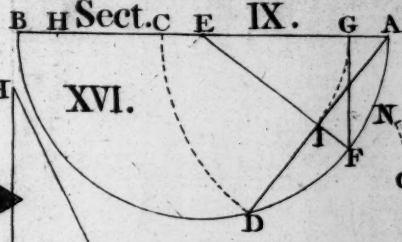
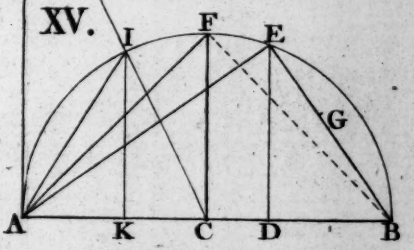
Tetraedron.



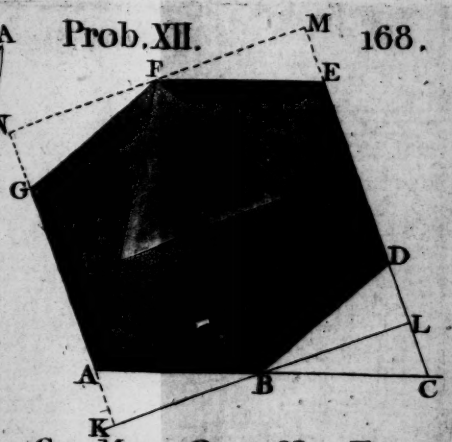


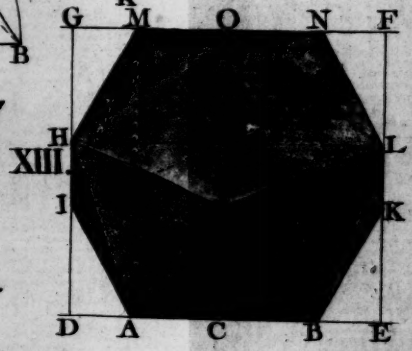
Hexaedron.

Octa- edron.

Dodecaedron.

Icosiedron.





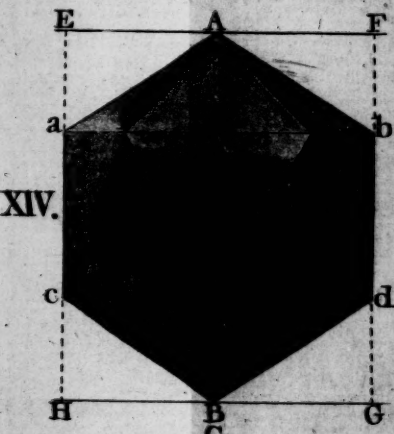
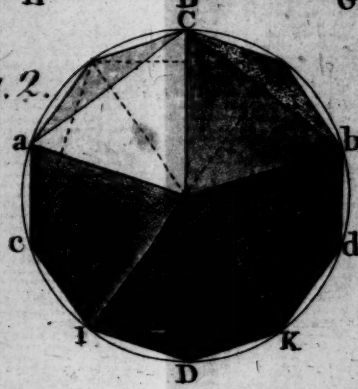


Fig. 2.



AN

APPENDIX.

On the Conſtruction of ELLIPSES; *of* proportional SCALES; *of the* LINE OF CHORDS, *and its uſe explained; with various other Matters, which could not, with proporiety, be otherwiſe introduced into this Tract.*

THE Ellipſis is not admited into Plane Geometry, the nature of its Curve, and the Properties peculiar to Ellipſes, being of no uſe in the Elements, for Inveſtigation, like the Circle; yet, both are generated by the ſection of a Cone, or Cylinder; one direct, the other is an oblique Section, in reſpect of its Baſe. But, as the Ellipſis is ſo uſeful a Figure to Mechanics, almoſt in general, and more particularly to the Architect and Builder, I could not diſpence with the omiſſion of it entirely. By refuſing it a Place, I ſhould deem the Work defective, and incomplete, as a practical Treatiſe; in which, elliptical, as well as circular Figures may, without any impropriety, be introduced.

Thoſe who would be thoroughly acquainted with the Properties of Ellipſes, and the nature of its Curve, muſt have recourſe to the Conic Sections; which, being a diſtinct Subject, would by no means be proper to touch on, here; nor have I any ſuch intention. I do not mean to inveſtigate all its Properties, or to develop the geneſis of Ellipſes, ſcientifically, but to ſhew its conſtruction, and how it may be deſcribed in any proportion required, in order to render it more generally uſeful; being well convinced, that 'tis not ſo well known to many, who have frequent occaſion for it, as it ought to be; therefore, no Apology need be made for obtruding it on the Public, in this Appendix.

Some of its Properties are demonſtrated, in the 1ſt Part; in which, the Affinity between the Circle and Ellipſis is obvious, as (comparatively, in figure) between the Square and Rhombus; an Ellipſis being conſidered as a Circle depreſſed, or drawn out at both extremes of a Diameter, to any length leſs than half the Circumference; as the Rhombus, from a Square, to any length of one Diagonal, ſhorter than two of its Sides.

DEFINITIONS.

DEF. I. An ELLIPSIS, or OVAL, is a Plane Figure bounded by a curved Line, falling into itſelf, which, is not uniform or circular in any part, but varying continually, being deſcribed on two Points.

The farther the Points are aſunder, on which an Ellipſis is deſcribed, the more it is drawn out and lengthened, and is then ſaid to be excentric, outcentred; and conſequently, the nearer they are together, the nearer its Curve approaches to that of a Circle; for, when they unite, it is not an Ellipſis, but a Circle.

DEF. II. PERIPHERY, or CIRCUMFERENCE, is the curved Line which bounds the Ellipſis; ACBD.

DEF. III. CENTER, of an ELLIPSIS, is the Point E, where any two Diameters interſect.

DEF. IV. DIAMETER, of an ELLIPSIS, is any Right line, AB, KM, &c. paſſing through its Center, E, and terminated by the Periphery.

For, every ſuch Line divides the Ellipſis equally in two; and is alſo biſected in the Center, E. Therefore, any two Diameters biſect each other; as in a Circle.

DEF. V. TRANSVERSE DIAMETER is the longeſt which can be drawn in an Ellipſis. As AB.

DEF. VI. CONJUGATE DIAMETER. This Term is generally confined to the ſhorteſt Diameter, CD; but, that is conjugate only in reſpect of the Tranſverſe, AB; which is alſo conjugate to CD.

Every Diameter has its Conjugate or Yokemate.

If Tangents to the Ellipſis be drawn, through the extremes, A and B, of the Tranſverſe, alſo, of C and D its conjugate Diameter, cuting each other in F, G, H and I, they will form a Rectangle; and if the Diagonals of the Rectangle, FH and IG, be drawn, they will paſs through the Center of the Ellipſis; the parts, KM and LN, which are terminated by the Curve, are Diameters of the Ellipſis; each of which, is Conjugate, in reſpect of the other, and are the only two Conjugate Diameters that are equal.

Note. The Tranſverſe Diameter, and its Conjugate, are the only two which are perpendicular to each other; and the moſt unequal. The Tranſverſe Diameter, AB, biſects the acute Angles, FEI and GEH; and the obtuſe Angles, FEG, IEH, are biſected by the Conjugate, CD.

Every Diameter, drawn within the acute Angles, is greater than its Conjugate; which, falls within the obtuſe Angles.

DEF. VII. ORDINATES are Right lines drawn parallel to the Conjugate of any Diameter, as MN, OP; which being biſected by the Diameter, AB; MN, OP, &c. are, therefore, double Ordinates to the Diameter AB.

DEF. VIII. A TANGENT is a Right line touching the Periphery, at the extreme of a Diameter, parallel to its Conjugate and Ordinates. As FI, FG, &c.

DEF. IX. AXES, of an ELLIPSIS, are the Tranſverſe and its Conjugate Diameter; AB and CD, Fig. 1ſt.

DEF. X. FOCI, of an ELLIPSIS, are the Points Q and R, on which it is deſcribed; as a Circle on its Center.

DEF. XI. LATUS RECTUM, or RIGHT PARAMETER, is that Ordinate to the Tranſverſe Axe which paſſes through either Focus; as MN; Fig. 1ſt.

The Latus Rectum, and every Parameter, is a third Proportional to the two conjugate Diameters, to which it is the Parameter.

Note. The Parameter to the Diameters KM and LN, Fig. 1ſt. coincides with them; for, being equal, there cannot be a third Propotional to them.

DEF. XII. ABSCISSA. If the Tranſverſe Diameter be cut, in any Point at pleaſure; the two Segments, made by that Section, are called *Abſciſſas*.

Note. An Ellipſis may be generated, by the Shadow of a Circle, or Ring, projected on a Plane, not parallel to the Ring; provided, the luminous Point be not in the Plane of the Circle.

PROBLEM I.

The Tranſverſe and its Conjugate Diameter being given, how to determine the Foci, on which an Ellipſis may be deſcribed; and how to deſcribe it, of the Proportion required.

Let X and Z be the meaſures given.

Draw, at right angles, two Right lines, AB and CD; make them equal, reſpectively, to X and Z, and biſecting each other, in the Point E.

On either extreme, C, or D, of the Conjugate Diameter, deſcribe an Ark, with the radius AE, or EB, half the Tranſverſe, cuting it in F and G; which are the Foci, or Centers, on which the Ellipſis may be deſcribed.

Then, proceed in the manner following.

Take a fine, ſmooth Cord or String, and having fixed two Pins, in F and G, carry the String round both Pins, and draw it, on both Sides, to the Point C, or D; where, fix a Pencil, and revolve it around, on the two Pins, keeping the String at full ſtretch; the point C will deſcribe a Curve, which will paſs through the four Points A, C, B, and D.

From which conſtruction it is evident, that any two Right lines, FH and GH, drawn from the Foci to any Point in the Circumference, are equal to FC and CG; that is, to the Tranſverſe Diameter, AB.

For, FG being common to both the Triangles FCG and FHG, the remaining Sides, added together, are equal; i. e. FC added to CG is equal to FH added to HG.

As it is difficult to keep the Point or Pencil true, in a String, this Method is not ſo eligible, for ſmall Ovals, or for any, which require the Curve to be exact; but, from what has been advanced, (being well conſidered and underſtood) it will be found practicable to deſcribe ſmall Ovals with tolerable exactneſs.

Having determined the two Foci, F and G, in the Tranſverſe Diameter, AB, whoſe Conjugate is CD (as above) as many Points, H, may be determined, in the Periphery, as are neceſſary; through which a Curve may be deſcribed, by a ſteady Hand, which will be a true Ellipſis; after the following manner.

With any Radius, at diſcretion, ſeting one Point of the Compaſſes in either Focus, as at F; with the other Point, make an Ark, at H.

Then, with the ſame meaſure, FH, ſeting one Point in either extreme of the Tranſverſe Diameter, as at A, cut the Tranſverſe, at E with the other.

Take the remaining Segment, EB, as Radius, and, on the Center, G, deſcribe another Ark, cuting the former in H; which will be a Point in the Periphery.

Thus, as many Points, H, or h, may be obtained as you pleaſe; which are all in the Periphery.

That the Point H is in the Periphery is manifeſt. For, ſeeing that the Foci are determined from the extremes, C and D, of the Conjugate Axe, making CF and CG each equal to AE, half the Tranſverſe; (by Prob. 1.) conſequently, FH and GH being made alſo equal to AB, the Point H is in the Periphery.

PROBLEM II.

To deſcribe an Ellipſis, by means of an Inſtrument called a Trammel, of a given Proportion.

The TRAMMEL is made of two pieces of wood, fixed together at right Angles, in the form of a Croſs; or, for ſmall Ovals, it is made of Metal, with two ſtreight Grooves, truly perpendicular to each other, as AB and CD.

Then, having provided a ſtreight Ruler, of wood or metal, fix a Point or Pencil at one end, as at E; and, let two ſhort, round Pins (equal, in Diameter, to the width of the Grooves in the Trammel, which will ſlide in them, freely) be fixed at F and G, in ſuch manner, that EG is equal to half the Tranſverſe, and EF half the Conjugate Diameter.

The Trammel and the Ruler being thus prepared, let the Trammel be placed, as in the Figure; exactly on the Tranſverſe and Conjugate Diameters.

Then, apply the Ruler, with the Pins in the Groove, along the Tranſverſe Diameter; G being in the Center, and the Point E in the extremity of the Tranſverſe.

When the Point E is moved towards C, the Pin at G, in the Center, falls into the Groove towards D, whilſt the Pin at F, moves towards the Center; which is continued till it falls into the Center, at G, and the Point E being arrived at C, has deſcribed, by its motion, the Curve

AC, a fourth part of the Periphery; which has all the variety of the whole, for every fourth part is the ſame; from C to B, and from A to D, it is inverted.

The Ruler being now in the poſition of the Conjugate Diameter; the Pin, at F, in the Center, and that at G fallen down to g; the motion is continued to B; the Pin at F (being now ſuppoſed to be in the Center) moves towards B, and the Pin at g goes back again to the Center; when the Point has deſcribed the Curve CB, another 4th part, the converſe of the former. Thus continuing the motion until the Curve is compleated, and the point E arrives again at A; the Pin, at F, ſtill moving, to and again, in the Tranſverſe, from F to f, and back again to F; and the Pin at G, to and again, in the Conjugate Groove; by which means a true Ellipſis is formed; which, it is evident, has no part of the Curve of a Circle in its compoſition, ſeeing it is deſcribed on two Centers.

The nearer the Foci are together, that is, the leſs the difference is between the Tranſverſe and Conjugate Diameters, the nearer it approaches to a Circle.

PROBLEM III.

To find the Center of an Ellipſis.

Draw, at pleaſure, two parallel lines, AB and CD, in the Ellipſis. Biſect them, in E and F; through which Points draw HI, which is a Diameter, of the Ellipſis.

Biſect the Diameter, HI, in G, which is the Center of the Ellipſis.

For, every Right line drawn through the Center is a Diameter; and every Diameter is biſected in the Center.

PROBLEM IV.

To find the Axes of an Ellipſis.

Having found the Center, by the laſt, with any Radius, leſs than the Tranſverſe Diameter and greater than the Conjugate, deſcribe a Circle, cuting the Ellipſis in four Points, A, B, C, and D.

Draw the Ordinates AD and BC, and biſect them, in E and F; through E and F draw HI, which is the Tranſverſe Axe.

Through the Center, G, draw KL, at right Angles with the Tranſverſe, or parallel to the Ordinates, AD and BC, and that is the Conjugate Axe.

PROBLEM V.

To find the Foci of an Ellipſis.

Having found the Center, and drawn the Tranſverſe and Conjugate Diameters, or Axes; take GH, half the Tranſverſe, for Radius, and on either extreme, K or L, of the Conjugate, as a Center, deſcribe an Ark, cuting the Tranſverſe in two Points, E, and F, which are the Foci, or Centers, on which the Ellipſis is deſcribed; as in Prob. 1.

In reſpect of the Foci, only, this Problem may be thought ſuperfluous; but, the firſt ſhews how to determine them, in order to deſcribe an Ellipſis, of a determined Proportion; this, to find the Foci, of an Ellipſis already deſcribed, for various other purpoſes.

PROBLEM VI.

To draw a Tangent to an Ellipsis at any Point given, in the Periphery.

If the Tangent were required at either extreme of the transverse or conjugate Axes, as B or C, it will be perpendicular to them, as in the Circle, and consequently makes equal Angles.

Let D be the given Point, through which a Tangent is required to be drawn.

Find E and F, the Foci of the Ellipsis, by the foregoing; and draw ED and DF.

Make DG equal DF, and draw GF; to which, if HI be drawn parallel, through the given Point D, it will touch or be a Tangent to the Ellipsis in that Point.

For, let KL be drawn, through C, parallel to the Transverse, AB; it will touch the Periphery in C.

Draw EC and CF; they are equal, by Construction; Pr. 4.
and, consequently, the Angle ECK, is equal LCF - 4. 1. El.
for the Angles CEF, CFE are equal. - - - 9. 1. El.
But, the Tri. GDF is Isosceles, and HI is par. to GF;
conf. the Angle HDG or E is equal to FDI. - 4. 1. El.
Therefore, HI touches the Ellipsis, in the Point D.

PROBLEM VII.

Any Diameter, as AB, being given, to find its Conjugate, CD.

Through either extreme of the Diameter, as A, draw the Tangent FG; through the Center E, draw CD, pa-

rallel to the Tangent; which is conjugate to the Diameter AB.

If Tangents are drawn, through the four Points A, B, C and D, of any two conjugate Diameters, meeting in F, G, H and I, the Parallelogram FGHI, is equal to a Rectangle under the two Axes.

PROBLEM VIII.

To make a Repreſentation of an Ellipſis, with Compaſſes.

Let AB be the given Tranſverſe Axe, and CD the Conjugate.

Biſect AE and EB, in F and G, and on the Centers F and G, deſcribe the Circles AIEK, & EHBL, touching at E.

Draw CA; and, with that Radius, on C, deſcribe the Ark AM, cuting the Conjugate Axe in M.

Make EN equal EM; on which Centers, M and N, with the Radius MC and ND, deſcribe the Arks HI and KL, falling into the Circles AE and EB, at H, I, K and L.

Or thus, when the difference between the Axes is leſs.

Divide the tranſverſe, AB, into three equal Parts, at C and D; on which, with the Radius AC, deſcribe two Circles, cuting each other in E and F; on which Centers, with the Radius AD, deſcribe the Arks GH and IK.

This laſt is nearer to an Ellipſis than the other; they may be lengthened or ſhortened at pleaſure. But, it is very obvious that they are not Ellipſes, but compounded of circular Curves; whereas, a true Ellipſis has no part of the curve of a Circle in its compoſition, the Curve, being

every where deſcribed on two Points, is continually varying; wherefore, no Oval, formed by Circles, can partake of the Properties of Ellipſes.

HAVING explained or defined all the Terms peculiar to the Ellipſis, and ſhewn how to fix the Points on which it is deſcribed, ſo as to conſtruct it of any given Dimenſions, there remains nothing more to be done in reſpect of the utility of it, to Mechanics, &c.

The Properties of the Ellipſis are really very extraordinary; many of which have ſo near an Affinity to thoſe of a Circle, that they are almoſt neceſſarily deduced from them. But, as it is requiſite to have ſome acquaintance with the Conic Sections, in order to a clear underſtanding of the Properties of Ellipſes, I ſhall ſay no more concerning them, here, having demonſtrated ſome of its peculiar ones, in the Appendix to the foregoing Work, the firſt Part of this.

THE CONSTRUCTION OF PROPORTIONAL SCALES; FOR DRAWING, &c.

PROPORTIONAL SCALE, or *Scale of equal Parts*, is ſo generally known, that it can ſcarcely be rendered more ſo, being applied to Maps, and geometrical Plans of all kinds; with the uſe of which, all are acquainted, who know the meaning of, and are converſant with Plans. Such Scales are nothing more than a Right line divided into ſome number of Parts, without any geometrical Proceſs, but merely manual; by dividing and ſubdividing, biſecting and triſecting, by trial; which may, indeed, be accurate enough for thoſe purpoſes; but into fifths, ſevenths, or eleventh parts, &c. the effecting it, by ſuch means, is a tedious proceſs, yet it may be done, though but rarely into *equal* Parts; which is done, geometrically, with eaſe, certainty, and expedition, the ſame as any even Diviſion whatever; and, by a diagonal Line, the ſmall Diviſions may be ſubdivided into any number of Parts, as minute as the Eye can diſcern (as in the following Problems) with the greateſt accuracy imaginable.

PROBLEM I.

To divide a Right line, of any determined Length, into Decimal Parts; that is, into tenth and hundredth Parts, &c.

AB is a given length, or Line to be divided, into a hundred equal Parts.

Firſt, divide AB into ten equal Parts, by ſome of the Methods (Pr. 12. 5.); which Parts being each ſubdivided into ten equal Parts, the buſineſs is done.

But, from the ſmallneſs of the tenth Parts, the ſame proceſs cannot be applied, for ſubdividing them, with any ſucceſs. Therefore,

Draw another Line, CD, indefinite; and, with any opening of the Compaſſes, at diſcretion, take ten equal Diviſions on that Line, from C to D, at 1, 2, &c.

Make DE equal to one of the Diviſions on AB, and perpendicular to CD; draw CE; and, at every Diviſion on CD, draw Perpendiculars, cuting CE.

The perpendicular Diſtances between CD and the diagonal Line CE are all the intermediate Diviſions, from one to ten, according to their Diſtance from C, begining at Unity; and may be continued to any number of places, unequal as well as equal, the Sum being taken on CD.

PROBLEM II.

To make a Scale, having the Integer divided into a hundred equal Parts, by diagonal Lines.

Let AB be a determined Meaſure, repreſenting one Foot, or any other meaſure required.

Produce AB, and repeat the meaſure of AB, to C and D, as often as you think neceſſary.

Draw AF and DE perpendicular to AD; on either of which, make, at pleaſure (i. e. with any opening of the Compaſſes) ten equal diviſions, as on AF; and, through them, draw Lines parallel to AD; as 1, 1, and 2, 2, &c. in all, eleven Lines.

Divide AB into ten equal Parts (12.5.); and, having made F1 one tenth, draw A1, 1 2, &c. diagonal-wiſe, as in the Figure; proceed in the ſame manner, to 9 G, parallel amongſt themſelves.

By which means, the length or meaſure given (AB) is divided into a hundred equal Parts; each interval, between AF and the firſt Diagonal A1, at 1, 2, 3, &c. expreſſing all the Units from 1 to 10.

Between each of the Diagonals, as A1, and 1 2, &c. the intervals are each ten of thoſe hundredth Parts, being every where equal (as it is evident) to the Diviſions on AB.

In the ſame manner, any Meaſure whatever, either greater or leſs than AB, may be accurately divided into Decimal Parts.

AFTER ſuch plain directions, in the Conſtruction, the application of it, in Practice, may ſeem unneceſſary; yet, probably, not ſo to all.

AB, BC, &c. repreſent the Integer, Feet, Yards, or other Meaſure, the firſt, or the laſt (AB) being ſubdivided into the ſmaller Parts required; as here into 100th Parts.

Whatever number of Feet, that is, of the Integer is required; ſuppoſe two, and 27 hundredth Parts, or other number. Apply one Point of the Compaſſes to 3, on the Line DE, and open them to the Interval, on the Line 3 3, to the Diagonal 7 8, which is the Meaſure wanted.

If thirty-ſeven were wanted, the Compaſſes muſt be extended to the next Diagonal; and ſo to forty, fifty or other tenth Parts, with the ſame Units, 7; but, according to the number of Units, application muſt be made to the Line on which that number of Units are, between BG and G9; the tenth Parts are the ſame on them all.

Perhaps it would be as well if the fractional Parts were made at the other extreme of the Scale; and better, being numbered as here; but that is at the diſcretion of each

Perſon, who makes a Scale for his own uſe, who will, with a little experience, ſoon diſcover which is moſt convenient.

PROBLEM III.

To make a Scale having the Integer divided into Duodecimals (12th Parts) as Feet and Inches.

Duodecimals is ſeldom practiſed, further than the diviſion of a Foot into Inches, which are the ſtandard Meaſures on all common Rules; therefore, to make a Scale of Feet and Inches, to any determinate meaſure, obſerve the following directions.

Let AB be the determined Meaſure of a Foot, according to the proportion you intend to make a Drawing, or Plan of any kind.

Produce AB, and repeat the meaſure of AB as often as you think neceſſary, as C, D, E.

Draw AG and EF, perpendicular to AE, and make ſix equal diviſions on AG, at diſcretion; through which, draw 1a and 2b, &c. parallel to AE, and alſo BH and CI, &c. perpendicular.

Biſect AB, or GH, and draw the Diagonals, A6 and B6; the given Meaſure, AB, is divided into twelve equal Parts, as they are numbered from the Perpendicular AG; on the Diagonals.

Note. It may be done with only four, or three Spaces, as between BI and CK; but I think it more applicable, in Practice, with ſix; that is ſubmited to every Perſon's diſcretion, who makes it for his own uſe; the Diviſions are the ſame in all.

The application of this Scale to uſe, and of all other, is the ſame as the preceding.

PROBLEM IV.

To make a Scale of Modules and Minutes, for proportioning the ORDERS, in Architecture.

THE Defect in the old Authors, Palladio, Vitruvius, and others, and more ſo, in the modern ones (Ware, Cole, &c.) who have republiſhed their Works, in not giving a Scale of the Module, and Minutes, by which the ſeveral Members are proportioned, is much to be lamented; nor have many of our modern Architects remedied that defect, in their own Productions. This gave occaſion for ſome pretenders to Architecture, to mutilate and deſtroy that harmony of Proportion which is in the Originals, in order to reduce them to a Scale of their own; by ſubdividing the large Parts, into the ſeveral Members of it, as if they muſt neceſſarily be ſome equal part; which has introduced many Innovations, according to the caprice of each Author, and thereby, the original beauty of the Grecian Orders has been much defaced, in order to render their Works more facile to the Artificer; who could not, for want of a Scale (which they knew not how to make) apply the Proportions of the old Authors, in Minutes of a Module.

The Ionians and Corinthians took the Diameter of the Shaft of their Columns (at the bottom) for the Module, which they divided into 60 equal Parts (as an Hour is divided into Minutes, or minute Parts); by which the ſmall Members are proportioned. Perhaps, 48 would have been a better Diviſion; the ſmall Fillets, which are but one Minute, being too ſmall for the adjoining Members. The Dorians took but half the Diameter for their Module, which they divided into 30 Parts; ſo that their Minutes

have the ſame proportion to the Diameter; the other is an uſeleſs diſtinction, to adhere to, at this time.

The Proceſs, for making a Scale of a Module into Minutes, is the ſame as into hundredth Parts (Prob. 2.); the Integer (which is no fixed Meaſure, but infinitely variable) being divided into ſix only, as the other into ten equal Parts.

AB is the meaſure given, for a Module; being the Diameter of the Columns, proportioned to the Drawing you are about to make.

Repeat the meaſure of AB as often as is neceſſary, at diſcretion; and at the two Extremes, draw A2 and DE perpendicular; take ten Diviſions on A2, with any opening of the Compaſſes; through each of which, draw Lines parallel to AD; alſo, draw B1, and CF, parallel to A2 and DE. Divide CD into ſix equal Parts, and draw E5; to which, through all the other Diviſions, draw Lines parallel, as in the Figure. So ſhall CD (equal AB) be divided into ſixty equal Parts, as required.

Some make twelve Diviſions on A2, others but ſix; conſequently, CD into five, or ten; but I think theſe Diviſions preferable to either; becauſe, the Spaces between the diagonal Lines being ten Minutes, every where, we are leſs liable to err, in taking the number of Minutes wanted; for, by applying to the Line of the Units, the tens are much eaſier to add (as 10, 20, 30, &c.) than twelves or ſixes. The Diviſions are the ſame.

In laying down and delineating the Plan of a large piece of Ground, or Building, it is uſual to take an Inch, or leſs, to repreſent ten Feet; a Scale, for which, is made the ſame as for Duodecimals (Prob. 3.); taking but ten Diviſions inſtead of twelve, as there, for Inches. The ſmaller Parts (Inches) in a Scale where half an Inch, or

three-fourths repreſents ten Feet, are ſeldom wanted with that accuracy, as to require a Foot to be divided in leſs than fourth parts, 3 Inches each; which may, nevertheleſs, be done, into any number, by Prob. 1. but thoſe minutias are gueſſed at.

THE LINE OF CHORDS,

and its Uſe, *explained.*

AS the Line of Chords, and its Uſe, is better known than underſtood; I preſume, a Deſcription of it may not be deemed unneceſſary.

PROBLEM V.

To conſtruct a line of Chords.

With any Radius at diſcretion, as AC, deſcribe the Arch ADB, of 90 Degrees; i. e. make ACB a Right angle; and, on the Center, C, with any Radius, AC, draw AEDB; which will be a fourth part of the circumference of a Circle.

Divide AEDB in nine equal Parts. Firſt, make AD equal AC, and divide DB into three equal Parts; AD will be equal ſix of thoſe diviſions, each of which will be ten Degrees of that Circle.

Draw the Chord line AB; and, from the extreme A, as a Center, transfer all the meaſures on the Ark, to the Chord AB, as in the Figure. Each Diviſion being ſubdivided, into ten, and transfered in the ſame manner, AB will then be a Scale for any number of Degrees, from one to ninety; which give a Right angle at C, the Center of the Circle, of which, AC is the Radius.

Note. This Line of Chords is upon every common Scale, in a Caſe of Inſtruments, for drawing; its Uſe is the ſame as the Protractor, viz. to meaſure, or lie down any Angle required, on a Plane, the number of Degeees, it contains, being known.

In applying the Line of Chords to uſe, take 60 Degrees from A, on the Line AB, with your Compaſſes; and, with that Radius, on any Point, C, of the Line AC (at which Point an Angle is required, with that Line, of ſome known meaſure) deſcribe the Ark AEDB, cuting AC in A; from which Point, ſet off the number of Degrees, on the Ark, which is the Meaſure of the Angle required.

e. g. Suppoſe an Angle of 35 Degrees be wanted, take 35 Degrees, from the Point A, on the Line of Chords, and ſet it off from A to E, draw EC; then, ACE is an Angle of 35 Degrees.

If an Angle of 60 Degrees were required; make AD equal to the Radius, and draw DC; then is ACD an Angle of 60 Degrees. For ADC is an equilateral Triangle, whoſe Angles are all equal; and conſequently, the Chord, AD, is equal to the Radius, AC. - - P. 12. 4. El.

If a Right angle be required; from A, make AB equal to the whole Line of Chords of 90 Degrees, and draw BC, which will be perpendicular to AC.

Thus, may any Angle, leſs than 90 Degrees, be readily obtained. But, if an obtuſe Angle be wanted, as 105 Degrees, with the Line AC, at the Point C; having firſt ſet off 90 Degrees, from A to B, or 60, to D; take the remainder, BF, 15 Degrees or DF 45, and draw FC, making the Angle ACF, 105 Degrees. Thus may any Angle whatever be deſcribed.

Or, an obtuſe Angle may be thus taken, at once, if there be room on your Paper.

If an Angle ACF, of 105, or ACH, 125, be required; ſubtract the number of Degrees from 180, the Sum of two Right angles, the Difference is 75, or 55.

Produce AC, towards G; with the Radius AC, equal AD (i. e. to 60 Degrees, on the Line of Chords) deſcribe the Ark GHF, and ſet off GF, 75, or GH, 55 Degrees, and draw CF, or CH, making the Angle required.

After the ſame manner an Angle may be meaſured; by deſcribing an Ark, on its Vertex, with the Radius of 60 Degrees, cuting both Sides; as AE, cuting AC and CE, of the Angle ACE; then, the meaſure of the Chord Line, AE (applied to AB) ſhews the number of Degrees that Angle contains.

The reaſon of all which is ſo very obvious, it needs no further explanation; but, be ſure that you always take your meaſure, or number of Degrees on the Line of Chords, begining at the Point A; becauſe, it is evident that they are continually diminiſhing, from A to B, although they are equal on the Ark; for the Diviſions on AB are the Chords of each Ark, ſet off from A towards B; each, of which, is leſs, in proportion to the Ark.

The Chord of 10 Degrees, it is evident, deviates very little from a Right line; the Chord of 30 Degrees deviates conſiderably, and AD, the Chord of 60, ſtill more; for the Chord AD, of 60, and the two Chords of 30 make a Triangle, AED; of which, it is manifeſt, that, the two Sides AE, ED, each of 30 Degrees, are greater than AD, the Chord of 60. (13. 1. El.)

If it ſhould be aſked, why 60 Degrees, particularly, is taken for Radius, when we begin the Operation of taking or deſcribing an Angle, by the Line of Chords? The reaſon is plain; becauſe, the Chord of 60 Degrees is always equal to the Radius; and, the Side, AD, of the equilateral Triangle ADC, is the Side of a regular Hexagon inſcribed in a Circle, of that Radius; and conſequently, it is equal to a ſixth Part of the Circumference, or 60 Degrees. Hence it is clear, that no other Radius can anſwer to that Line of Chords.

In conſtructing the Line of Chords, it is neceſſary to triſect an Ark of a Circle, in order to obtain the Chord, or the Ark of ten Degrees, for which no geometrical Proceſs is given; the following, therefore, may be acceptable to the young Student, but I muſt apprize him, that it is not ſtrictly geometrically obtained; the Demonſtration is truly ſo.

PROBLEM VI.

To triſect an Arch, or Ark of a Circle.

AB is the Portion required to be triſected, of the Circle ABD.

If the whole Circle were not given, it would be neceſſary to perfect it; as by Pr. 2. S. 3.

Through the Center (C) draw AF, indefinite; cuting the Circumference at D; to which, draw CE perpendicular. Draw BF, ſo, that EF be equal to the Diameter of the Circle, which will cut the Circumference, at G.

DG is equal to a third Part of AB. Draw GH, through the Center; and CI parallel to BG.

Dem. Becauſe CG is Radius of the Circle, and EF is equal to the Diameter, and ECF a Right angle, CG, EG, and GF are all equal amongſt themſelves; and the Angle EGC=2GCD (for GFD=GCD). 10. 1. El.

But, ACH=GCD (2. 1.); and, EGC (i. e. BGH) being at the Circumference, its meaſure is half the Ark BH, equal HI; for, the Angle HCI is equal BGH, which is double GCD. - - - Pr. 4 1. El.

Wherefore, AH (equal AI) which meaſures half the Angle BGH, is a 4th Part of the Arch BH; conſequently a third of AB. Therefore, AI, equal AH (equal DG) is a third part of AB.

If the Arch to be triſected be more than a Quadrant, triſect its Complement to 180 Degrees; a 3rd Part taken from 60, the remainder is a 3rd Part of the Arch given.

THE following arithmetical Problems are ingenious, and may be entertaining to ſome Readers.

PROBLEM VII.

The Sides and one Diagonal of a Trapezium being given; to determine the other, arithmetically.

ABCD is the Trapezium given, the meaſure of whoſe Sides and the Diagonal AC are known; to determine BD.

Square the Diagonal given, and all the Sides.

Then, to the Square of the Diagonal, add the Squares of the oppoſite Sides, ſeverally; from which Sums, ſubtract the Squares of the other Sides, lying on the ſame ſide of the Diagonal, ſeverally.

2nd. Divide half the Remainder of each by the known Diagonal, and from it ſubtract the Sums of both Quotients.

3rd. Subtract the Square of each Segment of the Diagonal, from the Square of the adjacent Side, reſpectively; the Square roots of the Remainders give the Perpendiculars, which add into one Sum.

4th. Square the Sum of both; to which, add the Square of what remained of the Diagonal, in the ſecond ſtep; the Square root of that Sum is the other Diagonal ſought.

ABCD is a Trapezium; the meaſure of whoſe Sides, and of the Diagonal AC, are given.

Draw the Perpendiculars, DE and BF, and produce the latter, indefinite; to which, draw DG perpendicular, i. e. parallel to AC.

DEM. $AC\square + BC\square - AB\square = 2ACF$. And, $AC\square = AF\square + FC\square + 2AFC$ (4. 2. El.); and $BC\square = BF\square + FC\square$ (20. 1.). Wherefore, $AC\square + BC\square = AF\square + BF\square + 2FC\square + 2AFC$, or $AF \times FC$.

But, $AB\square = AF\square + BF\square$; conſ. $AC\square + BC\square - AB\square$, $= 2FC\square + 2AFC$. And, $AFC\square + FC\square = ACF\square$ (3. 2.); th. $AC\square + BC\square - AB\square = 2AC \times CF$.

By the ſame, $AC\square + AD\square - DC\square = 2AC \times AE$.

Hence, by the 2nd ſtep, we have the Segments AE and FC, of the Diagonal AC, and conſequently, EF.

By the 3rd we have BF and DE; for, $BF\square = BC - FC\square$; and, $DE\square = AD\square - AE\square$; therefore, the Square roots of the remainders give them.

DG being parallel to AC, BG is the Sum of the two Perpendiculars, and DG = EF. But, the Angle BGD is right; wherefore, $BD\square = BG + DG\square$s; and conſequently, the ſquare of the two Perpendiculars, in one Sum, added to the ſquare of the intermediate part, EF (eq. DG) of the Diagonal AC, the Square root of that Product gives BD, the other.

PROBLEM VIII.

The three Sides of a Triangle being given; to determine the Diameters of a circumſcribing and of an inſcribed Circle, arithmetically.

Square the three Sides, and add either two into one Sum; from which, ſubtract the other, and divide half the remainder by either Side of the two whoſe Squares were added together. Square that Quotient; which, ſubtract from the ſquare of the other, of the two added.

Then, multiply the Side of this laſt, of the two whoſe Squares were added together, into the Side of that which was ſubtracted, and divide that Sum by the Square root of the laſt Remainder; the Quotient is the Diameter of a Circle which will circumſcribe the given Triangle.

Let A, B, and C be the three Sides. Square them, and add the ſquare of A to that of B; from which Sum ſubtract C ſquare, and divide half the remainder by A (or B). Square the Quotient, and ſubtract it from the ſquare of B (or A). Then, multiply B into C (or A into C) and divide the Product by the Square root of the laſt remainder; the Quotient is the Diameter ſought.

If A be 12, B 13, and C 14, the Diameter is 15.

Let ABC be the Triangle given, in which, draw a Perpendicular to any Side, as BD; which was the laſt Diviſor.

Let a Circle be circumſcribed (Pr. 4. 4); draw a Diameter, BE, and join AD.

DEM. $AC\square + BC\square = AB\square + 2AC \times CD$; conſequently, $AC\square + BC\square - AB\square = 2AC \times CD$ (13. 2. El.); half which Sum divided by AC gives CD.

But, $BC\square = BD\square + CD\square$ (20. 1.); wherefore, $BC\square - CD\square = BD\square$, the Square root of which gives BD, the Perpendicular.

The Triangles BCD, ABE, are ſimilar; for the Angle at D is right, and BAE is alſo right; 12. 3. El. and, the Angle at C = AEB (10. 3.); wherefore, BD : AB (or BC) :: BC (or AB) : BE; and conſequently, BE is a fourth Proportional, to BD, AB, and BC; which, is the Diameter of a circumſcribing Circle.

2nd. Having obtained a Perperpendicular (as BD, as above) find the Area of the Triangle given; Pr. 3. 6. double the Area, and divide that Sum by half the Sum of the Sides; the Quotient is the Diameter of an inſcribed Circle. Q. E. F.

For, drawing CA, CB, and CD, to the Sides, reſpectively, as in Fig. 2. being Radii, they are equal; and AE=AF, &c. wherefore, AE × EC + BE × EC + CF × DF =the Area of the Triangle (Ax. 2. 1.); (which was, the Radius multiplied into half the Sum of the Sides), conſequently, the Area divided by half the Sum of the Sides, gives the Radius; and therefore, double of the Area divided by the ſame, gives the Diameter.

THE Area of every Triangle is equal to a Rectangle, on half the Sum of its Sides multiplied into the Radius of a Circle inſcribed.

PROBLEM IX.

The Square, of the Hypothenuſe, of every right-angled Triangle, is equal to four Times the Area of the Triangle, added to the Square of the Difference between the Sides containing the Right angle.

If the Triangle be Iſoſceles (the Sides, containing the Right angle, being equal) it is manifeſt; for the ſquare of the Hypothenuſe is equal to two Squares of a Side (20. 1. El.); and, the Sides being equal, there is no Difference; therefore, four times the Area of the Triangle is equal to the ſquare of the Hypothenuſe.

This is not perfectly explicable, in Numbers; becauſe, the Hypothenuſe, in this caſe, is the Diagonal of a Square of the Side, which are incommenſurable with each other; from a geometrical Conſtruction of it, the thing is manifeſt, and evident on inſpection of the Figure; for ACDE, a Square conſtructed on AC, is equal to four Triangles, each equal to ABC, which is equal to half the Square, on a Side, AB. See Fig. 1.

Let the Sides containing the Right angle be 3 and 4, th Hypothenuſe will be 5; the Square of 5 being equal to the two ſquares of 3 and 4; for, $9+16=25$, the ſquare of 5.

The Area of the Triangle is 6 (half 3×4) and four times 6 make 24; to which, the ſquare of the Difference, 1, being added, the Sum total is 25, equal to the ſquare of the Hypothenuſe, 5×5.

That this holds in every right angled Triangle is manifeſt, and evident from inſpection of Fig. 2; although the meaſures of no other, than as above, can be taken with that accuracy as to produce equality, yet near enough to give conviction of it.

On the Hypothenuſe, AC, of the Triangle ABC, conſtruct the Square ACDE; and, on the three Sides, CD, DE, and AE, Triangles, CFD, DGE, and AEH, ſimilar to ABC, and alike ſituated on each Side; there will be formed the Square GB, having its Side equal to the Difference between AB and BC, equal BH; which, with the four Triangles, occupy all the Area of the Square ACDE.

FROM a Conſtruction ſomewhat ſimilar; it is evident and obvious, that the Square of the Hypothenuſe is equal to the two Squares conſtructed on the Sides containing the Right angle.

The Square ACDE being conſtructed, on the Hypothenuſe, AC, of the right angled Triangle ABC, and the two Squares, AG, and CH, on AB and BC, as in the Figure; let the Sides, FA and FG, IC and IH, be produced, meeting at K and L, forming the Square FKIL.

The two Rectangles, KB, and BL, are equal; for, AB=BG, and CB=BH; and, the four Triangles, AEF, ACK, &c. are equal among themſelves, for each is equal to half a Rectangle, BK, or BL; and conſequently to the Triangle ABC; wherefore, the four Triangles, AEF, &c. are equal to the two Rectangles, AB and BL.

But, the Square FI is equal to the two Squares, AB and BI, added to the two Rectangles, KB and BL. Ax. 2. 1. El. And it is alſo equal to the Square ACDE, added to the four Triangles, AEF, &c. Therefore, the Square ACDE (of AC) is equal to the two Squares, AB and BI, of AB and BC, as it was affirmed. Ax. 3.

That the Angle E, of the Square AG will neceſſarily cut FG, a Side of the Square AG; and, that IH, being produced, will paſs through the Angle D, of the Square AD, is demonſtrated, in Page 60, of the firſt Part; Th. 20. 1. in every proportion of the Triangle ABC.

THE Squares FB and BI, being cut by the Square AD, of the Hypothenuſe, into five Parts (three Triangles and two Trapezia, ABE and CH) they will cover the whole Area of the Square AD.

For, the Trapezium ABE, and Triangle Cdc are common to both; and, the Triangle AEF is congruous with ACB, and EaG with abD (Db being drawn perpendicular to BG); alſo, the Trapezium CIHd is congruous with Dbcd, which fills all the ſpace of the Square ACDE, and conſequently are equal to it, in Area; as it is demonſtrated in Th. 20. 1. El.

The two Squares being cut thus, with care, the Parts coinciding with the correſponding Figures in the Square AD will exactly cover it.

PROBLEM X.

In Parallelograms, the Difference between the Areas of thoſe which are about the Diameter is equal to the Difference of their Baſes, multiplied into the Sum of their Altitudes; or, the Difference of their Altitudes, multiplied into the Sum of their Baſes.

In the Parallelogram ABCD, DEFG, and HI are about the ſame Diameter, BD; I ſay, the Difference between the Parallelograms HI and EG is equal to the Difference of their Baſes, HF, FG, or ED, multiplied into the Sum of their Altitudes (CP); or, the Difference of their Altitudes, multiplied into the Sum of their Baſes, AD.

Take EK equal AE, equal HF; draw KL parallel to CD, and (through Q) MN parallel to AD; and draw the Perpendicular CR.

Then, the Parallelogram OP=HI, AO=EQ, equal QG; and QG=GL; conſequently, GL=AO. Wherefore, if from the greater Parallelogram, EG, there be taken the leſſer HI, or its equal OP, there is left the Gnomon EQG, equal to the Difference.

But, AO=QG; wherefore, the Parallelogram AN= the Gnomon EQG; i. e. equal to the Difference between HI and EG; alſo, as GL=QG, equal EQ, the Parallelogram KC is alſo equal to the Gnomon, EQG, and conſequently, to the Parallelogram AN.

But, KC is under the Difference of their Baſes, KD, and DC; and, AN is under the Sum of their Baſes, AD, and DN.

The Area of KC is KD×CR, and AN=AD×NR; i. e. the Difference of their Baſes multiplied into the Sum of their Altitudes, and the Difference of their Altitudes multiplied into the Sum of their Baſes; each of which, being equal, is equal to the Difference between HI and EG, the Parallelograms about the Diameter BD.

ON a retroſpection of the 9th Problem, Sect. 7. I find that, as it is there performed, it is applicable to Pentagons, or Trapezia, only; as the words, *from the two outer Angles*, will be liable to miſlead the Student, in Figures having more Sides; therefore, I think it neceſſary, or not improper, to give a more general application of it, to Figures of other Denominations, as follows.

IT is required, to divide the Heptagon ACF into three Parts, which ſhall have the Ratio, each to the other, between themſelves, as the Parts of the Side AG, divided at P and Q, at diſcretion.

From the 9th, Sect. 7. we might conclude, that C and F were the Angles meant, by the two *outer* Angles, particularly C, from the Diagonal AD; EF being parallel to DG, neither of them, or both, is implied thereby.

But, the Line (as there directed) muſt not be drawn from either, C or F, in this Figure, but from other Points, to be determined; by reducing the Trapezium ACBD to a Triangle, AHD, and drawing HI parallel to AD; and KM parallel to DG.

It is the ſame, whether the Triangle DKG, or DLG, be taken, equal to the Trapezium DEFG, as the Point M, it is evident, will be determined the ſame, by either; or by DNG; for, K and L are both in the Right line MN, parallel to DG.

The remaining Proceſs is there deſcribed (9. 7.); drawing IO and MO, through A and G, meeting at O; from which Point, Lines drawn through P and Q cuting IM, at R and S, divide the Trapezium AIMG in the Ratio, as AG is divided, at P and Q; by means of which, the given Figure is divided in the ſame Proportion, by the Lines OT and QU.

FINIS; OF THE APPENDIX.

AN

ADDENDA.

TO THE READER.

WHEN I undertook to write a Parallel and Critique, on the Engliſh Authors, in Perſpective, I was not apprehenſive, then, I could not conceive how arduous a taſk I had engaged myſelf in. I have, however, this ſatisfaction, that the Authors, then living, have never attempted, in any wiſe, to accuſe me of making uncandid Remarks on their Productions; and am on good terms, now, nor was ever otherwiſe, with one of them, not the laſt in merit. Another, a Gentleman of great literary Abilities, I am not acquainted with; a third (Noble) an ingenious Writer (ſince dead) I knew, and ſent him a Copy of the Critique on his Performance, before it was worked off. He came to me, on the occaſion, and had nothing to alledge againſt the Remarks I had made, of any moment, but expreſſed himſelf in theſe Words, as near as I can remember; "That he fared as well as others; if I gave him a rap on the knuckles in one place, I adminiſtered a plaiſter for a Sore, in another;" ſo that, after ſome trifling alterations, at his requeſt, he could not diſapprove of what I had ſaid, or impeach my Candour. He ſaid, further, "if I ſhould alter it ſo as he might wiſh for, and direct, it would be no longer a Critique of mine, but an Eulogium of his, on his own Work."

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I ſhould have done the ſame by the other two, ſtill living, had they lived in London, or near it; but they were at too great a diſtance from town; nor did I know where the learned Doctor reſided, then. Nothing on the ſubject has appeared ſince, that I have heard on, at leaſt, ſave a ſmall Pamphlet, of 79 Pages, in ſmall Octavo; which, from the manner of printing, might be comprized in 40 of theſe, and little more than half of it on Perſpective, with three Plates, the ſize of that annexed, by Mr. Bradberry; a Gentleman who has been making a figure, for ſome time, in the optical line; known, by the curious, in the ARTS of *Deception*. From the apparent inſignificance of the Work, on a curſory view of it, the Plates particularly, which are all of a piece with the ſample here given, I ſhould not have paid it, or the Author, ſo much Defference as to take *any* notice of ſuch a Production, had not the Preface breathed ſuch an Air of ſupercilious (what can I call it? not Ignorance, but) ſomewhat, bordering on Arrogance; which, giving it the moſt favourable conſtruction I can, may be termed ASSURANCE.

Being, at the time, employed in reprinting this Work, as a new Publication, in two Volumes, and was juſt begining this ſecond Part, when I firſt ſaw that Performance, although a ſecond Edition (as it ſays); and conceived it could not be improper, or deemed impertinent (except by the Author) to point out the merits of ſuch an extraordinary Production, in an Addenda to this Work. For as I had taken off a large Impreſſion of the Appendix to my Treatiſe on Perſpective, in which was the Critique on the various Authors thereon, it could not be added thereto, ſoon; and 'tis pity that ſo much modeſt merit ſhould be unnoticed: The Reviewers, I apprehend, have noticed and paid the attention due to its merits. I ſhall, therefore, make no Apology, for obtruding it on the

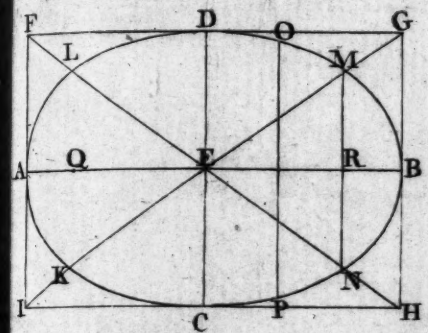
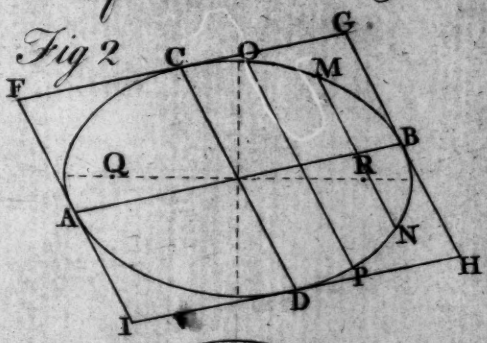


Fig 2



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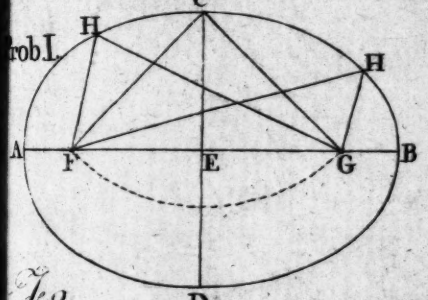

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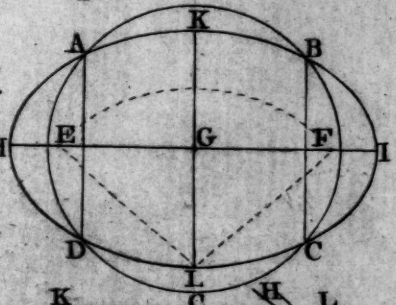
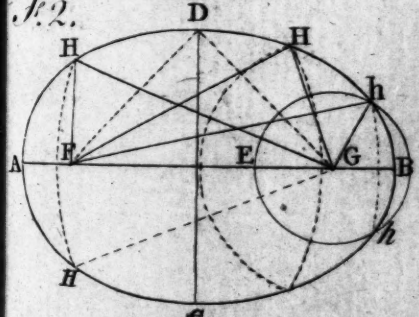


V.

Fig 2.



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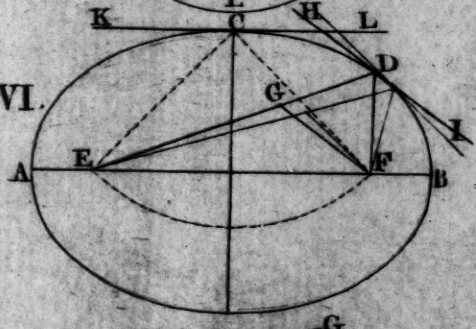

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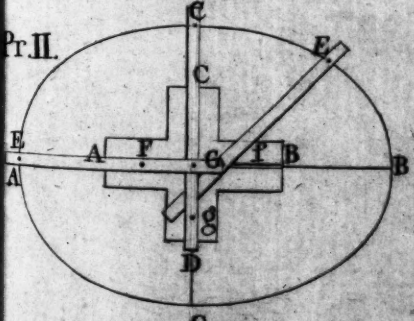

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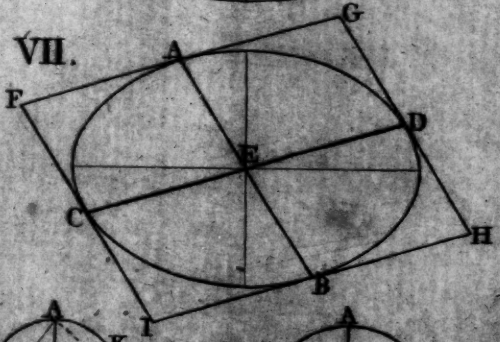

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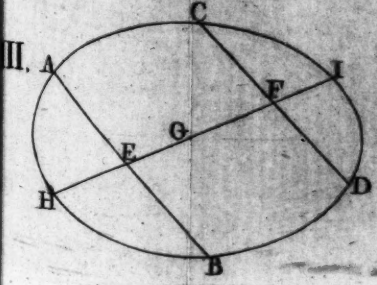

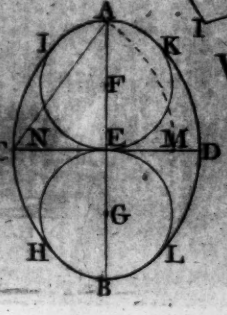

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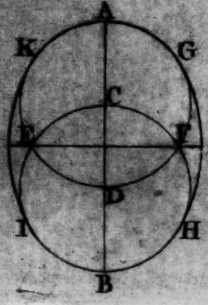
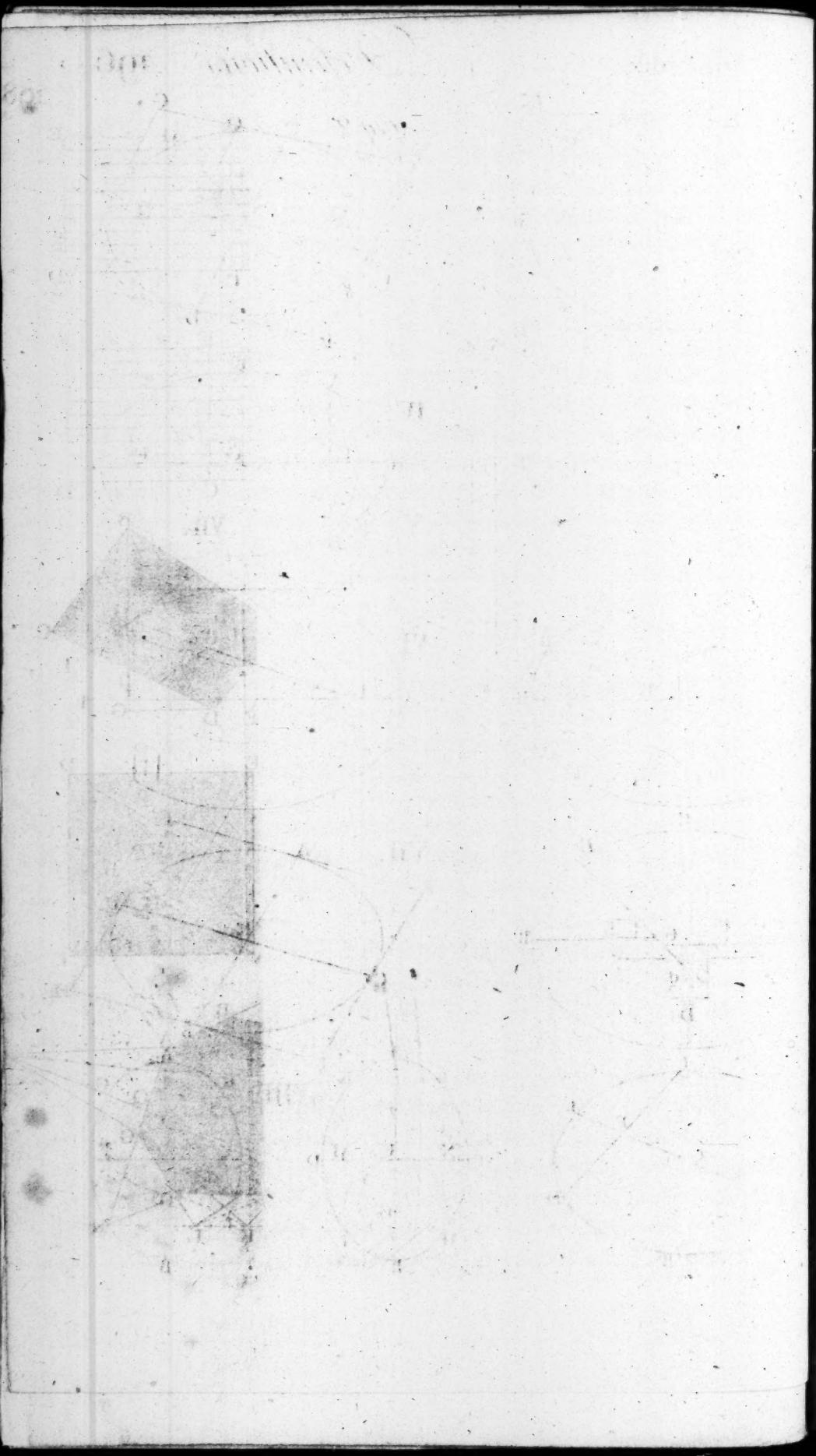
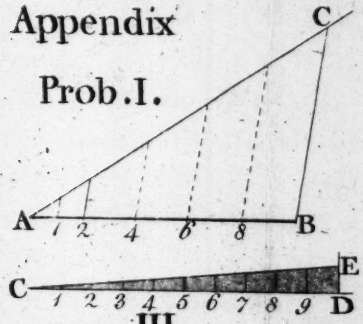




Appendix

Prob. I.



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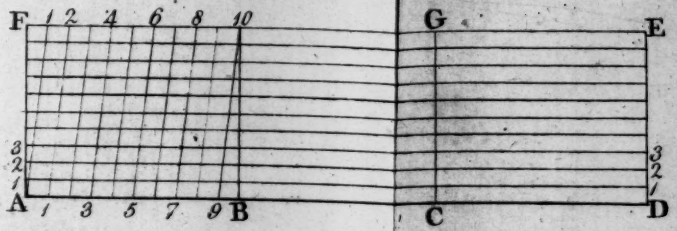

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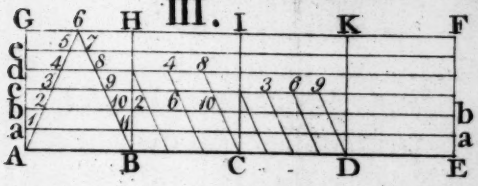

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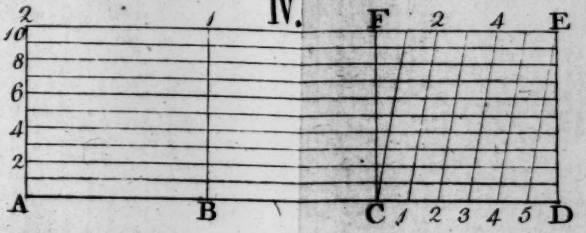

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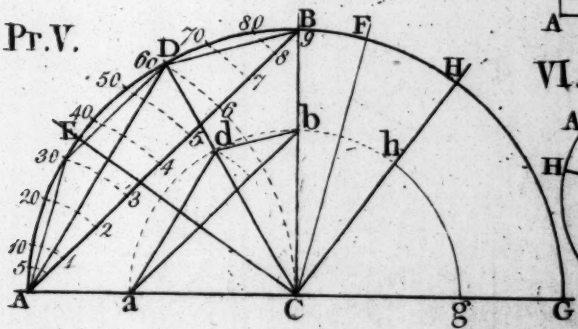

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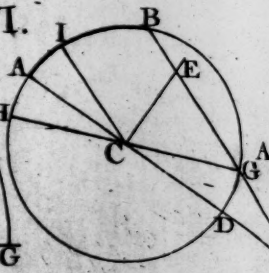

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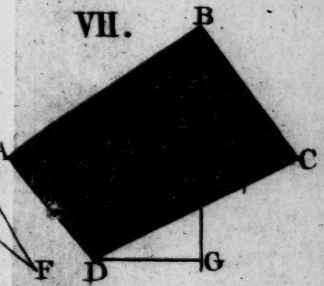

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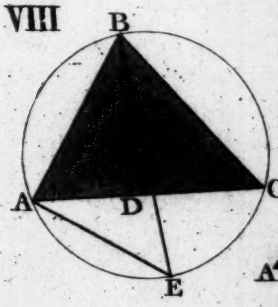
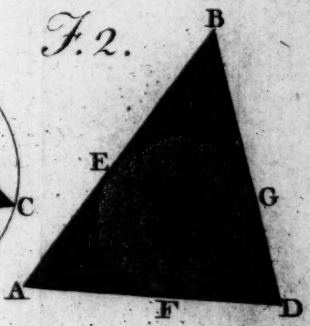


F. 2.



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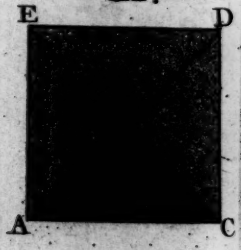

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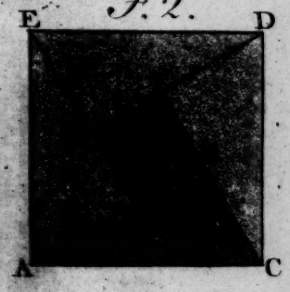

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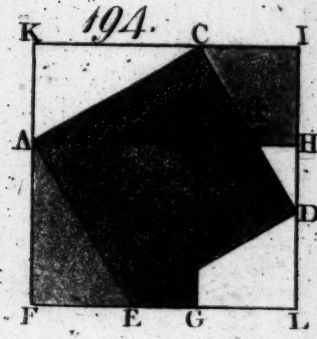
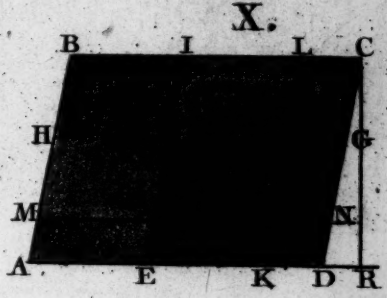


X.



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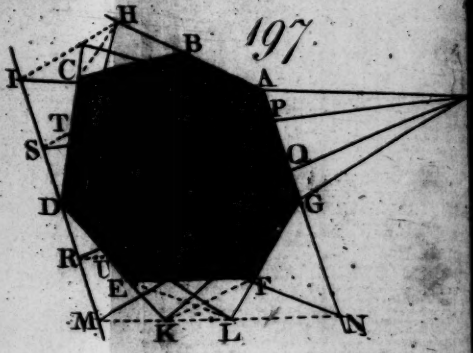

Public, here; in a Work, the Subject of which has so near Affinity with Perspective, the foundation of it, which must necessarily be known, before the other can possibly be understood; insomuch that, many Authors have made Geometry an Appendage to their Works, as being introductory to Perspective. That it is necessary to be understood is most certain, and it is equally so, in many other branches of the Mathematics; yet is never touched on, in the Introduction, or made an Appendage to Treatises on Astronomy, Projection of the Sphere, Conic Sections, &c. and yet, Geometry is previously necessary to a thorough understanding of those Subjects, as well as Perspective; but the Authors, on them, are content to refer to Euclid, for proof of what they advance, when it is requisite, in their Demonstrations; nor do I conceive it necessary to do more in a Treatise on Perspective. In small Tracts, merely practical, there is no impropriety in giving the elementary Principles of practical Geometry; as the practice of Perspective is so much dependent on it, that, without it, no progress can be made therein, being the Bases and Foundation, on which that elegant Superstructure, Perspective, is built.

THE Work now before me is entitled, "The Principles of Perspective, explained in a genuine Theory, and applied in an extensive Practice; &c." Surely, Mr. Bradberry was disposed to make a jest of it; if I thought he was serious, I would tell him, that *his* application of it is so very trivial, 'tis scarcely possible for any other Person to practice from it, at all, without other, and better assistance. "with the construction and uses of all such Instruments as are subservient to the purposes of this Science." The whole of which is a Parallel ruler, and a reticulated Frame, applied to Drawing, and taking

Views; one of them well known, to every Draughtſman; the other has been ſufficiently deſcribed, by others long ago, and, by ſome of them, much better.

In his Preface, he ſays; "No mathematical ſcience requires a *Theory* more than *Perſpective*, and perhaps no one has been leſs attended with it in publications of this kind."* The laſt Sentence is literally true, laying emphaſis on *this*, nor has he deviated from the reſt, that I can ſee; for, ſurely, Mr. Bradberry does not imagine, that HE has remedied that defect, in other Works, *of the kind*. "The complex manner† in which ſome Authors have treated the theoretical part has not been ſatisfactory to me;" probably ſo, nor to many more; who can help that?

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* I had no intention to cavil at, nor do I mean to criticiſe on Mr. Bradberry's literary Abilities, which are conſpicuous in this Work; but, there is ſomething ſo remarkable in the ſyntax, and grammatical conſtruction of this firſt Sentence, that I cannot but take notice of it. The whole Sentence is divided, by a comma, into two parts; the firſt is ridiculous and abſurd. What kind of Science is that, which has no Theory? the term SCIENCE, I think, implies *Theory*. Is not all Science ſpeculative? How then can a Science exiſt without Theory? But many Arts may be, and are practiced, without knowing the Theory of the Art; as is moſt notorious in the uſe and application of Numbers, which are ſo generally known.

But, the ſecond part of the Sentence is what I moſt admire; which ſhould have had another diviſion, cuting off the ablative part, *in publications of this kind*. It is hard to ſay, whether or no there was Deſign in this. What Mr. Bradberry meant is, in my opinion, complete without it; changing the Conjunction *and*, for *yet*; and *attended with it*, for *illuſtrated thereby*. But, if he meant the ablative Sentence to expreſs what it really does, why, I think him remarkably candid; for I do readily acquieſce with him, that no Publication ever was leſs attended with Theory (where it was requiſite) than ſuch as his. What elſe it implies, I cannot diſcover; 'tis in no wiſe neceſſary, to enforce or elucidate the foregoing; of *what* kind, then, can he mean? Publications on Perſpective would be worſe than Tautology, it would be, literally, moſt abſurd; for ſurely, Publications on *other* Subjects have nothing to do with a Theory on Perſpective.

† I preſume, he means rather, the *perplexed* manner.

But why, I aſk? what ſatisfaction does Mr. Bradberry require? Is the defect in the Theory given, or elſewhere? for I can aſſure him, that Perſpective has been as copiouſly, fully, and as ſatisfactorily treated as any Science, whatever; and as ſimply (for, Perſpective is by no means an abſtruſe Science) nor is any other more capable of the ſtricteſt Demonſtration, and particularly, of thoſe which are optical.

Obſerve what follows. "And as many others are of my way of thinking, I here preſent them with a *Theory of Perſpective*, which appears to me the moſt genuine, natural, and perſpicuous that poſſibly can be;" That may be; who can anſwer for, or object to its appearing ſo to him? but I think it will do ſo to none elſe. For, I apprehend that, thoſe who know what Theory is, and look for it in Mr. Bradberry's Work, will look for it in vain. Again; "as it conſiſts of the feweſt, and thoſe the moſt ſimple principles." Is it poſſible he can imagine, that this paltry Diagram (a very lame and imperfect Copy from Kirby's 53rd Figure; made more general in the 54th and 55th, and perfected in the 56th) contains the whole Theory of Perſpective? all *his* Theory it may, I grant. A Novice, in Perſpective, might be led to imagine, that Mr. Bradberry had diſcovered ſome defect in Brook Taylor's Principles; or, that he had found out ſome new ones, by which, the Proceſs might be facilitated, and, the whole comprized in leſs compaſs; in the proportion of *his* Work, to Dr. Taylor's. But I cannot find that he has made any new diſcovery, or the leaſt ſhadow of improvement in it; but greatly diſparaged the Science, and Art of Perſpective, in *what* he has done. And yet, I muſt ſay, that he ſeems to underſtand the Subject better than Ferguſon; whoſe Treatiſe on Perſpective, though more elaborate and ſtriking, to ſight, than this, is, of all other, the moſt puerile; and (ſave this)

the moſt contemptible production, on Perſpective, that has ever been expoſed to public view.

The following Paragraph is curious, and the moſt extraordinary of all. " The application of this Theory to practice, in the common way, is ſo eaſy and natural, that I think what is here ſaid may ſuffice for the purpoſe;" Is it poſſible for any Perſon to be ſo partial to his own productions and talents, as, by the magic of ſix lines, about " the motion of a point, and a viſual ray proceeding from it to the eye," to make any one elſe apply them to practice? What is meant by *the common way* is only to be underſtood by himſelf. " But, as there are many important particulars relative to the *Theory* and *Practice* of *Perſpective* not touched upon in treatiſes of this ſort, I have added as much on that head as I judged was neceſſary to give the ſtudent of the *Polite Arts* as juſt a notion of this ſcience as it deſerves, and as he ought to conceive of it." On *what* Head can he mean?

Was ever the like of this publiſhed before? it exceds all I have ever read, or heard of. I ſhould be glad to know what thoſe important particulars are, which have not been touched on by others, in Treatiſes *of this ſort*; 'tis, indeed, true enough, for none are to be found here, of any importance at all. It is much to be lamented, that Perſpective has been ſo ill treated, in general, as to make it ſo little deſerving the notice of Students in the Polite Arts, particularly; and juſtly may it be ſaid, if what Mr. Bradberry has publiſhed thereon be all that was neceſſary, to give them as juſt notions of the Science *as it deſerves.* But, ſurely, not as they *ought to conceive* of it; for, I think they ought to conceive more highly of Perſpective than, in general, they do; and apply themſelves, more aſſiduouſly, to the ſtudy of it, before they give too free ſcope to their volatile genius; by which, it might be kept within due bounds, and without being

ſhackled thereby, or tied down to Rule and Compaſs, in their Deſigns; their Judgment being corrected, by juſt notions, and a competent knowledge of Perſpective, though but in rationale, would prevent their runing into palpable and groſs abſurdities, in their Compoſitions; which they cannot account for, and reconcile to Reaſon, nor know how to correct; and therefore, they lay the whole blame on Perſpective, of which they have got a little ſmattering; juſt ſufficient to make them deſpiſe and ſlight its Rules, and treat that entertaining Science, and moſt valuable Art, the foundation of their own, with ſupercilious Contempt.

" Genius and inclination for a particular ſtudy may, by labour and aſſiduity, ſurmount many difficulties; yet to remove impediments from the path of ſcience is a grateful taſk, and genius will feel and acknowledge the obligation; for thoſe who are moſt capable of profiting by inſtruction are generally the moſt ſenſible of its value." Doubtleſs; but, I apprehend, few will profit from *his* labours, in Perſpective, though truly Herculean, in removing *Impediments* from the path of SCIENCE; which, with him, is a grateful taſk, being ſo able to encounter every difficulty of the kind, and ſo well diſpoſed to the undertaking. Real Genius muſt, and ever *will* acknowledge how much the fine Arts are indebted to Mr. Bradberry for his generous aſſiſtance, in lending his able hand, to facilitate their Studies; the Obligation can never be ſufficiently acknowledged.

" I have (he ſays) purpoſely avoided all kind of prolixity,"* (of difficulty rather) " and ſtudied one point only here,† viz. to make the way to ſcience as eaſy and

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* There are different degrees of it, but I never, before, heard of different *kinds* of Prolixity.

† As many others; to get Money, by any means; no matter how.

conciſe as poſſible, and with the leaſt expence of money and time." Yet, notwithſtanding his precautions, he has not, though with much Apparatus, of a perſpective Priſm, (a ridiculous term) a Viſual ray, and the imaginary motion of a Point; &c. communicated, and given ſo clear an Idea of what he intended, and that partially, as may be done generally, and more intelligibly conveyed, by imagining a Plane to paſs through any Right line, in an Object any how ſituated, and the Eye; which, cuting another Plane (the Picture) interpoſed, their interſection is the indefinite repreſentation of that Line, thereon; nor can the geneſis of a Right line on the Picture, be better and eaſier conceived, by a multiplicity of words and much circumlocution; the finite Line is determined by a Viſual ray from each extreme of the original; not its *Seat*, but the perſpective repreſentation of the Line.

Is this to pave the way to SCIENCE? and with the leaſt expence of Time and Money; I fear *both* will be thrown away, loſt, to thoſe who purchaſe this Production, and determine to ſtudy it, 'till they acquire a competent knowledge of Perſpective, either in Theory or Practice. I applaud his Gratitude to a *Generous Public* (in the following) for the ſupport they afforded him, in his "Exhibitions, of Optical and Philoſophical Effects," and his inſtructive Lectures; I would adviſe him to exert and employ all his great talents on *that* Work, which THOUSANDS are ſo anxious for; and not let that future Day be too remote; by which, the generous Public may (perhaps for ever) be deprived of the pleaſure and ſatisfaction, which muſt reſult from the moſt wonderful and beautiful effects (of his Art) that has hitherto appeared in *Optics*, ſuch as have never been thought of by any perſon but himſelf;" with what confidence he aſſerts what 'tis not poſſible for him to know. What a loſs to the World, for ever, ſhould he be called hence, before it be perfected.

His modeſty is commendable, in not requiring any advance on it, from the Subſcribers. I adviſe him not to be too nice, in that reſpect, having experienced, to my coſt, the ill effects of a generous Credulity; imagining that the Bubble, called Honour, was a ſufficient Pledge; well-a-day! that's as eaſily broke as a bubble of Air.

I HAVE, now, followed Mr. Bradberry entirely through his Preface; which is, indeed, genuine, and truly original. Next, I proceed with "The Genuine Theory of Perſpective, demonſtrated from its firſt Principles."

Firſt, then, I meant to examine his Definitions; as his Theory is genuine, one might reaſonably expect ſome new Terms to be defined. But, ſo far from that, I can find ſcarce *any*; a new and genuine Theory of Perſpective, almoſt without Definitions of the Terms made uſe of; and without one Axiom to ſupport the Theory. However, he defines Perſpective to be "the art of delineating on a plain ſurface the appearance of objects, ſuch as they have upon a plate of glaſs, placed between the eye and any object we look at, when the eye is placed at a given diſtance from the glaſs." I ſhall paſs over the ſenſe, and literal conſtruction of this Definition, ſuch as appearance of objects, *upon* a plate of glaſs, which would lead me into an extent that would be endleſs; and which, I have not, at this time, either leiſure or inclination for.

How greatly defective, and inadequate, this Definition of Perſpective is, ſhall be made evident. I ſhould imagine that, from Mr. Bradberry's optical ſtudies, he muſt, or ought to know, that the perſpective delineation of an Object is not its appearance; nor have Objects *any* Appearance on, or upon, a plate of Glaſs interpoſed; but, that a repreſentation, or apparent figure of the Object traced thereon, will communicate and convey an Idea

of the Object to the ſenſorium, or ſeat of Viſion, ſuch as it appears to the Eye, at a determined diſtance and poſition, or Point of view, in reſpect of the Object and of its Repreſentation. The Eye being placed at a given diſtance from the Glaſs is vague and undeterminate; as giving latitude for its being either higher or lower, to the right hand or to the left, to any diſtance, all around, from the only true point, which is invariable.

I find, ſcattered, here and there, without order, or form of Definitions, ſome few Terms explained; as, in the next Paragraph he ſays, "the point in which a viſual ray tranſits the tranſparent plane* is that which we call the *place* or *ſeat* of that point upon the *perſpective plane*." This is exceptionable, and *we* call it wrong; but, in a genuine Theory, who ſhall judge of the propriety, or preſcribe rules for the Author; who may call it what *he* pleaſes. I can only aſſure Mr. Bradberry that it has not uſually been ſo called, by other Authors; it *is* the perſpective *place*, or repreſentation of the Point thereon. The Seat of a Point, out of a Plane, is where a Perpendicular from the Point to the Plane cuts it, invariable; the other is infinitely variable, dependent on the place of the Eye.

"But this doctrine of the ſeat of a *point*, a *line*, a *ſuperficies*, and a *ſolid* muſt be demonſtrated by proper diagrams, that it may become eaſy and perſpicuous;" they ſhould be ſo; but, that they *muſt* is rather arbitrary, and cannot always be enforced, as is the caſe here. What fulſome pompoſity is in this *doctrine*, &c. "This method conſiſts in the conſtruction of a perſpective priſm, and its interſection by the perſpective plane;" ſection by,

* To tranſit a Plane is to paſs by or over it, not (as here meant) through the Plane; as a Planet is ſaid to tranſit the Sun's Diſk, an apparent Plane, when it paſſes over the Diſk.

it ſhould be, or interſection *with* the perſpective Plane. This perſpective Priſm of the Author's is, I believe, genuine. "The baſe of the priſm IGMK is the *ground plane*, in which all objects in the plane of the horizon are conſidered, and their perſpective ſeats determined in the plane AC." That is, the Baſes of ſuch Objects as ſtand on the Ground; their perſpective Seats determined in the Plane AC is abſurd, and as erroneous as of a Point. "The vertex Y of the priſm is the place or poſition* of the eye at its given diſtance Y*y* from the perſpective plane, which is here ſuppoſed to be poſited direct before the eye, or at right angles to the priſm." It is very difficult to conceive either, from this Diagram, which has more the appearance of the Vanes, or Sails of a Windmill, ſeen obliquely; but ſurely, the Plane of projection, that is, the Picture, may be direct before the Eye, when the *perſpective Priſm* is very oblique to it, as well as perpendicular thereto. The term Vertex is not applicable to Priſms, but to Cones and Pyramids; it is here, ſimply, the Vertex of a Triangle (GYI) *one* of the Baſes of the Priſm; for, what he calls its Baſe is not ſo, properly.

"The part of the plane AD, which ſtands on the ground, is called the *ground line*;" *I* ſhould have ſaid, the Line in which it *cuts* the Ground; "the point *(y)*† is called the point of ſight; and the line E*y*F (parallel to AD) the horizontal line." *y* is not the Point of Sight, but Y, the Vertex of his Priſm; the other is (according to Dr. B. Taylor) the Center of the Picture. Theſe are all I

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* The Eye, conſidered as a Point, merely, has *no* Poſition.

† Where is the propriety of puting *y* in a Parentheſis, as here? To leave it out in reading the Sentence (as, the point is called the point of ſight) *what* Point is not determined: This, I obſerve, is frequently done, in this Work; yet, the Vertex Y, which might be within Parentheſes, is not.

can find which can be called Definitions. The horizontal Line, and the Ground line, are certainly as he calls them; but, the former is arbitrarily ſo, he gives no reaſon *why* it is ſo called, as is reaſonable to expect.

This very extraordinary Plate contains the whole of Mr. Bradberry's Theory, and indeed, of his Practice too; for the firſt Figure, in this Plate (No. 3.) is taken from the ſecond of his, and the ſmall ones (No. 10. & 11.) from the 3rd; ſo that, here is the largeſt half of his three Plates, in this one, which contains *multum in parvo*; as, in it is comprized a genuine Theory of Perſpective, explained and demonſtrated from its firſt Principles (it is difficult to find out what thoſe Principles are) and applied in an *extenſive* Practice (without ever riſing out of, or above the Ground plane, a groveling Practice); which are almoſt ſecure from Criticiſm, they are *below* it. I am aſtoniſhed, that any Perſon can have the effrontery and aſſurance, as, with ſuch confined talents for ſcientific ſpeculations, apparently, and *no* genius in the Art of Delineation, ſhould attempt to impoſe on the Public an oſtentatious diſplay of Knowledge, on a Subject which requires both. Had his perſpective Priſm (as he calls it) been better deviſed, or more judiciouſly copied, from Kirby's, ſomething might be deduced from it, where there was a Capacity for the undertaking; but in this Work (alas! poor Yorick) others may find out, by it, what I cannot; and yet, extenſive as it is, I think I ſee, in it, all that the pompous Author of it aimed at, which is little enough.

In the firſt place, I think he intended, that his perſpective Priſm ſhould be an equilateral triangular one (but it does not appear ſo) or Iſoſceles; one Face horizontal, and its length parallel to the Plane it is projected on, but very oblique to ſight; and, the perſpective Plane cuting it perpendicularly. I would aſk Mr. Bradberry, whether, in this caſe, the two horizontal Lines, IG and KM, of the

Ends, or Baſes, are not parallel Lines, and parallel to the Ground line, AD (the part of the perſpective Plane which ſtands on the ground) which are perpendicular to the Plane they are projected on, and conſequently vaniſh in its Center; and how they can have parallel repreſentations here, with the Line CB, alſo? Seeing that, they are conſidered as receding, directly, from the Picture, or Plane of projection. And as OQRT is ſuppoſed to be a Square, whether, in this caſe, the two Sides OT and QR ſhould be equal, and the other two (OQ and RT) alſo equal and parallel Lines, and longer than the other, which are parallel to the Picture? I think, that, as a perſpective appearance of the whole Apparatus was aimed at, theſe things ſhould *not be ſo.*

I would alſo aſk him, if *y*X and *y*Z ſhould be equal here, as in Fig. 2? And if he means that they ſhould be each greater than Y*y*, the geometrical Diſtance? For I am inclined to think they ſhould not; he'll excuſe a Novice; if I am not clear in theſe matters, I ſhould be glad to be better informed. I alſo imagined, that the diagonal Lines, OX and QZ; on the perſpective Plane, would cut the Radials, O *y* and Q *y*, in the ſame Points where the Viſual rays *y*R and *y* T *tranſits* it; that is, at *r* and *t*; but, as they do not concur in thoſe Points, I am at a loſs to find out, by what means *r* and *t*, the *Seats* of R and T (as he calls them) on the perſpective Plane, are determined.* There are various ways of doing it, I imagine; but as there is no appearance of any other, here, I ſhould expect to have found a perfect coincidence, between OX and the Viſual ray YR, in the Point *r*, which is (as I apprehend) meant to repreſent the Angle R, of the Square OQRT.

* This is a juſt and correct Copy of Mr. Bradberry's Diagram.

As Mr. Bradberry has made the Science of Optics his peculiar Study; I mean, ſome Deceptions in Optics his *Profeſſion*, ſo long, it cannot be unreaſonable to ſuppoſe that he is thoroughly acquainted with the Subject he has attempted here, or, why did he attempt to inform others? Thoſe things may be mathematically demonſtrated; then ſurely, they ought by no means to be diſpenſed with, in this Diagram, which contains his whole genuine Theory, the moſt perſpicuous that *can poſſibly be*, as well as genuine, and natural. Genuine it is, though not original; and it *may* appear to *him*, the moſt perſpicuous poſſible; I apprehend to none beſide. And yet, he has taken no ſmall pains to appear well verſed in Perſpective, in Theory at leaſt; and uſes the term SCIENCE as familiarly, on every unneceſſary occaſion, as if he meant to inſinuate, that he is completely qualified to inveſtigate the Subject ſcientifically, or, in ſome degree, mathematically. He has, indeed, diſplayed ſome knowledge in the doctrine of Proportion; and ſpeaks of ſimilar Triangles with a degree of familiarity, indicating a thorough acquaintance therewith. But whether he has or not, I do aſſure him, that, that Doctrine is very eſſential in Perſpective; to underſtand it clearly, without a tolerable knowledge of ſimilar Triangles is not poſſible; 'tis the Eſſence of Perſpective, both in Theory and Practice.

But, I do not ſee the uſe, or neceſſity for conſtructing the ſimilar Triangles QTO, QZV (Fig. 1 and 2) by making OV equal to *y*Z, the diſtance of the Eye (See Page 20) ſeeing that, the Triangles already formed, QTO, *y*TZ (of which he ſays nothing) do all that was wanted, effectually, without any other; nor can fuller Demonſtration be given than is deducible from them. For ſince, by them, OT: T*y*:: QO: *y*Z; conſequently, OT: O*y*, i. e. to OT + T*y*, :: QO: QO + *y*Z (equal QV). In plain Words, at length, the repreſentation OT (or O*t*) of OT, has

that proportion to the whole indefinite repreſentation, O*y*, as QO, equal OT (or ſhould be) has to OT added to the diſtance of the Picture, or perſpective Plane, equal QV, by his unneceſſary Conſtruction; in which, he takes no notice of VZ being equal to O*y*, though an eſſential requiſite in *his* Demonſtration. As to the poſition of the whole, being horizontal, it is of no conſequence; becauſe 'tis all the ſame, being vertical, or otherwiſe poſited. As this is the whole of Mr. Bradberry's Theory, I would aſk him, if no other Writer, on Perſpective, has done the ſame thing, as clearly, and far more ſatisfactorily before? or if he imagines that there is any thing either original or novel, in *what* he has done?

I muſt here beg his pardon for having ſaid he gives no reaſon for calling EF the horizontal line; for, in another ſhort Paragraph, at the diſtance of four Pages, a reaſon is given, in theſe words. "It is manifeſt that any line QO, however large at the perſpective plane, yet *re*moved to an indefinite diſtance, viz. to the *horizon*, will there appear in the line EF, paſſing through the point (*y*) which is therefore called the *horizontal line*." But, why muſt it paſs *through* the point *y*? is it not enough to be *in* the Point? nor *can* it go through, while it is in the ground plane, at leaſt, nor if it were raiſed out of the ground plane; for, the Point *y* repreſents a Diſtance that is infinite; and therefore, as the Line cannot move beyond infinity, conſequently, it cannot appear to be paſſing through the point (*y*); and I am greatly miſtaken if this be not a ſatisfactory reaſon (*why*).

THIS Work is divided into eight Sections or Chapters; the firſt of which (the genuine Theory of Perſpective demonſtrated from its firſt Principles) I have examined, and find nothing in it which anſwers to that Title, nothing that can be called genuine, except the motion of

a Point be ſo, as applied here. For (as I have already obſerved) the Perſpective priſm is by no means genuine; and which, of all the Diagrams, I have ever ſeen, cut on Copper, is the moſt uncouth and diſtorted; and, except to thoſe who are acquainted with the Subject it relates to, the moſt unintelligible: I could not, for ſometime, find out what it meant, how then ſhould a novice in the Science. As to the motion of a Point, and Viſual ray tracing out the line on the Picture, which repreſents the Line it moves in, the Idea is a good one; but 'tis merely ideal, as is that of the line OQ moving in the direction of OK and QM, having one extreme always in each Line, until it vaniſhes in the point *y*, but never can paſs through it; yet, although I cannot, at this time, refer the Reader to the Author of that Idea, he may take it for granted, that it did not originate with Mr. Bradberry; it is not genuine, and never thought on before, by any but himſelf. Probably, it may not have been applied to Perſpective before; and if it never had, the Science would not, I am of opinion, be in any reſpect defective, for want of it; for the Subject is not, in the leaſt degree, rendered more perfect thereby; it neither elucidates the fundamental Principles, nor renders them more applicable to practice; as, a Line ever was drawn on the plane of the Picture, by the motion of a Point.

The ſecond Section he calls a Demonſtration of the foregoing general Rule of Practical Perſpective, &c. which has alſo paſſed examination: But what that general Rule is, I am at a loſs to diſcover; for I did not perceive that any Rules are there given. On a retroſpection, I find ſomething like a Rule, in theſe words; "All indefinite right lines perpendicular to the ground line of the plane, as *Od*, *Pe*, *Qc*, have their *perſpective* expreſſed by right

lines converging to the point of ſight (y) as O*y*, P*y*, Q*y*,* which are called *radial lines*." As by the old Authors; another Definition, which is grown obſolete, as not expreſſive, and ſignificant of what is meant by them. Lines radiating from the Eye only, are, by Dr. B. Taylor, called Radials; and, I think, with far more propriety than Lines tending to their vaniſhing Point, even though that vaniſhing Point be the center of the Picture; but, who ſhall decide when Doctors diſagree, in their Doctrines? I ſhall not preſume, being well ſatisfied to take them as I find 'em given by ſuch able and learned Men. Nothing more having the leaſt appearance of a Rule, either general or particular, is to be found in this Section; or elſewhere, the ſecond being a Demonſtration, only, of the foregoing; which Demonſtration I have examined, and find, that by an unneceſſary conſtruction of ſimilar Triangles, the Points *r* and *t* are the *ſeats*, on the Perſpective plane, of the Points R and T, on the Ground plane; yet, no Rule is given for determining thoſe Points, on the Picture, practically.

The third Section of this extraordinary Production is "The Principles of Scenographic Perſpective explained and demonſtrated." The Subject of this Section ſeems to be a fair field for critical diſquiſition; but, unfortunely, the Book I have (which was ſhewn me by mere accident) is defective there; that whole Section, with part of the foregoing, being torn out; and I have not been able to meet with another Copy of it; as it is printed for the

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* Surely, this is not his general Rule! I think it a very partial one; becauſe, not only Lines perpendicular to the Ground line; and in the Ground plane, but all which are parallel to them, as being perpendicular to the plane of the Picture (which is general) wherever ſituated.

Author, only, I can't hear of it at the Bookſellers. What he means by Scenographic Perſpective, I can't deviſe, which is there explained. I regret the want of its information; as I muſt have been under a miſtake, hitherto, reſpecting it, having always conſidered the Term Scenography as ſynonimous with Perſpective; or, the difference between them is not, to me, very obvious; and I find, that ſcientific Lexicographers have the ſame notion of it; ſome defining one with the other. But, here, it ſeems to be a diſtinct part, or branch of Perſpective, only; with which I may, probably, be unacquainted, and wiſh to be better informed; nor can I be ſatisfied until I know how this able Author has explained it, and, on what its Principles conſiſt; which, doubtleſs, he has rendered clear and perſpicuous as *poſſibly can be.* The loſs is to me the greater, in that, I cannot, at this time, communicate that branch of Perſpective to the Public, and my Treatiſe is defective therein; the notion I always had of it is, at full, in the ſecond Section of the ſecond Book, Page 47; which, without doubt, muſt be very erroneous, as Mr. Bradberry has demonſtrated it, on Principles in no wiſe congenial with my conceptions of it, as being a diſtinct ſpecies, or kind.

The next Section is called the Analyſis of Perſpective.

Having, in the foregoing, given the whole Theory and Practice of Perſpective (in which Mr. Bradberry has added as much on the head of *important particulars*, not touched upon in Treatiſes of this ſort, as he judged to be neceſſary for a Student in the Polite Arts, thereby rendering it copious and full) he is now diſpoſed to entertain them with an analization of Perſpective, as by a chymical Proceſs, reſolving it into the original and conſtituent Parts, of which it is compounded; a thing much wanted by them; who, not ſatisfied with knowing Per-

ſpective ſuperficially, from other Works, will be happy to find it analized in his; by which they may comprehend it clearly and ſatisfactorily, ſeeing into the very Eſſence of it, to the bottom. He tells us, "it conſiſts in the reſolution of a *perſpective picture* into its *prototype*, or original component parts, ſo that we may form a juſt idea of their ſituations, diſtances, dimenſions, &c." and, that "The firſt ſtep to be taken in this proceſs, is to conſtruct two *lines of meaſures*" (i. e. Scales) "one for taking the dimenſions, diſtances, &c. of objects upon the *ground plane*, and the other for thoſe which have an elevation above it." I ſhould think *one* ſufficient for both; anſwering, equally, for Plans and Elevations; but, to ſave trouble, one is fixed upright, the other lies, ready, on the Ground line.

The whole of this Chapter, or Section (which, I find, comprehends, or contains no *very* important particulars) is, of all the others, ſo futile and inſignificant, as renders it beneath my notice; and, as he applies it only to Lines parallel and perpendicular to the Picture, and that for the moſt obvious and cogent reaſon imaginable, it was in no wiſe neceſſary; as, all the Lines, on the Picture, whoſe originals are parallel thereto, are, every where, in the proportion to one another as the Originals; the others are dependant on the Diſtance wholly. It is impoſſible to do juſtice to the merits of this curious Analyſis, without quoting and remarking on every Sentence, which is unworthy of Cenſure; as I do not conceive any advantage to the Reader, or ſatisfaction to accrue from it, it would be treſpaſſing on his time, to no purpoſe whatever. In one place he ſays, "if Lines were drawn by the edge of a rule from the point (*y*) to each of thoſe diviſions, they would ſhew in any part of the ground plane, or ſide plane, the proper length of any perſpective lines parallel to the ſaid ſcales, and, of courſe the form of

walks, *canals*, *viſtas*, *&c.* which are perpendicular to the ſaid ſcales." This is altogether incomprehenſible, that their *forms* ſhould be ſhewn by the application of a Ruler, or *Rule*; it muſt be more expreſſive then, than that given. In another place, " But now to reſolve and meaſure any *parallelogram* in perſpective parallel to the ground line, &c." A Parallelogram may be parallel to the Picture, but, to the Ground line is not determinate. All his Parallelograms are ſo, and (to avoid difficulty) right angled, chiefly Squares.

" To analize a building, or other object, in any part of the landſcape, you are to obſerve, that all the ichnography of it is reſolved and meaſured by the method juſt mentioned; and the ſcenography is determined by meaſuring the upright lines thus." I had not, before this, any conception, what part of a perſpective Draught is diſtinguiſhable from the reſt, by the appellation of Scenography, which is here made evident; for, as he uſes the term Ichnography (as being more pompous) for Plans, and Figures on the Ground plane, he applies Scenography to Elevations above it; ridiculous! ſo, here is an explanation of his third Section; and as to the demonſtration of it, it may be very well diſpenſed with. I always underſtood, before, that Scenography included the whole, not applicable to particular parts of a perſpective View, only, as here.

The three laſt lines of this 4th Section are worth all the foregoing, as having ſome ſenſe, and meaning in them. " Thus birds diminiſh to our view, and apparently deſcend towards the horizon, when they fly from us in a direction parallel thereto;" and even *this* is exceptionable; for we may be, and are frequently, ſo ſituated, that they will ſeem to aſcend, as they ſkim along the ſurface of the Ground, or Water, in a direction parallel to the Horizon, below us; which needs no demonſtration.

The Analyſis of Perſpective, in the laſt Section, is ſo very puerile, ſo trifling and uſeleſs, as not to deſerve further comment on it; being, from its inſignificance, below Criticiſm; otherwiſe, much more might be ſaid, for it is exceptionable in almoſt every part of it. I therefore proceed to the fifth, and laſt Section on which I mean to expatiate, and do honour to, in making remarks on it; the ſixth being a Deſcription of "the conſtruction and uſe of a Perſpective Table (as he calls it) for readily drawing in true Perſpective, &c." invented by Sir Chriſt. Wren, ſince, much improved, and perfected by our Author; as *his*, "he thinks, admits of the greateſt facility and certainty of motion, which is the utmoſt excellency of this ſort of inſtruments." This Inſtrument (which he deſcribes in a moſt puerile manner, telling us of what materials each part is formed, how fixed or moved, like a School boy) is, ſimply, a large double parallel Ruler; which he deſcribes, more fully, in the 7th Chapter.

One Sentence, in this Section, I *muſt* remark on, as my peculiar Province is to detect Error. He ſays (P. 53) "ALSO, *it is evident, that for the ſame height of the Eye y P, the landſcape will increaſe or decreaſe as the diſtance of the Eye from the perſpective plane decreaſes or increaſes.*" As the whole of this Sentence is in Italics, I apprehend, he meant to have it particularly noticed; I adviſe him, however, in the *third* edition (if not too late) to reverſe it, leaſt his judgment in Optics ſhould be queſtioned, which might hurt him, in the wonderful effects of his optical genius, which he has propoſed to produce, ſometime; it will be eaſily done, by changing the firſt Syllables in the two laſt Words, making them *in*creaſes, and *de*creaſes; as the Ratio is not reciprocal, but direct.

But, what has the height of the Eye to do with increaſing or decreaſing the Landſcape? which, previous to this, he ſays, "will be *larger* as the height of the Eye is

greater, at the *ſame* Diſtance;" and in continuation of the foregoing Paſſage; "Hence, when the eye is at a great height, and the diſtance of the plane or table from it very ſmall, the perſpective of a given ſquare will be *enormouſly great*, and almoſt of a *triangular form*." How ridiculouſly abſurd and nugatory is all this; the height of the Eye is not productive of increaſe of the Objects on the Picture, but is the cauſe of diſtorted repreſentations of them, by draging out the receding Lines in the Object, too much. No Perſon can be ignorant, or inſenſible, that the higher the Eye is raiſed, the Horizon of the Landſcape is raiſed, in proportion; and conſequently, the Ground is raiſed to a greater height, in the Picture; yet, the Objects are not *larger*, but we ſee *more* of them from an Eminence, than we can do on a level with them.

THE fifth Chapter is, as *he* terms it, "The Mechaniſm of Perſpective; or Mechanical Conſtruction of the Theory thereof, for a Demonſtration to the Sight." Which, he tells us, "is the next thing neceſſary for the young deſigner to be acquainted with, and to have often before his eyes." The Man is, ſurely, beſide himſelf, with his deep ſtudy of Optics; or intoxicated with the ſmall ſhare of knowledge he has acquired in that branch of it called Perſpective. The Mechaniſm of Perſpective, he ſays, "conſiſts in conſtructing, or building up the theory of perſpective into a real or material form, that the rationale of the ſcience may, as it were, become the object of ſight, and thereby eaſier conceived by the mind."

It is well the ingenious Author has ſo clearly explained what the mechaniſm of Perſpective is; as *I* could never have formed an Idea of it elſe. "In order to this, it will be neceſſary to have a rectangular piece of wood (mahogany) about twenty four inches long, and ſix or eight wide; in this, at about ten inches from the end, is to be made a dove-tail grove, half an inch deep, and wide; and then a frame muſt be made to move or ſlide freely in

the ſaid grove, yet ſo as to be always upright or perpendicular to the ſurface on which it ſtands." I am at a loſs how to ſpeak, or what to ſay of this curious deſcription of his Mechaniſm of Perſpective; a material Fabric, conſtructed of Mahogany, Braſs, Ivory, Horn, Paſtboard, &c. with Threads, repreſenting Viſual rays; an Apparatus to aſſiſt the Imagination, in contemplating the Theory of Perſpective, not the Theory itſelf, which is the Soul of the Apparatus, and exiſts without the Body, or material form, I preſume.

It is not neceſſary, nor is it poſſible to point out the particular excellence of this Conſtruction, which is literally and truly mechanical; Inſtructions for an Artificer how to make it, and of what Materials, as if *that* were material. Suppoſe the whole Fabric of *one* kind, of Wood (Deal, or Wainſcot, no matter which) or of Paſtboard; is it the Materials or the Form, from which we are to gain Information? or, are the dimenſions of the parts neceſſary thereto? If ſo, he has left us in the dark; for, although he is ſo minutely exact in the width and depth of the Groove, he has neither given the dimenſions of the Frame, which is to ſlide in it, nor told us of what materials it muſt be made. The Square OQRT is to be a thin piece of Braſs, having fine holes drilled in the angles and ſides, to faſten Threads to. Now, I ſhould imagine, that if good Paper were paſted over the whole Board, and Lines drawn on it, forming a Square, having the Threads faſtened at each Angle, &c. by ſmall pins or pegs of Wood, in holes made with a fine Awl, the ſame end would be anſwered. The Lines GM and IK might alſo be drawn on the Paper, without any auxiliary help of either Ivory or Paſteboard; but, as the edges of the mahogany Board form theſe Lines, alſo GI and MK, I don't conceive that any other are neceſſary, or can be drawn, there.

"All that has been hitherto ſaid relates to the mechanical conſtruction of the theory; but the practical part of

perſpective may alſo be proved and confirmed by experiments made in the ſame machine, with a little variation;" for which, he ſays, "it ſhould have a round plate of braſs, inſtead of the ſtile HY in form of a wire, with a ſmall hole in the middle, to look through, like a common ſight vane to quadrants, &c." It is, I ſuppoſe, abſolutely neceſſary to be of Braſs, and round, with the Hole in the middle; yet, I ſhould imagine, that if a piece of tin plate, ſquare, and with the Hole ſomewhat on one ſide were at hand, it might anſwer every purpoſe of the other, and full as well; unleſs ornament, rather than utility, is the object aimed at.

I am afraid I have quite exhauſted the Patience of the Reader, for mine is, I freely own; but one quotation more, and I have done; I will follow him no farther. "Then if ſeveral pieces of horn (very ſmooth and tranſparent) were fitted to the frame AC, ſo as to be taken in and out at pleaſure, and upon theſe ſeveral perſpectives of objects drawn, as of a chair, table, inſide of a room, doors open, &c. Then if theſe are ſucceſſively and oppoſitely placed by the ground plane of the original, and viewed through the ſight vane at Y, the prototype and its perſpective will be ſeen perfectly to coincide upon the plate of horn in every point and part; and conſequently the truth of the draught is eſtabliſhed."

'Tis very true, ſo it is, *if* they do coincide; but ſuppoſe they ſhould not, what is to be done? where are your Rules, Mr. Bradberry, for making thoſe Perſpectives of Chairs and Tables, Doors open, &c. upon the horn? I don't find any in your Book for them, nor for any thing but ſquare Figures, on the Ground plane: They are, I ſuppoſe, firſt traced on the Horn, from the Original; and, the truth of it is eſtabliſhed by this coincidence, as ſeen through the Sight hole from which they were drawn, which is, with him, Demonſtration; of *what?*

In the foregoing, then, and in the application of a double parallel Ruler, to his perſpective Table (which, I am of opinion, is not ſo applicable to the purpoſe as he imagines) with the uſe of Reticulation, applied to Drawing and taking Views, deſcribed, conſiſts the whole of Mr. Bradberry's extenſive Practice; which, he tells us, "is univerſal for all objects remote or near." So, here is a genuine Theory of Perſpective (which is the moſt natural and perſpicuous that poſſibly can be) contained in a puerile deſcription of an Apparatus for illuſtration; and applied in an extenſive Practice, without any Rules for drawing, at all, but by mechaniſm; and all this for the inconſiderable price of five Shillings; though embelliſhed with three Plates, of Diagrams only, which might, probably, coſt him little ſhort of one Guinea; very moderate indeed. If merely to get Money be not the motive, it is ſcarcely poſſible to conceive what inducement any perſon could have for publiſhing ſuch a Work; and yet, I ſhould not have imagined, that the name of BRADBERRY is ſo famed as to depend on *that*, for the ſale of ſuch a Work as this: HE is not a *Ferguſon*, or a *Walker*, yet.

When ſuch mercenary and ſordid motives, added to the vanity of becoming an Author, are ſo predominant, as to ſtimulate Men to publiſh to the World their ill-digeſted performances; it is not to be wondered at, that the Preſs teems with ſo many and ſuch monſtrous Productions. But, how any Man can be affected with the empty Vanity of writing on, and publiſhing ſuch Subjects as he is by no means competent in, and but very little converſant with, or has abilities for (as is ſo eminently conſpicuous in this Eſſay) is moſt unaccountable, and entirely beyond my comprehenſion.

The firſt Figure in the Plate, being wholly practical, its meaning is obvious, at one glance. But, I am at a loſs to deviſe why the large Circle, ſeen obliquely, is reticulated thus, in preference to Squares; as I don't

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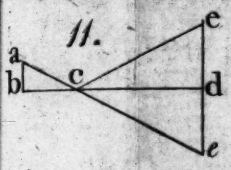
ſee any uſe in it. (It is a juſt copy). Is the diſtance (yX) which is uſed here, ſuch as Mr. Bradberry approves, and would recommend, for drawing theſe Objects, at the height yF? in which, the indefinite repreſentation of O3, a ſide of the Square, is (3X) a perpendicular Line; although the original is inclined to the Ground line, AD, in an Angle of 45 Degrees; if not, why has he not taken a greater diſtance, having room enough on the Plate?

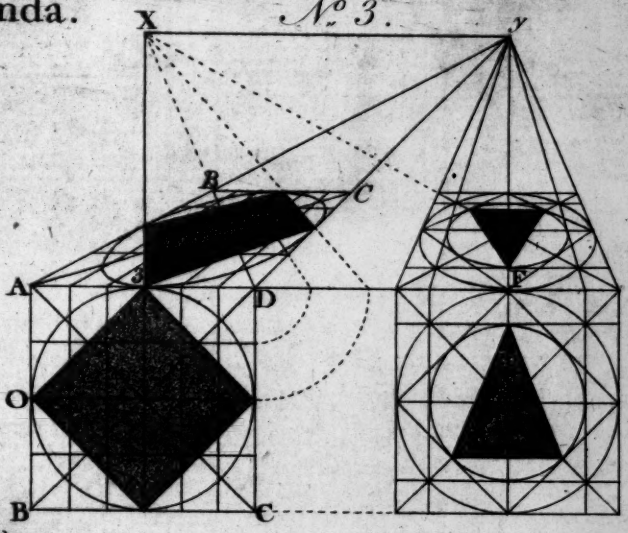
The ſmall theoretic Figure (10) has ſome glaring imperfections. Is KH meant for a horizontal Line? what does EH, cuting it, mean? and IG, cuting FH? I apprehend, that E1, in the Horizontal plane, is meant to be paralel to the bottom line of the Picture, BP; yet they don't appear as ſuch; a hint is enough to Mr. Bradberry. Fig. 11. is intended, I ſuppoſe, to demonſtrate (or to ſhew, at leaſt) that the Angle of Incidence of a Ray of light, is equal to that made by the reflected Ray; for, unfortunately, I have no deſcription of it, in my imperfect Copy. I have alſo to lament, that, of the laſt Section (The Principles of Spherical Perſpective, demonſtrated) I have no more than three Pages, out of ſixteen; ſo that I can ſay nothing of it; in which, he ſeems to diſplay a greater depth of Erudition than I ſhould have imagined, from the foregoing ſpecimen; he ſeems to be quite at Home, among the Stars. I can make no better Apology to my Readers for the omiſſion, in not extending theſe Strictures to the end of the Book; which, is the moſt extraordinary Production, *of the kind*, I have ever ſeen.

There are two coloured ſketches, Ovals ($4\frac{1}{4}$ by $3\frac{1}{4}$ Inches) paſted on blank Leaves, by way of Frontiſpiece, intended for Views of South Bridge Street, and George Street, Edinburgh, merely for ſhow; what elſe Mr. Bradberry could have in view, from ſuch paltry Scrawls prefacing his Work, I cannot deviſe. They ſhew, however, what great *little* things may be effected by the drawing Apparatus; not by the ſtudy of Perſpective, from his Book, as nothing can be done from it, by Rule.

THE END.

Addenda. *No. 3.* *End.*





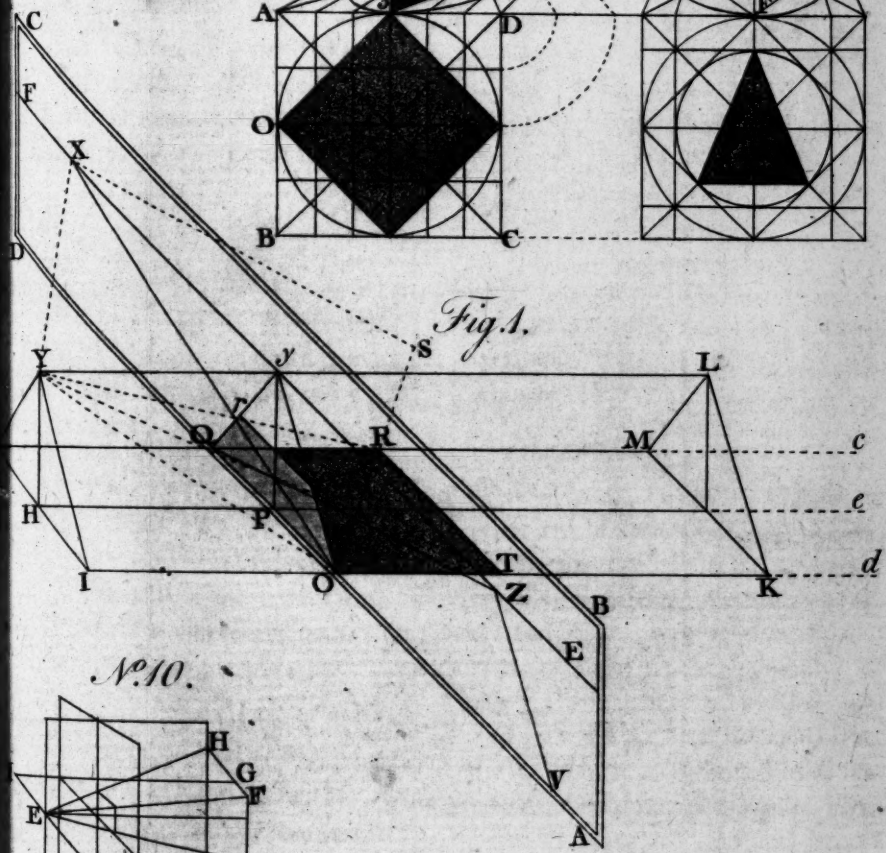


Fig. 1.

No. 10.

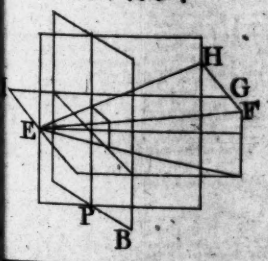
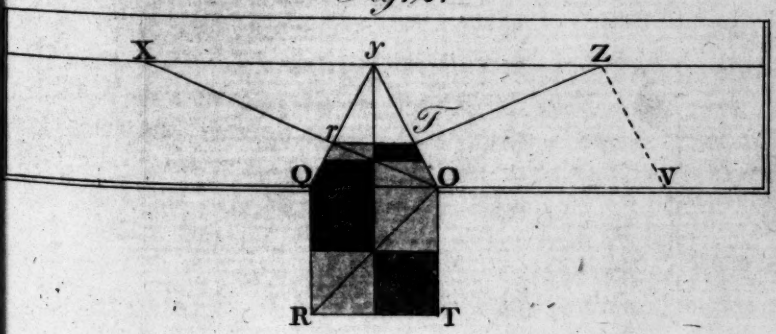


Fig. 2.



POSTSCRIPT.

SINCE the whole of the Addenda was printed off, ſundry impediments protracted the publication of this Work; until, by accident, turning over ſome ſmall Tracts, and Eſſays, on various ſubjects, I chanced to look into one of Martin's, Optician, on Perſpective. On peruſing the Title, it immediately ſtruck me, to be the ſame with Bradberry's. I was ſomewhat ſurprized; but, turning over the leaf, I read the Preface, which is alſo, verbatim, in Mr. Bradberry's, he having omited the laſt Paragraph, of five Lines, only; which tells us, that that work was only the recreation of his (Martin's) leiſure hours, which he knew not how to turn to better account; together with a greek Proverb (in the foregoing Paragraph, that, "a great Book is a great evil;" and, the latin adage, "*Verbum ſat ſapienti*" (a word to the wiſe) and, "an Engliſh genius can ſee Day at a little hole;" which, Mr. Bradberry's modeſty would not ſuffer him to add, to *his* Preface: There is ſome addition of his own. I was amazed; but, on looking further, I found the Contents the ſame, verbatim, and in the ſame order; Martin having nine Chapters, Bradberry (omiting the eighth) has *but* eight, and he calls them Sections; with the Appendix, concerning, the difference between optical and perſpective planes; and of the appearances of Objects upon them. The eighth Chapter which Bradberry has omited, wholly, is a Demonſtration that Perſpective is only a branch of the ſcience of Optics, &c; almoſt the principal Section, on which, being ſo famous in Optics, he ſhould have deſcanted largely. Why it is omited, I leave the public to judge of, I ſhall not preſume; though I think it is not very difficult to be accounted for.

Examining further, I found that he had copied the whole Work, almoſt verbatim; only, here and there, omiting a ſentence; as, after theſe words (in the Preface) "I have purpoſely avoided all kind of Prolixity, and ſtudied one Point only, here, *as in all my other Writings*; *viz.*" Mr. Bradberry's delicate, bluſhing Modeſty not permitting him to inſert thoſe words; it being, I apprehend, his firſt Eſſay, as an AUTHOR. On the only Point ſtudied, I have added a Note (ſee Page 205) which this diſcovery has verified, and confirmed; being well informed, that he had the unparalled aſſurance to ſolicit and teaze his Acquaintance to ſubſcribe for the Work, at the very reaſonable price, of *five Shillings*; the whole Work being, not pirated from, but, a literal Copy of Mr. Martin's. I have often heard talk of Plagiariſm, but never of

ſuch an one as this, before. What kind of ſentiment and feeling muſt ſuch Men have; and how do ſuch Impoſters deſerve to be treated? The whole of the Letter, and the Diagrams, alſo, being exactly copied, without the leaſt variation or tranſpoſition; no varying the Expreſſions, to give an appearance of being his own; the very references are by the ſame Letters, and every, even the moſt palpable Error truly and faithfully copied. Martin has but one Plate, Bradberry has made three of it, without an additional Diagram, of his own. Mr. Martin being ſomewhat famous, as an Optician, he imagined (like Mr. Kirby, reſpecting Hogarth's Line of beauty) that "he could not treat the Subject better than *he* has done;" and therefore, thought it ſafeſt not to riſque any thing of his own, but to publiſh the whole of Martin's tract for his. I had given him credit to know ſomething; but am now well convinced, that this wonderful Man knows nothing at all of Perſpective; and, I apprehend, as little of Optics. Poor Ferguſon; I aſk his Manes pardon, for rating his abilities, in Perſpective, ſo *very* low, as to put Bradberry in competition with him. See Page 203.

The attentive and critical Reader may think it ſtrange, that, having wrote Strictures on all the Engliſh Authors, on Perſpective; and that, Martin being in repute, as an Author, on various Sciences, particularly in Optics; and having treated on Perſpective, as a branch of Optics, I ſhould be ſo impoſed on, as to criticize, on Bradberry's production, as his own. I muſt own, that it may ſeem ſomewhat ſtrange; but, let them conſider, that my ſtrictures are on ſuch Authors, only, as had wrote Treatiſes on the ſubject, not trifling Eſſays; of which, thoſe who have my Appendix (which, has been made public, though never advertiſed, ſome years) may be ſatisfied. Although I had Martin's tract by me, when I wrote thoſe Strictures, I thought it ſo inſignificant a work as to deſerve not four Lines to be ſaid on it, in theſe words. "The late, ingenious Mr. Martin publiſhed a ſmall Eſſay; with only one plate, in order to explain the Principles of Perſpective. He was a voluminous writer; and therefore, it cannot be expected that he ſhould be thoroughly verſed in every Subject; of which, this Eſſay is a manifeſt proof." Several other ſmall Tracts, by Painters, and even Mathematicians, of great repute, as Emerſon, Muller, Ozanam, are noticed only in the ſame manner.

Wherefore, as I thought ſo ſlightly of Martin's Tract, at that time, and having never ſeen, nor thought of it ſince, 'tis not ſtrange that it ſhould make ſo little impreſſion on my memory, as not to retain any Idea of it,

ſo long. Nor ſhould I have taken notice of Mr. Bradberry (as the Author of it) now, but for the arrogance of the language, in ſundry expreſſions, and treating it as a ſubject of little conſequence, even to the Artiſt; that, I thought 'twas incumbent on me to defend it from the obloquy of ſuch pretenders to Science. From Mr. Martin, indeed, who was in repute, as an Author, and a ſcientific Man, it might come with a better grace, than from Mr. Bradberry; yet, I muſt ſay, that the language of *that* work could not redound to his credit; and, the manner in which he has treated the ſubject evinces that he was by no means competent therein; and therefore, he was highly deſerving of cenſure. Moſt probably, had I noticed thoſe paſſages in the work, which are ſo very exceptionable, I ſhould have taken more notice of it, then, as being more dangerous from a man of Mr. Martin's known abilities. However, I have made him amends, now, for that neglect, and have treated him (through Mr. Bradberry) as he deſerves; and, in ſo doing, have verified the old adage, of killing two Birds with one ſtone.

A quarto Work has been publiſhed, ſince the commencement of the year 1793, by a Mr. Sheraton, Cabinet-maker, called the Cabinet-maker and Upholſterers Drawing Book; which ſeems to intimate its being of no uſe to any other; and is, indeed, encouraged chiefly by them, Subſcribers for it. A large part of the Book is on Perſpective, applied to Furniture, chiefly; without any improvement propoſed, either by rendering the ſubject clearer, or more eaſily applicable.

I ſhould not have taken notice of this production, had not the Author given ſome glaring proofs of his incompetency, in Perſpective, by dogmatically taking up, and endeavouring to ſupport an argument on a point which had, long ſince, been debated on, and ſettled; viz. that the repreſentation of a Circle, ſeen obliquely, is not a regular Ellipſis; and attempts to ſhew, that Mr. Noble (who ſettled that point, with the Critical Reviewers, ſatisfactorily) is under a miſtake; and for the ſame reaſon that others have alledged, viz. that the repreſentation of the hither Semicircle is larger than the other; and ſo, concludes that they are not of equal curviture. What ſeems to miſlead this Author (as it has done others) is, that the Axes of the Ellipſis are not the repreſentations of any Diameters of the Circle; nor the center of one, the center of the other. It would be ſtrange if they were, and contradictory to the principles of Perſpective; but that is no proof that 'tis

not a perfect Ellipſis, in every caſe, and in every ſituation in which a Circle can be ſeen, but when the Eye is perpendicularly over its center; and, when the ſection of the Optic cone, by the Picture, is ſubcontrary; though, in the former, only, it appears a Circle, in the latter, an Ellipſis.

But he goes further; and boldly objects to the oblique ſection of a Cone, being a true Ellipſis; becauſe, the curviture of a Cone is continually various, from the Baſe to the Vertex; and thence imagines, that the lower half of the Section muſt be flatter in its curviture than the other, nearer to the Vertex. "A Cone (he ſays, Page 51) is a ſolid which terminates to a point at the top; and therefore any ſection oblique to its baſe muſt produce an oval, broader at one end than the other;" not conſidering, that the plane of the Section is more inclined to the ſide of the Cone, at the lower end, than at the upper, and occaſions an equal curviture, at the extremes of the tranſverſe Axis. Probably, the ſeveral writers on Conic ſections have not well conſidered this matter; future writers would do well to reconſider it, and benefit themſelves of this Author's obſervations thereon. Or, that Mr. Sheraton, *re*-conſiders it, with attention, before he publiſhes a ſecond Edition; and, if he ſhould find himſelf under a ſmall miſtake, that he will not be aſhamed to acknowledge it, candidly; it would redound more to his credit than to perſiſt and perſevere in error, though it may not appear quite clear to *him*; but, as acquieſcing in the general concurrence of ſo many ſcientific Men, of greater repute in the world of Science.

There are ſome few inaccuracies in this work; particularly, in taking Mouldings, for Pediments (Plate 5, fig. 35); in which, the Author, I am perſuaded, is ſomewhat miſtaken. Alſo, in the 11th figure, (Pl. 17) there are ſome unneceſſary lines, in order to determine the Vaniſhing line of a Plane inclined, obliquely, to both the Horizon and Picture. In Plate 26 (the laſt figure) I apprehend, that the reflected image of the inclined poſt (a b) ſhould be meaſured from the ſurface of the Water, not the Ground; as it would be various, according as the ground is more or leſs above the water.

If Mr. Sheraton had done what is propoſed, in Page 71, to find any number of mean proportional lines, between two given ones, at equal diſtances from each other, it would be a curious and moſt extraordinary problem, indeed; but he ſhews only, how, by an unneceſſary proceſs, to divide one of them into nine equal parts from the other, which is to be firſt divided into ten.

MVSEVM BRITAN NICVM